Limits in the homogeneous setting
Selected methods of potential theory–Probability mini-course, Part 4

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Plan: Discuss [BKLP23] and related ideas

1. Fractional Laplacian $\Delta^{\alpha/2} = -(-\Delta)^{\alpha/2}$
2. Dirichlet problem
3. Cones and limits
4. Ornstein-Uhlenbeck semigroup [BJKP], [BKLP23]

We fix $d \geqslant 1$ and $0 < \alpha < 2$. Let

$$\nu(y) = c|y|^{-d-\alpha}, \quad y \in \mathbb{R}^d \,.$$

Here c is such that

$$\int_{\mathbb{R}^d \setminus \{0\}} [1 - \cos(\xi \cdot y)]\, \nu(y) dy = |\xi|^\alpha \,, \quad \xi \in \mathbb{R}^d \,.$$

Define $\quad p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t|\xi|^\alpha} d\xi \,, \quad x \in \mathbb{R}^d \,,\; t > 0 \,.$

Scaling: $p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x)$.

Let $P_t f = f * p_t$. We have, e.g., for $f \in C_0^2(\mathbb{R}^d)$,

$$\lim_{t \to 0^+} \frac{P_t f(x) - f(x)}{t} = \lim_{\varepsilon \to 0+} \int\limits_{|y|>\varepsilon} [f(x+y) - f(x)] \nu(dy) =: \Delta^{\alpha/2} f(x).$$

Let Ω be the class of càdlàg functions $X : [0, \infty) \to \mathbb{R}^d$.

For $x \in \mathbb{R}^d$ we define probability $\mathbb{P}_x$ on Ω by

$$\mathbb{P}_x(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) = \int\limits_{B_1} p_{t_1}(x_1 - x) \cdots \int\limits_{B_n} p_{t_n - t_{n-1}}(x_n - x_{n-1})\, dx_n \cdots dx_1.$$

Expectation: $\mathbb{E}_x := \int_\Omega d\mathbb{P}_x$. Let $D \subset \mathbb{R}^d$ be open,

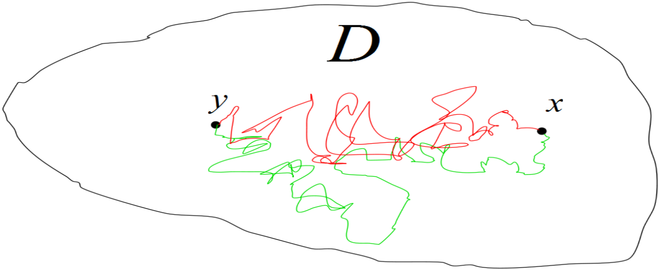
$$\tau_D := \inf\{t > 0 : \ X_t \notin D\} \qquad \text{(exit/ruin time)},$$

$$P_t^D f(x) := \mathbb{E}_x[t < \tau_D;\ f(X_t)] = \int_{\mathbb{R}^d} f(y) p_t^D(x, y) dy,$$

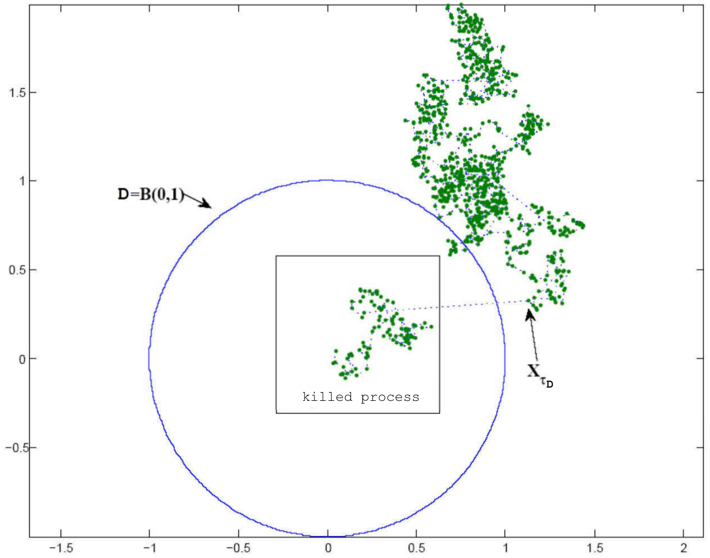
$$G_D(x, y) := \int_0^\infty p_t^D(x, y) dt.$$

