Economathematics

Problem Sheet 2

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1. Calculate Ee^X where X is a gaussian random variable with mean μ and volatility $\sigma > 0$.

2. Verify that
$$\int_0^t W_s^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s ds,$$
where W is a Wiener process.

3. Find the stochastic differential equation (SDE) which is satisfied by the process
$$X(t) = e^{\sigma W_t}$$
for the volatility $\sigma > 0$.

4. Show that the process
$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s}, \qquad t \in [0,1],$$
is the solution of
$$dX_t = \frac{-X_t}{1-t}dt + dW_t, \qquad X_0 = 0.$$

5. Show that $X_t = \sinh(t + W_t)$ is the solution of
$$dX_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)dt + \sqrt{1+X_t^2}dW_t, \qquad X_0 = 0.$$
Recall that $\sinh(x) = \frac{e^x - e^{-x}}{2}$.

6. Find the value μ, such that the process $X_t = \mu t + W_t$, where W_t is the Wiener process, is a martingale.

7. (30 points) Assets of a company can be described by the Brownian motion $X_t = X_0 + \mu t + \sigma W_t$ with $\mu = 1,5$ and $\sigma = 4$. Find the initial capital X_0 such that the probability of bankruptcy of the company in the first year is less than $0,05$. We say that bankruptcy happens whenever X_t gets below 0 any time between 0 and 1.

8. Assume that the price of one-year future contract on gold is $F = 500$USD for ounce, spot price $S = 450$USD, interest rate $r = 7\%$, storage costs $U = 2$USD per year. How much can an investor earn?

9. (20 points) Prove that the stock price governed by the binomial tree converges to the stock prices of Black-Scholes model. What kind of assumptions should be made? Are they unique?

10. Suppose that X is an integrable random variable and $\{\mathcal{F}_t\}$ is chosen filtration. Prove that $M(t) = E[X|\mathcal{F}_t]$ is uniformly integrable martingale.

11. Find dZ_t when:

    (i)
$$Z_t = e^{\alpha W_t};$$

    (ii) $Z_t = X_t^2$, where X solves the following SDE:
$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

12. Using Feynman-Kac formula solve the following PDE:
$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0;$$
$$F(T,x) = x^2.$$

13. Prove that
$$X_t = e^{\alpha t} x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW_s$$
solves the following SDE:
$$dX_t = \alpha X_t dt + \sigma dW_t,$$
$$X_0 = x_0.$$

14. Consider the standard Black-Scholes model. An innovative company, Z, has produced the derivative "Logarithm", henceforth abbreviated as the L. The holder of a L with maturity time T, denoted as L(T), will, at time T, obtain the sum $\log S_T$. Note that if $S(T) < 1$ this means that the holder has to pay a positive amount to Z. Determine the arbitrage free price process for the L(T).

15. Consider the standard Black-Scholes model. Find the arbitrage free price for $X = (S_T)^\beta$ where T is a maturity date.

16. A so called binary option is a claim which pays a certain amount if the stock price at a certain date T falls within some prespecified interval $[\alpha, \beta]$. Otherwise nothing will be paid out. Determine the arbitrage free price.

17. Find the arbitrage free price of $X = S_T / S_{T_0}$ for Black-Scholes market with expiry date T.

18. Do the same for $X = \frac{1}{T-T_0} \int_{T_0}^{T} S_u \, du$.

19. Consider corporation Ideal inc., whose stocks in Euro are given by the following SDE:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t^1.$$

The exchange rate Y_t PLN/Euro is described by:

$$dY_t = \beta Y_t dt + \delta Y_t dW_t^2,$$

where W^1 and W^2 are independent Wiener processes. Broker Ideal Inc. creates the derivative

$$X = \log \left[Z_T^2 \right]$$

with maturity date T, where Z is the stock price given in złoty. Find the arbitrage free price X (in PLN) assuming that r is a spot rate of złoty.

20. Consider the following financial market:

$$dB_t = rB_t dt, \qquad B_0 = 1,$$
$$dS(t) = \alpha S_t dt + \sigma S_t dW(t) + \delta S_{t-} dN_t,$$

where N is a Poisson process with intensity λ which is independent of the Wiener process W.

(i) Is this market arbitrage free ?

(ii) Is it complete ?

(iii) Does exist unique martingale measure ?

