Economathematics

Problem Sheet 3

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1. Prove that if X has a gaussian distribution $N(m, \sigma^2)$ then

$$E\left(e^{aX}\mathbf{1}_{\{X\geqslant k\}}\right) = e^{am+a^2\sigma^2/2}\Phi(d),$$

where $d = \sigma^{-1}(-k + m + a\sigma^2)$ and $\Phi(x)$ is a distribution function of the standard gaussian random variable.

2. Using above problem find the price of an European call option in the Black-Scholes market.

3. Prove that if X has a gaussian distribution $N(m, \sigma^2)$ then the random variable $Y = e^X$ has a mean $e^{m+\sigma^2/2}$ and variance $e^{2m+\sigma^2}(e^{\sigma^2} - 1)$.

4. In the binomial lattice model (BLM), the price of asset at time n equals $S_n = S_0 \prod_{i=1}^{n} Y_i$ where Y_i are i.i.d. r.v.s. distributed as $P(Y = u) = 1 - P(Y = d) = p$ for $d < 1+r < u$ and r being an interest rate. Check that for any fixed time t we can re-write Black-Scholes continuous time asset price S_t as a similar i.i.d. product by dividing the interval (0, t] into n equally sized subintervals $(0, t/n], (t/n, 2t/n], \ldots, ((n-1)t/n, t]$. Defining $t_i = it/n$, and $L_i = S_{t_i}/S_{t_{i-1}}$ the random variable L_i can be approximated by Y_i (give some arguments based on CLT). What u, d, p should we choose (assume that additionally $ud = 1$)? Recall how the risk-neutral probability p looks like. How is it related with SDE defining S_t in Black-Scholes model under the martingale measure?

5. Find the expression for Δ in the Black-Scholes market.

6. Find the expression for Γ in the Black-Scholes market.

7. Find the expression for $\mathcal{V}$ in the Black-Scholes market.

8. Find the expression for ρ in the Black-Scholes market.

9. We give here heuristic (imprecise, original Black-Scholes) proof of Black-Scholes differential equation. Consider portfolio with one option (long position) and some amount Δ (in practice later is not fixed in time !) of underlying asset (short position). Its price we denote as Π. It is equal to

$$\Pi = V(S, t) - \Delta S,$$

where $V(S, t)$ is option price for asset S, and S denote price of underlying asset. Using Itô formula check that random part in formula for $\mathrm{d}\Pi$ is $(\frac{\partial V}{\partial S} - \Delta)\mathrm{d}S$. We can delete risk if $\Delta = \frac{\partial V}{\partial S}$ (*delta hedging*). Show that, assuming no-arbitrage condition on market, price of our hedging portfolio satisfy $\mathrm{d}\Pi = r\Pi \mathrm{d}t$, where $r > 0$ is interest rate. Derive the *Black-Scholes equation*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

if $\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}W$, where W is a standard Brownian motion.

10. Derive equation for option price on stock S which pays dividend D continuously (e.g. in the same, short way as above).

11. Derive equation for option price on currency assuming continuous interest rates of level r and r'. That is, in holding the foreign currency we receive interest at the foreign rate of interest r'.

12. Derive equation for option price on row material assuming constant cost of storage (*cost of carry*) which equals to q. To be precise, for each unit of the commodity held an amount $qS_t\, dt$ will be required during short time dt to finance the holding.

13. Derive equation for option price on futures contract. Recall that the future price of a non-dividend paying equity F is related to the spot price by

$$F = e^{r(T-t)} S_T$$

where T is the maturity date of the futures contract.

14. Find formula for option price on asset with continuous dividend D. To do that, substitute $S'_t = S_t e^{Dt}$ and observe that the European call option price is the price for the basic call option substituting $S'_t e^{-D(T-t)}$ in the place of S_t.

15. Assume that some bank sells 10^6 European call options. Assume that the starting price of underlying asset is $S_0 = 50$, strike is $K = 52$, $r = 2,5\%$, $T = 1/3$ and $\sigma = 22,5\%$. Calculate Black-Scholes price for this option. Consider two positions, described below, which bank can take and for each calculate its net premium at maturity T:

    Covered position: At time 0 bank buys 10^6 underlying assets by price S_0. When this position can be profitable?