Dirichlet heat kernel=transition density of the process killed off D

$$p_t^D(x,y) := p_t(y-x) - \mathbb{E}_x\left[\tau_D \leqslant t;\ p_{t-\tau_D}(y - X_{\tau_D})\right]$$



Simulated trajectory $t \mapsto X_t$, for $\alpha = 1.8$, $d = 2$



Connection to generator:

For $s \in \mathbb{R}$, $x \in \mathbb{R}^d$, $\phi \in C_c^\infty(\mathbb{R} \times D)$, we have

$$\int_s^\infty \int_D p_{t-s}^D(x,z) \left(\partial_t + \Delta_y^{\alpha/2}\right) \phi(t,y)\, dy\, dt = -\phi(s,x)\,.$$

Approximating $\phi(t,y) := \varphi(y) \in C_c^\infty(D)$, we get

$$\int_D G_D(x,y) \Delta^{\alpha/2} \varphi(y) dy = -\varphi(x)\,.$$

Glossary of formulas for Dirichlet conditions (killing/stopping on D^c):

$G_D(x,y) := \int_0^\infty p_t^D(x,y)dt$

$G_D f(x) := \int_D G_D(x,y)f(y)dy = \mathbb{E}_x \int_0^{\tau_D} f(X_t)dt$

$\omega_D^x(A) := \mathbb{P}_x[X_{\tau_D} \in A] = \int_A \int_D G_D(x,y)\nu(z-y)dydz$ (etc.)

$u(x) := \mathbb{E}_x g(X_{\tau_D})$ is harmonic/a solution to Dirichlet problem:

$$\Delta^{\alpha/2} u = 0 \text{ on } D, \quad u = g \text{ on } D^c.$$

$u(x) := \mathbb{E}_x g(X_{\tau_D}) - G_D f(x)$ solves inhomogeneous Dirichlet problem:

$$\Delta^{\alpha/2} u = f \text{ on } D, \quad u = g \text{ on } D^c.$$

If $u(x) = \mathbb{E}_x g(X_{\tau_D}) - G_D f(\cdot, u(\cdot))(x)$, $x \in D$, then

$$\Delta^{\alpha/2} u(x) = f(x, u(x)), \quad u = g \text{ on } D^c, \text{ etc.}$$

Glossary of formulas for Dirichlet conditions (continued).

$$u(t,x) := \int_D p_t^D(x,y)f(y)dy = \mathbb{E}_x\left[t < \tau_D;\, f(X_t)\right]$$

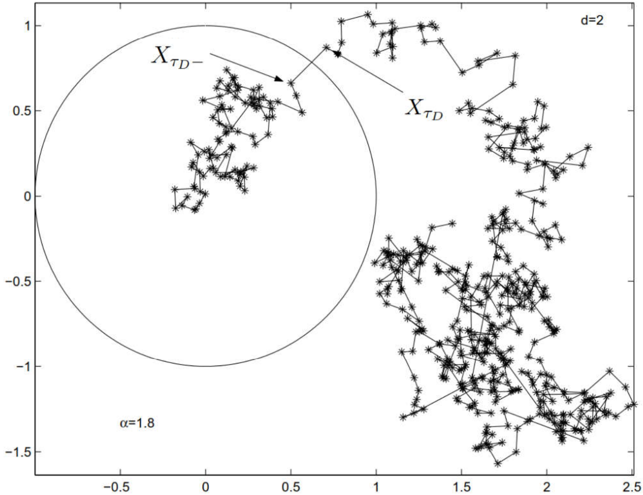
solves $\partial_t u = \Delta_x^{\alpha/2} u$ on $(0,\infty)\times D$, $u(0+,\cdot) = f$ on D, $u(t,\cdot) = 0$ on D^c.

