PROBLEMS FOR THE COURSE ATDG

1. Consider a DTMC on the state space \( X = \{0, 1, 2\} \) with the transition matrix

\[
P = \begin{bmatrix}
0.7 & 0.2 & 0.1 \\
0 & 0.6 & 0.4 \\
0.5 & 0 & 0.5
\end{bmatrix}.
\]

Determine the conditional probabilities \( \Pr(X_3 = 1 | X_0 = 0) \) and \( \Pr(X_4 = 1 | X_0 = 0) \).

If it is known that the process starts in state \( X_0 = 1 \), determine the probability \( \Pr(X_2 = 2) \).

2. Consider a DTMC on the state space \( X = \{0, 1, 2, 3\} \) with the transition matrix

\[
P = \begin{bmatrix}
0.4 & 0.3 & 0.2 & 0.1 \\
0.1 & 0.4 & 0.3 & 0.2 \\
0.3 & 0.2 & 0.1 & 0.4 \\
0.2 & 0.1 & 0.4 & 0.3
\end{bmatrix}.
\]

Suppose that the initial distribution is \( \alpha_i = 1/4 \) for \( i = 0, 1, 2, 3 \). Show that \( \Pr(X_n = k) = 1/4 \) for all \( k = 0, 1, 2, 3 \) and all \( n \).

3. Suppose that \((X_n)\) is a DTMC on \( X = \{0, 1\} \) and with the transition matrix

\[
P = \begin{bmatrix}
\alpha & 1 - \alpha \\
1 - \beta & \beta
\end{bmatrix}.
\]

Then, \( Z_n = (X_{n-1}, X_n) \) is a DTMC having four states: \((0, 0)\), \((0, 1)\), \((1, 0)\) and \((1, 1)\). Determine its transition matrix.

4. The weather in Koszalin is classified as sunny, rainy, cloudy. Suppose, that tomorrow’s weather depends only on today’s weather as follows: if it sunny today it is cloudy tomorrow with probability 0.3 and rainy with probability 0.2; if it is cloudy today it is sunny tomorrow with probability 0.5 and rainy with probability 0.3; if it is rainy today, it is sunny tomorrow with probability 0.4 and cloudy with probability 0.5. Model the weather process as an DTMC. Is this chain is irreducible, aperiodic? Does it exist a stationary distribution? Does it exist a limiting distribution? If yes, find them.

5. Mr. and Mrs. Nowak have planned to celebrate their wedding anniversary in Koszalin. Counting today as the first day they are supposed to be there on 7th and 8th days. They are thinking about buying insurance that promises to reimburse them for the entire vacation package cost 2500 PLN if it rains on both days and nothing is reimbursed otherwise. The insurance costs 200 PLN. Suppose the weather in Koszalin changes according to the above-mentioned model. Assuming that it is sunny today in Koszalin should they buy the insurance?

6. ComputerNet stocks a wide variety of PCs for retail sale. It is open for business Monday through Friday 8 a.m. to 5 p.m. It uses the following operating policy to control the inventory of PCs. At 5 p.m. Friday the store clerk checks to check how many PCs are still in stock. If the number is less than two, then he orders enough
PCs to bring the total in stock up to five at the beginning of the business day Monday. If the number in stock is two or more, no action is taken. The demand for PCs during the week is a Poisson random variable with mean three. Any demand that cannot be immediately satisfied is lost. Provide a state space and transition matrix $P$ for this Markov chain. What happens when $n \to \infty$?

7. The Gadgets company has a manufacturing setup consisting of two distinct machines, each producing one component per hour. Each component is tested instantly to be identified as defective or non-defective. Let $a_i$ be the probability that a component produced by machine $i$ is non-defective. The defective components are discarded while the non-defective components are stored in separate bins. When a component is present in each bin, the two are instantly assembled together and shipped out. Each bin can hold at most two components. When the bin is full, the corresponding machine is turned off. It is turned on again when the bin has space for at least one component. Model this manufacturing system as a DTMC.

8. Suppose that both bins are empty at time 0 at the beginning of 8-hour shift. What is the probability that both bins are empty at the end of the 8-hour shift? Set $a_1 = 0.99$ and $a_2 = 0.995$.

9. Is this chain irreducible, aperiodic? Does it exist a stationary distribution? Does it exist a limiting distribution? If yes, find them.

10. Mr. Mac is a coffee addict. He keeps switching between three brands of coffee, say A, B or C, from week to week according to a DTMC with the transition matrix

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.1 & 0.5 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}.$$ 

If he is using brand A this week (i.e., week 1), what is the probability distribution of the brand he will be using in week 10?