(iv) Suppose that we want to replicate European call option with the maturity date T. Is it possible to hedge it using portfolio consisting of the risk-free instrument B, the basic instrument S and European call option with expiry date $T - \delta$ for fixed $\delta > 0$?

21. Prove the following theorem.
Consider the following financial market:

$$dB_t = rB_t dt, \qquad B_0 = 1,$$
$$dS_t = \alpha(t, S_t) S_t dt + \sigma(t, S_t) S_t dW(t) + \delta S_{t-} dN_t$$

and the claim

$$X = \Phi(S_T, Z_T)$$

with the expiry date T and

$$Z_t = \int_0^t g(u, S_u) \, du.$$

Then X can be replicated in the following way:

$$\phi_t^1 = \frac{F(t,S_t,Z_t) - S_t F_s(t,S_t,Z_t)}{F(t,S_t,Z_t)},$$
$$\phi_t^1 = \frac{S_t F_s(t,S_t,Z_t)}{F(t,S_t,Z_t)},$$

where F solves the following boundary problem:

$$\begin{cases} F_t + srF_s + \frac{1}{2}s^2\sigma^2 F_{ss} + gF_z - rF & = & 0, \\ F(T,s,z) & = & \Phi(s,z). \end{cases}$$

The value process equals

$$F(t,s,z) = e^{-r(T-t)} E_{t,s,z}^{Q} \left[\Phi(S_t, Z_t) \right],$$

where Q-dynamics is described by the following SDEs:

$$\begin{aligned} & dS_u = rS_u du + S_u \sigma(u,S_u) dW_u, \\ & S_t = s, \\ & dZ_u = g(u,S_u) du, \\ & Z_t = z. \end{aligned}$$

22. Consider the Black-Scholes model and the derivative asset:

$$X = \begin{cases} K & S_T \leqslant A, \\ K + A - S_T & A < S_T < K + A, \\ 0 & S_T > K + A. \end{cases}$$

Replicate this derivative using portfolio consisting of bond, asset S and European call option. Find the arbitrage free price for X.

23. Do the same for

$$X = \begin{cases} K - S_T & 0 < S_T \leqslant K, \\ S_T - K & K < S_T. \end{cases}$$

24. Do the same for

$$X = \begin{cases} B & S_T > B, \\ S_T & A \leqslant S_T \leqslant B, \\ A & S_T < A. \end{cases}$$

25. Do the same for

$$X = \begin{cases} 0 & S_T < A, \\ S_T - A & A \leqslant S_T \leqslant B, \\ C - S_T & B \leqslant S_T \leqslant C, \\ 0 & S_T > C, \end{cases}$$

where $B = (A + C)/2$.

26. Consider the Black-Scholes model. A two-leg ratchet call option has the following features. At time $t = 0$ an initial strike K_0 is set. At time T_1, the strike is reset to K_1 being the asset value at time T_1. At the time $T > T_1$, the holder receives the payoff of a call with strike K_1 and the amount $(K_1 - K_0)^+$. Find the value option at time t depending if $t \leqslant T_1$ or $t > T_1$.

27. Let the stock prices S^1 and S^2 be given as the solutions to the following system of SDEs:

$$\begin{aligned} dS_t^1 &= \alpha S_t^1 dt + \delta S_t^1 dW_t^1, \qquad S_0^1 = s_1, \\ dS_t^2 &= \beta S_t^1 dt + \gamma S_t^2 dW_t^2, \qquad S_0^3 = s_2. \end{aligned}$$

The Wiener processes W^1 and W^2 are assumed to be independent. The parameters $\alpha, \delta, \beta, \gamma$ are assumed to be known and constant. Your task is to price *a minimum option*. This claim is defined by

$$X = \min \left[S_T^1, S_T^2 \right].$$

The pricing function for a European call option in the Black-Scholes model is assumed to be known, and is denoted by $C(s, t, K, \sigma, r)$ where σ is the volatility, K is the strike price and r is the short rate. You are allowed to express your answer in terms of this function, with properly derived values for K, σ and r.

28. Consider two dates, T_0 and T, with $T_0 < T$. *A forward-start call option* is a contract in which the holder receives, at time T_0 (at no additional cost), a European call option with expiry date T and exercise price equal to S_{T_0}. Write down the terminal payoff, i.e. the payoff at time T, of a forward-start call option and then determine its arbitrage free price at time $t \in [0, T_0]$.