$$u(t,x) := \mathbb{E}_x\left[t < \tau_D;\, f(X_t)\right] + \mathbb{E}_x\left[t \geqslant \tau_D;\, g(X_{\tau_D})\right]$$

solves $\partial_t u = \Delta_x^{\alpha/2} u$ on $(0,\infty)\times D$, $u(0+,\cdot) = f$ on D, $u(t,\cdot) = g$ on D^c.

$$u(t,x) := \mathbb{E}_x\left[t < \tau_D;\, f(X_t)\right] + \mathbb{E}_x\left[t \geqslant \tau_D;\, g(X_{\tau_D})\right] + \mathbb{E}_{t,x}\int_t^{\tau_D} k(s,X_s)ds$$

solves $\partial_t u - \Delta_x^{\alpha/2} u = k, \quad u(0+,\cdot) = f, \quad u(t,\cdot) = g$, etc.



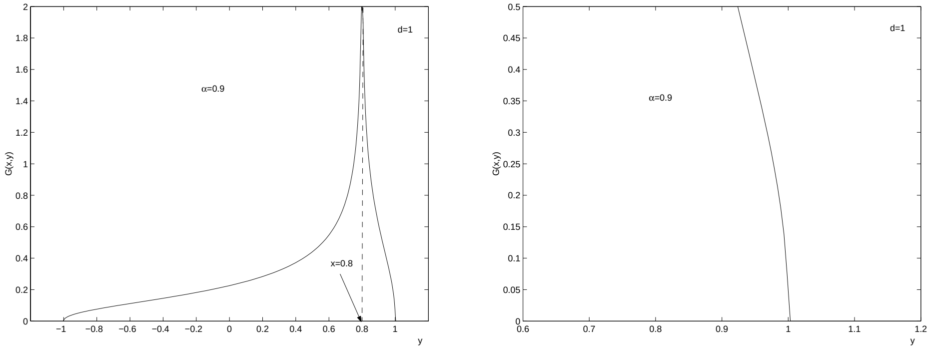
Ikeda-Watanabe formula:

$$\mathbb{P}_x[X_{\tau_D-} \in dy, X_{\tau_D} \in dz, \tau_D \in dt] = p_t^D(x,y)\,\nu(z-y)dydzdt.$$

M. Riesz' formula for the ball

M. Riesz 1938; Blumenthal, Getoor, Ray 1961:
Let $x, y \in B(0,1) \subset \mathbb{R}^d$, and $w = (1-|x|^2)(1-|y|^2)/|x-y|^2$. Then,

$$G_{B(0,1)}(x,y) = \mathcal{B}_{d,\alpha}\,|x-y|^{\alpha-d}\int_0^w \frac{r^{\alpha/2-1}}{(r+1)^{d/2}}\,dr\,.$$



[BG10, BGR10, BGR14]:

For many (jump processes, times t, points x, y and) sets $D \subset \mathbb{R}^d$,

$$p_t^D(x,y) \approx \mathbb{P}^x(\tau_D > t)\ p_t(y-x)\ \mathbb{P}^y(\tau_D > t).$$

Note that

$$P^x(\tau_D > t) = \int_D p_t^D(x,y)dy,$$

and

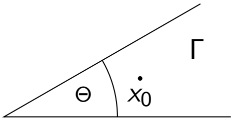
$$p_t(x) \approx \frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}, \quad t > 0,\ x \in \mathbb{R}^d.$$

History:
BHP,
estimates of the Green function,
estimates of the heat kernel,
subordinated BM, unimodal Lévy processes, Markov jump processes, ...