11. Classify the DTMCs with the transition matrices as reducible or irreducible:

a) $P = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.15 & 0.25 & 0.35 & 0.25 \end{bmatrix}$  

b) $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  
c) $P = \begin{bmatrix} 0.1 & 0 & 0.9 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.25 & 0 & 0.75 \end{bmatrix}$

12. Classify the irreducible DTMCs from the above example as periodic or aperiodic. Classify the following irreducible DTMCs as periodic or aperiodic:

a) $P = \begin{bmatrix} 0 & 0.2 & 0.3 & 0.5 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$  
b) $P = \begin{bmatrix} 0 & 0 & 0 & 0.4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

13. Compute a normalised solution to the balance equation for the DTMC in points a), b), d), e). When possible compute the limiting distribution, stationary distribution, occupancy distribution.
14. Find the mean time to reach state 3 starting from state 0 for the DTMC on the state space \{0, 1, 2, 3\}, if the transition matrix is as follows: 

\[ P = \begin{bmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.07 & 0.2 & 0.1 \\ 0.0 & 0.9 & 0.1 \\ 0.0 & 0 & 0 & 1 \end{bmatrix}. \]

15. Consider a DTMC whose transition matrix is 

\[ P = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}. \]

Determine the probability that the Markov chain ends in 0, if the initial state is 1. Determine the mean time to absorption.

16. Consider a DTMC whose transition matrix is 

\[ P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

Determine the probability that the Markov chain ends in 0, if the initial state is 1. Determine the mean time to absorption.

17. A coin is tossed repeatedly until two successive heads appear. Find the mean number of tosses required, if the first coin toss results in a tail. \textbf{Note:} Let \( X_n \) be the cumulative number of successive heads at time period \( n \).

18. A coin is tossed repeatedly until either two successive heads appear or two successive tails appear. Suppose the first toss coin results in a head. Find the probability that the game ends with two successive tails.

19. Consider a DTMC whose transition matrix is 

\[ P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

Determine the mean time that the process spends in state 1 prior to absorption and the mean time that the process spends in state 2 prior to absorption. Verify that the sum of these is the mean time to absorption.

20. Consider a DTMC with the following transition matrix:

\[ P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.3 & 0.3 \end{bmatrix}. \]

Determine the probability that the process never visits state 2, if it starts in state 1.
21. A white rat is put in compartment 4 of the following maze (see above). It moves through the compartment at random (if there are \( k \) ways to leave a compartment, it chooses each of these with probability \( 1/k \)). What is the probability that it finds the food in compartment 3 before feeling the electric shock in compartment 7?

22. Let \( X_n \) be a DTMC with transition probabilities \( p_{ij} \). We are given a discount factor \( \beta \in (0, 1) \) and a cost function \( c(i) \) paid in state \( i \). We wish to determine the total expected discounted cost starting from state \( i \), defined by

\[
h(i) = E \left[ \sum_{n=0}^{\infty} \beta^n c(X_n) | X_0 = i \right].
\]

Using a first step analysis show that \( h(i) \) satisfies the system of linear equations

\[
h(i) = c(i) + \beta \sum_j h(j)p_{ij}.
\]

23. Consider a DTMC whose transition matrix is

\[
P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}.
\]

It is known that the process starts in state \( X_0 = 0 \). Eventually the process will end up in state 2. What is the probability that when the process moves into state 2, it does from state 1?
24. Find all the Nash equilibria for the following game. Draw the best response diagram and then conclude the Nash equilibria.

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<tr>
<td>U</td>
<td>6,0</td>
<td>5,3</td>
</tr>
<tr>
<td>D</td>
<td>6,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

25. Find all Nash equilibria for the following game:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-4,5</td>
<td>3,-3</td>
<td>2,-5</td>
<td>-1,4</td>
<td>1,1</td>
</tr>
<tr>
<td>2</td>
<td>3,-3</td>
<td>-6,7</td>
<td>6,4</td>
<td>0,0</td>
<td>3,-2</td>
</tr>
</tbody>
</table>

Calculate the expected payoffs for the players when they play equilibrium.

26. (Finite dynamic game I) The game is played with three cards: King, Ten and Deuce (Two). Ann chooses one and lays it face down on the table. Peter calls 'High' or 'Low'. If he is right (King=High, Deuce=Low), he wins 3 from Ann (Ann loses 3, she earns -3), but if he is wrong he loses 2 (and Ann gets 2). If the card is Ten he wins 2 if he called Low, but Ann must choose between the King and Deuce if he called High. This time after she lays her card face down on the table and he calls High or Low, he wins 1, if he is right, but loses 3 if he is wrong. Draw the tree for this game. Find the pure strategies for the players and the payoff matrix. Find equilibrium strategies, the payoffs in equilibrium strategies.

27. (Finite dynamic game II) Two Russians guards buy a black market American sports kit. It consists of a carton of Larks, a Colt 45 six shooter, one bullet and rules for Russian roulette. Each player antes1 a pack of cigarettes. As senior officer A plays first, he may add two packs to the pot and pass the gun to B or he may add one pack, spin the cylinder, test fire at his own head and then hand the gun to B. If and when the B gets the gun, he has the same options; he may handed back duly adding two packs or he may add one pack and try the gun. The game is now ended. Each picks up half a pot, if feasible; else the bereaved takes it all. Draw the tree for this game. Find the pure strategies for the players and the payoff matrix. Find equilibrium strategies, the payoffs in equilibrium strategies.

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1 a bet made before any action