# MINIMAL NON-UNIQUELY ERGODIC TOPOLOGICAL SYSTEMS

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This presentation is based on two papers: [D2, D3].

Let $(X, \Sigma, \mu)$ be a standard Borel probability space and let $T$ be a measurable measure-preserving transformation from $X$ into itself, i.e., such that $\mu(A) = \mu(T^{-1}(A))$ for every $A \in \Sigma$. Then $(X, \Sigma, \mu, T)$ is called a measure-theoretic dynamical system or an endomorphism. An invertible $T$ is often called an automorphism. A measure-theoretic dynamical system is called ergodic if all $T$-invariant sets (i.e., $A \in \Sigma$ satisfying $T(A) \subset A$) have either measure 1 or 0. Two measure-theoretic dynamical systems $(X, \Sigma, \mu, T)$ and $(X', \Sigma', \mu', T')$ are said to be isomorphic if there exists a bimeasurable bijection $\psi$: $X_0 \to X'_0$, where $X_0 \in \Sigma$, $X'_0 \in \Sigma'$, $\mu(X_0) = \mu'(X'_0) = 1$, which sends the measure $\mu$ to $\mu'$ (i.e., $\mu(A) = \mu'(A')$ whenever $A' = \psi(A)$, $A \in \Sigma$), and which is equivariant, i.e., $\psi \circ T = T' \circ \psi$ $\mu$-almost everywhere. A system isomorphic to an ergodic one is ergodic.

**KEY DEFINITION**
By an **assignment** we will mean a function $\Psi$ defined on an abstract metrizable Choquet simplex $K$, whose "values" are measure-theoretic dynamical systems, i.e., for $p \in K$, $\Psi(p)$ has the form $(X_p, \Sigma_p, \mu_p, T_p)$. Two assignments, $\Psi$ on a simplex $K$, and $\Psi'$ on a simplex $K'$, are said to be **equivalent** if there exists an affine homeomorphism of Choquet simplexes $\pi$: $K \to K'$ such that for every $p \in K$ the systems $\Psi(p)$ and $\Psi'(p')$, where $p' = \pi(p)$, are isomorphic.

By a topological dynamical system we shall mean a pair $(X, T)$, where $X$ is a compact metric space and $T$ is a continuous map of $X$ into itself. A topological dynamical system $(X, T)$ is minimal if for every $x \in X$ the orbit $\{T^n(x) \colon n \in \mathbb{N}\}$ is dense in $X$. In the context of a topological dynamical system $(X, T)$, by a "measure" we will always mean a probability measure on the Borel sigma-field $\mathcal{B}_X$. By $\mathcal{P}_T(X)$ we will denote the collection of all $T$-invariant measures on $X$, i.e., measures $\mu$ preserved by $T$, in other words such that $(X, \mathcal{B}_X, \mu, T)$ becomes a measure-theoretic dynamical system. It is well known that $\mathcal{P}_T(X)$ is a nonempty compact, for the weak* topology of measures, metrizable Choquet simplex whose extreme points are precisely the ergodic invariant measures. A topological dynamical system $(X, T)$ determines a natural assignment on the simplex $\mathcal{P}_T(X)$ "by identity", i.e., by the rule: $\mu \mapsto (X, \mathcal{B}_X, \mu, T)$.

An assignment is called **topological** (**minimal**) if it is equivalent to a natural assignment arising from a topological (minimal) dynamical system.

It is known ([D1]) that minimal (hence topological) assignments exist on every metrizable Choquet simplex. We are interested in the following abstract problem:

*Given a simplex $K$, characterize the topological (minimal) assignments on $K$.*

The renowned Jewett-Krieger theorem solves this problem for the trivial (one-point) simplex and automorphisms; every assignment of an ergodic automorphism can be equivalently realized by a minimal (strictly ergodic) invertible zero-dimensional topological system. A. Rosenthal [R] proved an analogous theorem also for ergodic endomorphisms. Thus we restrict our investigations to nontrivial simplexes.

In this generality there is no known characterization of the minimal assignments. Likewise, there is no characterization of the topological assignments. For example, the following elementary question did not have an answer (until the results presented in this talk have been established):

**Motivation question.** *Does there exist a minimal system with two ergodic measures isomorphic to two irrational rotations, or to the same irrational rotation, or one being for example a rotation, and the other being for example Bernoulli?*

Though, there are some obvious restrictions on such assignments. Let us begin with the most general and obvious ones:

(R1) $\Psi$ assigns ergodic systems to extreme points of $K$;
(R2) $\Psi$ obeys the **ergodic decomposition rule**: if $K \ni p = \int e\, d\xi_p(e)$, where $\xi_p$ is the unique probability measure with barycenter at $p$, supported by the extreme points $e$ of $K$, and $\Psi(p) = (X_p, \Sigma_p, \mu_p, T_p)$, then $\mu_p$ admits a decomposition $\mu_p = \int \mu_e\, d\xi_p(e)$ with each $\mu_e$ ergodic, preserved by the transformation $T_p$, and such that $(X_p, \Sigma_p, \mu_e, T_p)$ is isomorphic to $\Psi(e)$.