Program in progress:

Calculate *limits*, when $x \to \partial D$...

Consider open cone $\Gamma \in \mathbb{R}^d$.



Let $x_0 \in \Gamma$.

[BB04]: For $y \in \mathbb{R}^d$, the following limits exist

$$M(y) = \lim_{\Gamma \ni x \to \infty} \frac{G_\Gamma(x,y)}{G_\Gamma(x,x_0)} ,$$

$$K(y) = \lim_{\Gamma \ni x \to 0} \frac{G_\Gamma(x,y)}{G_\Gamma(x,x_0)} .$$

Example: If $\Gamma = (0,\infty) \subset \mathbb{R}^1$, then $M(y) = y_+^{\alpha/2}$ and $K(y) = y_+^{\alpha/2-1}$.

All cones: $M(y) = |y|^\beta M(y/|y|)$ and $K(y) = |y|^{\alpha-d-\beta} K(y/|y|)$.

[BSS15]: $\beta = \alpha - c\Theta^{d-1+\alpha} + l.o.t.$, when $\Theta \to 0$.

Heat kernel of the cone [BG10, BGR10] (Example):

For the circular cones $\Gamma \subset \mathbb{R}^d$:

$$p_t^\Gamma(x,y) \approx \frac{\left(1 \wedge \frac{\delta_\Gamma(x)}{t^{1/\alpha}}\right)^{\alpha/2}}{\left(1 \wedge \frac{|x|}{t^{1/\alpha}}\right)^{\alpha/2-\beta}} \; p_t(y-x) \; \frac{\left(1 \wedge \frac{\delta_\Gamma(y)}{t^{1/\alpha}}\right)^{\alpha/2}}{\left(1 \wedge \frac{|y|}{t^{1/\alpha}}\right)^{\alpha/2-\beta}}\,.$$

Here:
$t > 0$, $x, y \in \mathbb{R}^d$,
$\delta_\Gamma(x) = \mathrm{dist}(x, \Gamma^c)$,
$\beta \in (0, \alpha)$ is the homogeneity exponent of the Martin kernel M for Γ.

Theorem (Yaglom limit fo Lipschitz Γ) [B., Z. Palmowski, L. Wang, 2018]

$$\mathbb{P}_x\left(t^{-1/\alpha}X_t \in A \,|\, \tau_\Gamma > t\right) \to n(A) \quad \text{as } t \to \infty.$$

This is how *scaling* works:

$$\mathbb{P}_x\left(t^{-1/\alpha}X_t \in A \,|\, \tau_\Gamma > t\right) = \frac{\mathbb{P}_x\left(\tau_\Gamma > t,\ t^{-1/\alpha}X_t \in A\right)}{\mathbb{P}_x(\tau_\Gamma > t)}$$

$$= \frac{\mathbb{P}_{t^{-1/\alpha}x}\left(\tau_\Gamma > 1,\ X_1 \in A\right)}{\mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1)}$$

$$= \int_A \frac{p_1^\Gamma(t^{-1/\alpha}x, y)dy}{\mathbb{P}_{t^{-1/\alpha}x}(\tau_\Gamma > 1)} \to \int_A n(y)dy =: n(A).$$

Step 1

If f is bounded and integrable, then

$$\lim_{\Gamma \ni x \to 0} \frac{G_\Gamma f(x)}{M(x)} = C \int_\Gamma K(y) f(y) dy < \infty.$$

Step 2

For $x \in \Gamma$, let $\kappa_\Gamma(x) = \int_{\Gamma^c} \nu(y-x)dy$. We get

$$\mathbb{P}_x(\tau_\Gamma > 1) = \int_1^\infty \int_\Gamma \int_{\Gamma^c} p_t^\Gamma(x,y)\nu(z-y)dzdydt$$

$$= \int_0^\infty \int_\Gamma \int_\Gamma p_t^\Gamma(x,w)p_1^\Gamma(w,y)dw\,\kappa_\Gamma(y)dydt$$

$$= G_\Gamma P_1^\Gamma \kappa_\Gamma(x).$$

Step 3

$P_1^\Gamma \kappa_\Gamma$ is bounded.