    Naked position: At maturity T bank buys 10^6 underlying assets and sells them to options holders. When this position can be profitable?

16. Consider uncertain but fixed parameters. Derive the bounds in the Black-Scholes formula when volatility lies within the band $\sigma \in [\sigma^-, \sigma^+]$.

17. Similarly, derive the bounds when interest rate $r > 0$ lies within the band $r \in [r^-, r^+]$.

18. Derive the bounds for the option on currency when foreign interest rate $r_f > 0$ lies within the band $r_f \in [r_f^-, r_f^+]$.

19. Derive the bounds when asset pays dividend D in continuous way and D lies within band $D \in [D^-, D^+]$.

20. Consider American put option without maturity (*perpetual American put*) - see e.g. chapter *9 Early exercise and American option* from P. Wilmott book 'Paul Wilmott on Quantitative Finance' mentioned during lecture. Let V be the price function. Why we can assume that V does not depend on time? Why function V must satisfy following condition
$$V(S) \geqslant \max(E - S, 0),$$
where E is strike price?

21. Let V be price of *perpetual American put*. Prove that V satisfy following equation
$$\frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 V}{\mathrm{d}S^2} + rS\frac{\mathrm{d}V}{\mathrm{d}S} - rV = 0 \tag{2}$$
if S follows Black-Scholes model. General solution of (2) is given by $V(S) = C_1 S + C_2 S^{-2r/\sigma^2}$, where C_1, C_2 are constants. Show that for perpetual American put we have: $C_1 = 0$ and $C_2 = \frac{\sigma^2}{2r}\left(\frac{E}{1+\sigma^2/2r}\right)^{1+2r/\sigma^2}$ (to do this find point S^* which $V(S^*) = \max_{S \geqslant S^*} V(S)$).

22. Consider *perpetual American call* with price function V. Assume the continuous dividend D. Show that function V satisfies
$$\frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 V}{\mathrm{d}S^2} + (r - D)S\frac{\mathrm{d}V}{\mathrm{d}S} - rV = 0. \tag{3}$$
Show that general solution of (3) is $V(S) = AS^{\alpha^+} + BS^{\alpha^-}$ for constants A, B and
$$\alpha^{\pm} = \frac{1}{\sigma^2}\Big(- (r - D - \frac{1}{2}\sigma^2) \pm \sqrt{(r - D - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}\Big).$$
For perpetual American call $V(S) = AS^{\alpha^+}$. Find A and optimal time to exercise S^*. From that notice that for dividend equal to zero the optimal time is infinity.

23. Show that price for *perpetual American put* with continuous dividend D is given by $V(S) = BS^{\alpha^-}$. Find constant B and point for optimal exercise S^*.

24. Show that for price C of American call option with maturity T and strike E the following inequality is satisfied
$$C \geqslant S - Ee^{-r(T-t)}.$$

25. Consider American options put and call with prices P and C, with the maturity T and strike E. Prove that
$$C - P \leqslant S - Ee^{-r(T-t)}.$$
Additionally show that $S - E \leqslant C - P$ (here we ignore influence of interest rate).

26. Let $V = V(t;T)$ be a price of an obligation with the deterministic interest rate $r = r(t) > 0$ and the maturity date T. Additionally we assume that the bond has coupon payments with respect to the function $K(t)$. If the bond at time T pays X, what is the value of $V(T;T)$? If we have one bond in our portfolio, the change in time $\mathrm{d}t$ is

$$\Big(\frac{\mathrm{d}V}{\mathrm{d}t} + K(t)\Big)\mathrm{d}t.$$

Using no-arbitrage condition show, that V fulfills the following equation:

$$\frac{\mathrm{d}V}{\mathrm{d}t} + K(t) = r(t)V.$$

Additionally, using the boundary condition, prove that the solution is of the following form:

$$V(t;T) = e^{-\int_t^T r(\tau)\mathrm{d}\tau}\Big(X + \int_t^T K(s)e^{\int_s^T r(\tau)\mathrm{d}\tau}\mathrm{d}s\Big). \tag{4}$$