These two restrictions apply to all topological assignments and follow from the basics of ergodic theory. They allow us to focus on assignments defined only on the extreme points of simplexes, and associating ergodic measure-preserving transformations.

In minimal assignments another restriction is obvious:

(R3) The assigned measure-theoretic dynamical systems are nonatomic.

Indeed, the atomic part of an invariant measure is supported by finitely many periodic points, so that the only assignments involving atomic measures and realizable in minimal systems are those on trivial simplexes assigning a measure supported by a single periodic orbit. But we have agreed to exclude trivial simplexes from our considerations.

It is obvious that the above list of restrictions is incomplete. There must be some kind of regularity ("measurability" or even "semicontinuity") of the assignment involved, but due to lack of a natural topology or measurable structure in the "class of classes" of measure-theoretic dynamical systems modulo isomorphisms, they seem extremely difficult to capture. A manifestation of the existence of such type of restriction is seen in the following condition, valid for all topological assignments:

(R4) The entropy function $p \mapsto h(\Psi(p)) := h_{\mu_p}(T_p)$ must be a nondecreasing limit of upper-semicontinuous functions (see [D-S]).

In this talk we exploit the following approach:

1. A topological assignment determined by a non-minimal topological dynamical system should possess all the "mysterious" regularity properties. Does minimality impose any restrictions other than (R3)? In other words, if $\Psi$ is a topological assignment determined by an arbitrary topological dynamical system $(Y, S)$ having no periodic points (this is (R3) for such assignments), does there exist a minimal dynamical system $(X, T)$ whose assignment is equivalent to $\Psi$?
2. Suppose we have a topological assignment $\Psi$ on a simplex $K$. Let $K'$ be a **face** of $K$, i.e., a subsimplex of $K$ whose extreme points are extreme in $K$. Is the restriction of $\Psi$ to $K'$ also topological? It should be so, because the restriction should inherit all the "mysterious" regularity properties from $\Psi$.

We will answer the first question affirmatively in the case of $Y$ zero-dimensional and the second question if $Y$ is zero-dimensional and, in addition, the restriction $\Phi|_{K'}$ contains no periodic measures.

**Theorem 1.** *If $Y$ is zero-dimensional and $(Y, S)$ has no periodic points then the assignment determined by $(Y, S)$ is equivalent to an assignment determined by some minimal system $(X, T)$.*

**Theorem 2.** *Let $(X, T)$ be a zero-dimensional dynamical system, and let $K'$ be a face in the simplex $K$ of the invariant measures of $(X, T)$. Assume that $K'$ contains no periodic measures. Then there exists another zero-dimensional dynamical system $(Y, S)$, whose natural assignment is equivalent to the identity assignment on $K'$.*

The superposition of the above two results characterizes all minimal assignments arising in Cantor minimal systems:

**Every nonperiodic face of any zero-dimensional topological assignment is itself a minimal zero-dimensional assignment** (and obviously vice-versa).

Although it seems to be merely a reduction statement (because it characterizes minimal assignments using topological assignments), it has very strong consequences and allows for practical classification of a large class of assignments as minimal (and Cantor). We will provide examples soon.

Recall that Cantor minimal systems are among the most extensively studied in topological dynamics. They appear naturally in many areas; first of all in symbolic dynamics, also in smooth dynamics, for example as attractors for unimodal maps (and interval maps in general), they have a well developed theory of Bratteli-Vershik diagram representations and the theory of orbit equivalence.

Let us discuss briefly the consequences of the above theorems.

1. Theorem 1 above allows one to immediately see that if $K$ has finitely many extreme points, then any assignment of ergodic nonperiodic measure-theoretic systems to the extreme points is minimal. Here is why: such assignment is obviously topological: take the disjoint union of Jewett-Krieger or Rosenthal realizations for the assigned ergodic measures. Moreover, these realizations can be made zero-dimensional. By nonperiodicity, Theorem 1 provides a Cantor minimal model for the same assignment. We have answered affirmatively the "motivation question" posed at the beginning of the exposition.
2. One can ask the same question for simplexes with countably many extreme points: is every nonperiodic ergodic assignment on the extreme points of such a simplex minimal? Here, the situation is much less trivial, because the (no longer discrete) topology of the set of extreme points may impose additional restrictions.

However, the restriction (R4) turns out to be void on such simplexes. Thus such a question remains reasonable. And indeed, using some elementary facts concerning the so-called universal zero-dimansional system (the full shift on the Cantor alphabet) one deduces that every such assignment appears as a face of the natural assignment of this universal system, hence by Theorems 1 and 2 it is minimal.