$$P_1^\Gamma \kappa_\Gamma(x) \approx \mathbb{P}_x(\tau_\Gamma > 1) \int_\Gamma \mathbb{P}_y(\tau_\Gamma > 1)\,(1+|y|)^{-d-\alpha}\,\kappa_\Gamma(y)dy, \quad |x| \leqslant 1\,.$$

Step 4

$$\lim_{\Gamma \ni x \to 0} \frac{\mathbb{P}_x(\tau_\Gamma > 1)}{M(x)} = \lim_{\Gamma \ni x \to 0} \frac{G_\Gamma\left(P_1^\Gamma \kappa_\Gamma\right)(x)}{M(x)} = C \int_\Gamma K(y) P_1^\Gamma \kappa_\Gamma(y)dy\,.$$

Step 5

Let $\phi \in C_0^\infty(\Gamma)$ and $u_\phi = -\Delta^{\alpha/2}\phi$. Then $G_\Gamma u_\phi = \phi$ and

$$P_1^\Gamma \phi = P_1^\Gamma G_\Gamma u_\phi = G_\Gamma P_1^\Gamma u_\phi\,.$$

Step 6

$$\lim_{\Gamma\ni x\to 0}\frac{P_1^\Gamma\phi(x)}{M(x)}=\lim_{\Gamma\ni x\to 0}\frac{G_\Gamma P_1^\Gamma u_\phi(x)}{M(x)}=C\int_\Gamma K(y)P_1^\Gamma u_\phi(y)dy\,.$$

Step 7

Let $y\in\Gamma$ and $\phi(\cdot)=p_1^\Gamma(\cdot,y)\in C_0^\infty(\Gamma)$. By Chapman-Kolmogorov:

$$p_2^\Gamma(x,y)=\int_\Gamma p_1^\Gamma(x,z)p_1^\Gamma(z,y)dz=P_1^\Gamma\phi(x)\,,\quad x\in\Gamma\,.$$

Step 8

$$\frac{p_2^\Gamma(x,y)}{\mathbb{P}_x(\tau_\Gamma>1)}=\frac{P_1^\Gamma\phi(x)}{\mathbb{P}_x(\tau_\Gamma>1)}=\frac{G_\Gamma P_1^\Gamma u_\phi(x)}{\mathbb{P}_x(\tau_\Gamma>1)}\text{ converges as }\Gamma\ni x\to 0.$$

Theorem (Summary)

$$n_t(y) := \lim_{\Gamma \ni x \to 0} \frac{p_t^\Gamma(x,y)}{\mathbb{P}_x(\tau_\Gamma > 1)} < \infty \quad \text{for all} \quad t > 0,\ y \in \Gamma,$$

$$n_1(y) \approx \frac{\mathbb{P}_y(\tau_\Gamma > 1)}{(1+|y|)^{d+\alpha}},$$

$$n_t(y) = t^{-(d+\beta)/\alpha} n_1(t^{-1/\alpha} y),$$

$$n_{t+s}(y) = \int_\Gamma n_t(z) p_s^\Gamma(z,y) dz.$$

Remarks: the Yaglom limit measure is $n(dy) = n_1(y)dy$
$n_t(y)$ is the entrance law density for the killed process on Γ;
$n_t(y)$ is a self-similar solution of the (Dirichlet) heat equation in Γ.