27. Consider that we have now a zero-coupon bond, i.e. $K(t) = 0$. From the equation (4) we have that

$$V(t;T) = Xe^{-\int_t^T r(\tau)\mathrm{d}\tau}.$$

Assume that the function $V(t;T)$ is differentiable with respect to T. Prove that

$$r(T) = -\frac{1}{V(t;T)}\frac{\partial V}{\partial T}.$$

What we can deduce from above equation?

28. Assume that we would like to hedge price of bond A, which YTM is equal to y_A, using another bond B, with YTM y_B. Assume that change in time of y_A imply proportional change of y_b, i.e. $\mathrm{d}y_A = c \cdot \mathrm{d}y_B$, for some constant c. Assume that we have bond A and some Δ bonds B in our portfolio

$$\Pi = V_A(y_A) - \Delta V_B(y_B).$$

How we should choose Δ to hedge against YTM's changes?

29. Prove that the solution for the short term return rate r_t in the Vasicek model:

$$dr_t = (a - br_t)dt + \sigma dW_t$$

has the following form:

$$r_t = r_s e^{-b(t-s)} + \frac{a}{b}\left(1 - e^{-(bt-s)}\right) + \sigma\int_s^t e^{-b(t-u)}dW_u. \tag{5}$$

30. Prove that in the Vasicek model (5) the conditional law of r_t with respect of the natural history $\mathcal{F}_s$ is gaussian with the conditional expectation:

$$E[r_t|\mathcal{F}_s] = r_s e^{-b(t-s)} + \frac{a}{b}\left(1 - e^{-b(t-s)}\right)$$

and the conditional variance:

$$\mathrm{Var}[r_t|\mathcal{F}_s] = \frac{\sigma^2}{2b}\left(1 - e^{-2b(t-s)}\right).$$

31. Prove that

$$\lim_{t\to\infty} E[r_t|\mathcal{F}_s] = \frac{a}{b}$$

and

$$\lim_{t\to\infty} \mathrm{Var}[r_t|\mathcal{F}_s] = \frac{\sigma^2}{2b}.$$

32. Using partial differential equations find the price of zero-coupon bond in the Vasicek model.

33. Let the volatility coefficients $b(\cdot, T)$ and $b(\cdot, U)$ of the zero-coupon bonds be bounded functions. Prove that for $0 \leqslant t < T$ the arbitrage price of European call option with expiration time $T > 0$ and strike price $K > 0$ on the bond with maturity date $U \geqslant T$ is given by:

$$C_t = B(t,U)\mathrm{N}(h_1(B(t,U),t,T)) - KB(t,T)\mathrm{N}(h_2(B(t,U),t,T)),$$

where

$$h_{1/2}(b,t,T) = \frac{\log(b/K) - \log B(t,T) \pm \frac{1}{2}v_U^2(t,T)}{v_U(t,T)}$$

for

$$v_U^2(t,T) = \int_t^T |b(u,U) - b(u,T)|^2 du.$$

34. Assume that the asset price under the spot martingale measure spot has the following evolution:

$$dS_t = S_t(r_t dt + \sigma(t) dW_t),$$

where σ is a bounded function. Prove that if the volatility $b(\cdot, T)$ is bounded then the arbitrage price of call option is given by:

$$C_t = S_t\mathrm{N}(h_1(S_t,t,T) - KB(t,T)\mathrm{N}(h_2(S_t,t,T))),$$

where

$$h_{1/2}(b,t,T) = \frac{\log(b/K) - \log B(t,T) \pm \frac{1}{2}v_S^2(t,T)}{v_S(t,T)}$$

for

$$v_S^2(t,T) = \int_t^T |\sigma(u) - b(u,T)|^2 du.$$

35. Assume that we can take derivative under the expectation sign. Prove that forward return rate is related with short rate via:

$$f(t,T) = \frac{E_{P^*}[r(T)\exp\{-\int_t^T r_s ds\}]}{E_{P^*}[\exp\{-\int_t^T r_s ds\}]},$$

where P^* is the spot martingale measure. Hence indeed we have $r_t = f(t,t)$.