Let us remark that I. Kornfeld and N. Ormes [K-O] have proved a beautiful theorem, implying the above statement whenever all the systems assigned to the countably many extreme points of the simplex are (in addition to being nonperiodic) invertible. The authors are even able to construct a minimal realization of an arbitrary such assignment on a simplex $K$ within any topological orbit equivalence class of Cantor minimal systems whose simplex of invariant measures is affinely homeomorphic to $K$ (the affine-topological "shape" of this simplex is an invariant of the orbit equivalence relation, so this requirement cannot be skipped). The methods used by the authors are completely different from those of [D2] and rely on multitowers constructions and manipulations of the order of the floors. These orbit techniques seem impossible to be used in the noninvertible situation, also, any applications to simplexes with uncountable sets of extreme points seem rather hard. Our methods, based on symbolic representations and codes, allow to deal with nonivertible systems and uncountable extreme sets, but in turn, are completely unfit for exploration of the orbit equivalence classes. Unfortunately, neither methods allow excursions beyond dimension zero, further than some direct consequences of theorems on zero-dimensional modeling.

3. Finally, let us provide an example of a class of assignments on simplexes with uncountably many extreme points, which were hitherto unknown to be minimal, and for which our theorems provide an evidence for minimality.

**Example.** For any pair of positive numbers $a < b$ there exist a minimal Cantor system whose all ergodic measures are one-sided (or two-sided, if one wishes) Bernoulli and form a topological arc parametrized by their entropies ranging linearly from $a$ to $b$. Likewise, we can construct minimal models with all ergodic measures being Bernoulli and arranged topologically as any preassigned metrizable compact, and with entropy varying continuously following any preassigned positive continuous function on this compact.

The above follows immediately, because such an arc (or compact set) of Bernoulli measures is easily found in the simplex of invariant measures of a full (one or two-sided) shift over a finite or countable alphabet. The spanned simplex is then a face in the simplex of all invariant measures of the full shift. Because Bernoulli measures are nonperiodic we can thus obtain a minimal realization.

Let us conclude this presentation with a number of open questions and relevant comments. The list below is copied from the preprint [D3].

*Question 1.* Are the families of all topological and of all minimal assignments essentially larger than the families of topological zero-dimensional and of minimal zero-dimensional assignments, respectively?

*Comment.* There are many possibilities to replace a system by its zero-dimensional extension without changing the assignment (so-called *small boundary property*, see the work of E. Lindenstrauss [L]), and for many examples of higher-dimensional systems at least one of these possibilities is available (see [D2], the list preceding Theorem 2). But not for all of them. Examples are presented also in [L]. On the other hand, it seems that dimension zero imposes the weakest possible topological constraints, allowing the highest flexibility for realizations of measure-preserving systems. So, it is hard to expect that there exist assignments realizable only on some connected or partly connected spaces.

*Question 2.* Is Theorem 1 true for any topological nonperiodic assignment (not necessarily zero-dimensional)?

*Comment.* This is automatically true if Question 1 has a negative answer. In [D2], Theorem 2, we indicate a class of higher-dimensional systems for which Theorem 1 still holds (extensions of systems with the small boundary property, in particular extensions of zero-dimensional systems). Of course, there is no indication that the class of topological assignments on such spaces is essentially larger than that on zero-dimensional ones.

*Question 3.* Is Theorem 2 true for any topological nonperiodic assignment (not necessarily zero-dimensional)?

*Comment.* Again, this is true if Question 1 is false. By the applied "marker methods", it is not hard to see that Theorem 2 extends to at least the same class of higher-dimensional systems as indicated in Theorem 2 of [D2].

*Question 4.* By a *factor* of an assignment $\Psi$ on $K$ we will mean an assignment $\Psi'$ on some $K'$, such that there exists an affine continuous surjection $\pi : K \to K'$ and for every $p \in K$ a factor map of measure preserving transformations from $\Psi(p)$ to $\Psi'(\pi(p))$. Is every factor of a topological (minimal) assignment topological (minimal)?

*Comment.* Every topological dynamical system admits a zero-dimensional extension. Its assignment is hence a factor (in the above sense) of the assignment of this extension. Thus, it suffices to answer the question for topological zero-dimensional assignments. If in addition, one could prove that a factor of such an assignment is again realizable in dimension zero, this would answer (negatively) the Question 1, hence resolve all the other questions formulated above; minimal assignments would coincide with topological nonperiodic assignments. This direction seems to be the most promising for further investigations.

*Question 5.* Is Theorem 2 true without assuming nonperiodicity of the face?

*Comment.* Then the composition with Theorem 1 is not possible, but for Theorem 2 alone there are no immediate reasons why it should not hold.

*Question 6.* ([K-O]) Is any topological nonperiodic (perhaps zero-dimensional) invertible assignment on a simplex $K$ minimally realizable within the orbit equivalence class of an arbitrary Cantor minimal system whose simplex of invariant measures is affinely homeomorphic to $K$?

*Comment.* Answering this question requires refinements of the methods used in [K-O], allowing to deal with uncountably many ergodic measures. See [K-O] for more comments on such attempts.

## References


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