Event: AMS-PTM MEETING
Session: "Ergodic Theory and Topological Dynamics"
Organizers: M. Lemanczyk & D. Rudolph

Speaker: Tomasz Downarowicz
Title: *Classes of large sets in* $\mathbb{Z}$ *and their connection with mixing properties*
(joint work with Vitaly Bergelson)

Abtract: We study notions of large sets in $\mathbb{Z}$ such as syndetic sets, IP*-sets, Central*-sets, Δ^*-sets, etc. in the context of sets of "fat intersectons" $\{n \in \mathbb{Z} : \mu(A \cap T^n B) > \mu(A)\mu(B) - \epsilon\}$. This allows for a characterization of mixing, mild mixing and weak mixing, as well as to identify a new intermediate class (between weak and mild mixing).

In Furstenberg's book one can find the following theorem: a measure-preserving system (X, Σ, μ, T) is *mildly mixing* if and only if for any two sets $A, B \in \Sigma$ and $\epsilon > 0$ the set of times of *ϵ-independent intersections*

$$Q^{\epsilon}_{A,B} = \{n \in \mathbb{Z} : |\mu(A \cap T^n B) - \mu(A)\mu(B)| < \epsilon\}.$$

is IP*.

Also, it is not hard to deduce (and it is also written in Furstenberg's book) that if (X, Σ, μ, T) is *weakly mixing* then every set $Q^{\epsilon}_{A,B}$ has positive density.

Finally, directly from the definition, a system is mixing if and only if every set $Q^{\epsilon}_{A,B}$ is cofinite.

We will discuss a hierarchy of notions of "large" sets in $\mathbb{Z}$ and relate them to various notions of mixing. Instead of ϵ-independent intersections we will use *sets of times of ϵ-fat intersections*

$$R^{\epsilon}_{A,B} = \{n \in \mathbb{Z} : \mu(A \cap T^n B) > \mu(A)\mu(B) - \epsilon\}.$$

There will be no time for any details, just a list of definitions and results.

Definition 1.
(1) $\mathcal{I}$ denotes the collection of all infinite sets. Accordingly, $\mathcal{I}^*$ is the collection of all cofinite sets.

(2) A subset $F \subset \mathbb{Z}$ is called a $\triangle$-set or we say that F belongs to the family $\varDelta$, if there exists an injective sequence of integers $S = (s_n)_{n\geq 1}$ such that the difference set $\triangle(S) = \{s_i - s_j : i > j\}$ is contained in F.

(3) The collection $\mathcal{IP}$ (of IP-sets) is the union of all nonzero idempotents $0 \neq p \in \beta\mathbb{Z}$. Accordingly, $\mathcal{IP}^*$ is the intersection of all nonzero idempotents.

(4) The collection $\mathcal{D}$ (of D-sets) is the union of all idempotents $p \in \beta\mathbb{Z}$ such that every member of p has positive upper Banach density. Accordingly, $\mathcal{D}^*$ is the intersection of all such idempotents.

(5) The collection $\mathcal{C}$ (of C-sets or *central sets*) is the union of all minimal idempotents. (An idempotent is *minimal* if it belongs to a minimal right ideal in $\beta\mathbb{Z}$). Accordingly, $\mathcal{C}^*$ is the intersection of all minimal idempotents.

(6) A set $F \subset \mathbb{Z}$ is *thick* if it contains arbitrarily long intervals $[a, b] = \{a, a+1, a+2, \ldots, b\}$. The collection of all thick sets will be denoted by $\mathcal{T}$. The dual family $\mathcal{T}^*$ is easily seen to coincide with the collection of all syndetic sets (i.e., sets having bounded gaps).

We have the following inclusions

$$\text{cofinite} = \mathcal{I}^* \subset \varDelta^* \subset \mathcal{IP}^* \subset \mathcal{D}^* \subset \mathcal{C}^* \subset \mathcal{T}^* = \text{syndetic}$$

Definition 2. For a given family $\mathcal{F}$, we define

$$\mathcal{F}_+ = \bigcup_{k\in\mathbb{Z}} (\mathcal{F}+k) \qquad \text{and} \qquad \mathcal{F}_\bullet = \bigcap_{k\in\mathbb{Z}} (\mathcal{F}+k).$$

The families $\mathcal{F}_\bullet$ and $\mathcal{F}_+$ are always *shift invariant*.

When applying these operations to a dual family $\mathcal{F}^*$, we will write $\mathcal{F}^*_+$ and $\mathcal{F}^*_\bullet$, skipping the parentheses in what should formally be $(\mathcal{F}^*)_+$ and $(\mathcal{F}^*)_\bullet$.
Note that in general, $\mathcal{F}^*_+$ is not a dual family (to anything). On the other hand, the family $\mathcal{F}^*_\bullet$ is the dual of $\mathcal{F}_+$.

Now we have the following hierarchy of notions:

$$\begin{array}{ccccccccccc}
\mathcal{I}^*_\bullet & \subset & \Delta^*_\bullet & \subset & \mathcal{IP}^*_\bullet & \subset & \mathcal{D}^*_\bullet & \subset & \mathcal{C}^*_\bullet & \subset & \mathcal{T}^*_\bullet \\
\| & & \cap & & \cap & & \cap & & \cap & & \| \\
\mathcal{I}^* & \subset & \Delta^* & \subset & \mathcal{IP}^* & \subset & \mathcal{D}^* & \subset & \mathcal{C}^* & \subset & \mathcal{T}^* \\
\| & & \cap & & \cap & & \cap & & \cap & & \| \\
\mathcal{I}^*_+ & \subset & \Delta^*_+ & \subset & \mathcal{IP}^*_+ & \subset & \mathcal{D}^*_+ & \subset & \mathcal{C}^*_+ & \subset & \mathcal{T}^*_+
\end{array}$$

FACT: only the visible inclusions (and their compositions) hold.

Theorem 1. *Let* (X, Σ, μ, T) *be an invertible probability measure preserving system. Then:*

(1) *For any* $A \in \mathcal{B}$ *and any* $\epsilon > 0$ *the set* $R^{\epsilon}_{A,A}$ *is a* $\triangle^*$*-set.* (also proved in [K-Y])

(2) *If* (X, Σ, μ, T) *is ergodic and has discrete spectrum then all sets* $R^{\epsilon}_{A,B}$ *are* $\triangle^*_+$.

(3) (X, Σ, μ, T) *is ergodic if and only if all sets* $R^{\epsilon}_{A,B}$ *are* D^*_+.

(4) (X, Σ, μ, T) *is weakly mixing if and only if all sets* $R^{\epsilon}_{A,B}$ *are* $\mathrm{D}^*_\bullet$.

(5) ([F]) (X, Σ, μ, T) *is mildly mixing if and only if all sets* $R^{\epsilon}_{A,B}$ *are* $\mathrm{IP}^*_\bullet$.

(6) ([K-Y]) (X, Σ, μ, T) *is mixing if and only if all sets* $R^{\epsilon}_{A,B}$ *are* $\triangle^*_\bullet$.

Theorem 2. *There exists a weakly mixing probability measure preserving system* (X, Σ, μ, T)*, sets* $A, B \in \mathcal{B}$ *and* $\epsilon > 0$ *such that the set* $R^{\epsilon}_{A,B}$ *is not* IP^*_+.

Theorem 3. *There exists a weakly mixing but not mildly mixing probability measure preserving system* (X, Σ, μ, T)*, such that all the sets* $R^{\epsilon}_{A,B}$ *are* IP^*_+ *(but not all of them are* IP^**).*