36. Consider two sides: A and B, that signed the following contract. A invests K in the financial instrument that gives return rate R. After time T A pays B the amount $K_A - K$ where K_A is a investment value of A after time T. Similarly, B invests K in the financial instrument with stochastic return rate r_t and pays its value after time T to A. Find the swap rate R.

37. Find stationary distribution of the interests rate in the Vasicek model. Prove that is density solves invariant Fokker-Planck equation (without increment with $\partial/\partial t$).

38. In consolidated bonds we pay a unit at time dt. In other words, its price can be described as follows:

$$C(t) = \int_t^\infty B(t,u)\, du.$$

Assume that bond price solves the following SDE:

$$dB(t,T) = B(t,T)r_t dt + B(t,T)b(t,T)dW_t.$$

Prove that C solves:

$$dC(t) = (C(t)r_t - 1)dt + \sigma_C(t)dW_t\,,$$

where $\sigma_C(t) = \int_t^\infty B(t,u)b(t,u)\, du$.

39. Consider the national (PLN) and foreign (EUR) bonds $B_d(t,T)$ and $B_f(t,T)$. Assume that both satisfy HJM model with forward rates f_d and f_f:

$$df_d(t,T) = \alpha_d(t,T)dt + \sigma_d(t,T)dW_t,$$

$$df_d(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dW_t.$$

Let the exchange rate X (PLN/EUR) has the following dynamics:

$$dX(t) = \mu(t)X(t)dt + X(t)\sigma_X(t)dW_t.$$

Prove that under the martingale measure of the national currency (PLN) the foreign forward rate satisfies the following drift condition:

$$\alpha_f(t,T) = \sigma_f(t,T)\left(\int_t^T \sigma_f(t,u)du - \sigma_X(t)\right).$$

40. Assume, that dynamic of interest rate r is given by following stochastic differential equation:

$$\mathrm{d}r = u(r,t)\mathrm{d}t + w(r,t)\mathrm{d}W, \tag{6}$$

where W is standard Brownian motion, u and w are some set functions. Let $V(r,t;T)$ denote price of bond at time t with interest rate r and with maturity T. Consider portfolio Π of bond with maturity T_1 and $-\Delta$ of bond with maturity T_2:

$$\Pi = V_1 - \Delta V_2,$$

where V_i is the price at T_i ($i = 1,2$). Using no-arbitrage property ($\mathrm{d}\Pi = r\Pi\mathrm{d}t$), Itô formula and choosing appropriate Δ show that

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2}{\frac{\partial V_2}{\partial r}}. \tag{7}$$

Assume, that left and right hand side of equation (7) do not depend on T, so we can eliminate indexes and write

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}} = a(r,t)$$

for some function a. Show that we can rewrite a to $a(r,t) = \lambda(r,t)w(r,t) - u(r,t)$ for some function λ. Taking this a we can rewrite BS formula to

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0. \tag{8}$$

41. Consider portfolio with only one bond with price $V(r,t;T)$. Calculate $\mathrm{d}V$ from Itô formula and using equation (8) show, that

$$\mathrm{d}V - rV\mathrm{d}t = w\frac{\partial V}{\partial r}(\mathrm{d}X + \lambda\mathrm{d}t).$$

How we can interpret the $\lambda(r,t)$ function (it is *market price of risk*)?

42. When deriving Black-Scholes formula we construct portfolio with option and $-\Delta$ of asset. This time consider portfolio with two options (with prices $V_1(S,t)$, $V_2(S,t)$) and different maturities (or different strikes). We have $\Pi = V_1 - \Delta V_2$. Using the same argument as before show that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (\mu - \lambda_S\sigma)S\frac{\partial V}{\partial S} - rV = 0. \tag{9}$$

Note that it has the same form like equation (8). Function $V = S$ has to fulfill equation (9) (why?). Rewriting the formula for $V = S$ we get

$$\lambda_S = \frac{\mu - r}{\sigma}.$$

This is *market price of risk for asset*. What we will get when we use this λ_S in equation (9)?