Example 1

For $\Gamma = (0,\infty) \subset \mathbb{R}$ we have $\beta = \alpha/2$, $\mathbb{P}_y(\tau_\Gamma > 1) \approx y_+^{\alpha/2} \wedge 1$ and $n_t(y) = t^{-1-\alpha/2} n_1(t^{-1/\alpha} y)$, where

$$n_1(y) = n(y) \approx (y_+^{\alpha/2} \wedge 1)/(1+|y|)^{1+\alpha}, \quad y \in \mathbb{R}.$$

Example 2

Let $d = \alpha = 1$ and $\Gamma = (0,\infty)$. Then X_t is the symmetric Cauchy process on $\mathbb{R}$. Let, as usual, $\tau_\Gamma = \inf\{t \geqslant 0 : \ X_t < 0\}$ (ruin time). For $x > 0$, let

$$r(x) = \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{t}{(1+t^2)^{5/4}} \exp\left(\frac{1}{\pi}\int_0^t \frac{\log s}{1+s^2} ds\right) e^{-tx} dt$$

and $\psi(x) = \sin\left(x + \frac{\pi}{8}\right) - r(x)$. Then the Yaglom limit μ has the density

$$n(y) = \lim_{x\to 0+} \frac{p_1^\Gamma(x,y)}{\mathbb{P}_x(\tau_\Gamma > 1)} = \sqrt{\frac{\pi}{2}} \int_0^\infty \lambda^{1/2} \psi(\lambda y) e^{-\lambda} d\lambda\,, \quad y > 0\,.$$

Auxiliary semigroups (1): Doob's conditioning

Theorem ([BJKP])

For inner-fat cones the Martin kernel M is invariant for P_t^Γ:

$$\int_{\mathbb{R}^d} p_t^\Gamma(x,y)M(y)dy = M(x), \qquad t>0,\ x\in\Gamma\,.$$

We define Doob-conditioned (renormalized) kernel:

$$\rho_t(x,y) := \frac{p_t^\Gamma(x,y)}{M(x)M(y)}, \quad t>0, \quad x,y\in\Gamma.$$

With the weight M^2 we get:

$$\int_\Gamma \rho_t(x,y)M^2(y)dy = 1, \quad x\in\Gamma, \quad t>0,$$

$$\int_\Gamma \rho_s(x,y)\rho_t(y,z)M^2(y)dy = \rho_{t+s}(x,z), \quad x,z\in\Gamma,\ s,t>0.$$

We note the scaling, too: $\rho_{sr}(r^{1/\alpha}x, r^{1/\alpha}y) = r^{\frac{-2\beta-d}{\alpha}}\rho_s(x,y)$.

Auxiliary semigroups (2): Ornstein-Uhlenbeck semigroup

We define a *transition density* with respect to $M^2(y)dy$:

$$\ell_t(x,y) := \rho_{1-e^{-t}}(e^{-t/\alpha}x, y), \qquad t > 0, \quad x, y \in \Gamma.$$

Indeed, by Chapman-Kolmogorov and scaling of ρ,

$$\begin{aligned}
&\int_\Gamma \ell_s(x,y)\ell_t(y,z)M^2(y)dy \\
&= \int_\Gamma \rho_{1-e^{-s}}(e^{-s/\alpha}x, y)\rho_{1-e^{-t}}(e^{-t/\alpha}y, z)M^2(y)dy \\
&= \int_\Gamma \rho_{1-e^{-s}}(e^{-s/\alpha}x, y)(e^{-t})^{\frac{-2\beta-d}{\alpha}}\rho_{e^t-1}(y, e^{t/\alpha}z)M^2(y)dy \\
&= (e^{-t})^{\frac{-2\beta-d}{\alpha}}\rho_{e^t-e^{-s}}(e^{-s/\alpha}x, e^{t/\alpha}z) = \rho_{1-e^{-s-t}}(e^{-(s+t)/\alpha}x, z).
\end{aligned}$$

For suitable functions (densities) f we define (O-U semigroup):

$$L_t f(y) = \int_\Gamma \ell_t(x,y)f(x)M^2(x)dx.$$

Stationary density

Theorem

There is a unique stationary density φ for the operators L_t, $t > 0$.