43. Assume that the solution of (8) is of form $V(r,t;T) = e^{A(t;T)-rB(t:T)}$. Use this function in (8) to rewrite the Black-Scholes formula:

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \frac{1}{2}w^2B^2 - (u - \lambda w)B - r = 0. \tag{10}$$

Taking second order derivative with respect to r show that

$$\frac{1}{2}B\frac{\partial^2(w^2)}{\partial r^2} - \frac{\partial^2(u-\lambda w)}{\partial r^2} = 0.$$

Prove that

$$\begin{cases} \frac{\partial^2(w^2)}{\partial r^2} &= 0 \\ \frac{\partial^2(u-\lambda w)}{\partial r^2} &= 0. \end{cases}$$

Solve above equations. Show that

$$\begin{aligned} u(r,t) - \lambda(r,t)w(r,t) &= \eta(t) - r\,\gamma(t) \qquad (11) \\ w(r,t) &= \sqrt{r\,\alpha(t) + \beta(t)}. \qquad (12) \end{aligned}$$

for some functions α, β, γ i η.

44. Using (11) and (12) in (10) derive formulas for A and B:

$$\begin{aligned} \frac{\partial A}{\partial t} &= \eta(t)B - \frac{1}{2}\beta(t)B^2 \qquad (13) \\ \frac{\partial B}{\partial t} &= \frac{1}{2}\alpha(t)B^2 + \gamma(t)B - 1. \qquad (14) \end{aligned}$$

From the boundary condition deduce that $A(T;T) = B(T;T) = 0$.

45. Assume that α, β, γ i η are constant. Solve equations (13) and (14).

46. Consider the asset price process of the following form:

$$\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}W_1,$$

where σ is volatility with dynamics given by

$$\mathrm{d}\sigma = p(S,t,\sigma)\mathrm{d}t + q(S,t,\sigma)\mathrm{d}W_2,$$

where W_1, W_2 are two standard Brownian motions and $\mathbb{E}(\mathrm{d}W_1\mathrm{d}W_2) = \rho\,\mathrm{d}t$ (correlation is ρ). Consider portfolio with two options V, V_1:

$$\Pi = V - \Delta S - \Delta_1 V_1.$$

Using no-arbitrage property, using Itô formula and taking appropriate Δ i Δ_1 derive Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma Sq\frac{\partial^2 V}{\partial S\partial\sigma} + \frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2} + rS\frac{\partial V}{\partial S} + (p - \lambda q)\frac{\partial V}{\partial\sigma} - rV = 0. \tag{15}$$

The function λ is called *market price of volatility risk*.

47. (*Heston model*)
Consider model

$$\mathrm{d}S = \mu S \mathrm{d}t + \sqrt{\nu} S \mathrm{d}W_1$$

where

$$\mathrm{d}\nu = (\theta - \nu)\kappa\, \mathrm{d}t + c\sqrt{\nu}\mathrm{d}W_2, \tag{16}$$

with parameters μ, κ, θ, c and assume that $\mathbb{E}(\mathrm{d}X_1 \mathrm{d}X_2) = \rho\, \mathrm{d}t$. Show that

$$\frac{\partial V}{\partial t} + \mathcal{L}V - rV = 0, \tag{17}$$

where

$$\mathcal{L}V = \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho c \nu S \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2} c^2 \nu \frac{\partial^2 V}{\partial \nu^2} + rS\frac{\partial V}{\partial S} + \Big((\theta - \nu)\kappa - c\sqrt{\nu}\lambda(S, t, \nu)\Big)\frac{\partial V}{\partial \nu}.$$

48. Derive the Black-Scholes formula for $X = \ln S$ for S given in the previous problem.