"Proof": Let $t = 1$. Note

$$\rho_1(x, y) \approx (1 + |y|)^{-d-\alpha} \frac{\mathbb{P}_y(\tau_\Gamma > 1)}{M(y)}, \quad |x| \leqslant R,\ y \in \Gamma.$$

Consider functions of the form

$$f(y) = \int_{\Gamma_1} \rho_1(x, y)\, \mu(dx), \quad y \in \Gamma,$$

with sub-probabilities μ concentrated on Γ_1 of bounded support. We get

$$f(y) \lesssim (1 + |y|)^{-d-\alpha} \frac{\mathbb{P}_y(\tau_\Gamma > 1)}{M(y)}, \quad y \in \Gamma.$$

We use uniform integrability, Schauder's fixpoint and positivity of ℓ. □

Asymptotic stability

By results of Kulik and Scheutzov [KS15, Theorem 1 and Remark 2], for every $x \in \Gamma$ we get the L^1-convergence

$$\int_{\mathbb{R}^d} |\ell_t(x,y) - \varphi(y)|\, M^2(y)dy \to 0 \text{ as } t \to \infty.$$

By cosmetics, we obtain:

Lemma

We have $\int_\Gamma |\rho_1(x,y) - \varphi(y)|\, M^2(y)dy \to 0$ *as* $x \to 0$.

We bootstrap this to pointwise convergence and continuity of φ.

Continuous extension of $\rho_t(x,y)$

Theorem

$\rho_t(x,y)$ has a continuous extension to $(0,\infty)\times\Gamma\cap\{0\}\times\Gamma\cap\{0\}$.

Proof.

By scaling, Chapman-Kolmogorov, for $\Gamma\ni y, x\to 0$,

$$\begin{aligned}
\rho_1(x,y) &= 2^{\frac{d+2\beta}{\alpha}}\rho_2(2^{1/\alpha}x, 2^{1/\alpha}y)\\
&= 2^{\frac{d+2\beta}{\alpha}}\int \rho_1(2^{1/\alpha}x, z)\rho_1(z, 2^{1/\alpha}y)M^2(z)dz\\
&\to 2^{\frac{d+2\beta}{\alpha}}\int \varphi(z)\rho_1(z, 2^{1/\alpha}y)M^2(z)dz\\
&= \int \varphi(z)\rho_{1/2}(2^{-1/\alpha}z, y)M^2(z)dz = L_{\ln 2}\varphi(y) = \varphi(y).
\end{aligned}$$

Thus, $\rho_1(0,y) := \lim_{x\to 0}\rho_1(x,y) = \varphi(y)$, for $y\neq 0$, etc. □

The self-similar solution $\Psi_t(x)$ (aka Quasistationary distribution/Yaglom limit)

By the theorem, for $t > 0$, $x \in \mathbb{R}^d$,

$$\Psi_t(x) := \lim_{y \to 0} \frac{p_t^\Gamma(y,x)}{M(y)} = \rho_t(0,x)M(x) = t^{\frac{-2\beta-d}{\alpha}} \varphi(t^{-1/\alpha}x)M(x).$$

In particular,

$$\Psi_t(x) = t^{\frac{-\beta-d}{\alpha}} \Psi_1(t^{-1/\alpha}x), \quad t > 0, \quad x \in \mathbb{R}^d.$$

Since $\rho_{t+s}(0,x) = \int_\Gamma \rho_t(0,y)\rho_s(y,x)M^2(y)dy$, we get

$$\int_\Gamma p_s^\Gamma(y,x)\Psi_t(y)dy = \Psi_{t+s}(x).$$

Further, $\int_{\mathbb{R}^d} \Psi_t(x)M(x)dx = \int_{\mathbb{R}^d} \Psi_1(x)M(x)dx = 1$.

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