Event: AMS-PTM MEETING
Session: "Real Dynamics"
Organizers: S. Hurder, M. Misiurewicz & P. Walczak

Speaker: Tomasz Downarowicz
Title: *Entropy structure in* C^∞

Abtract: In 1989 (Annals of Math.) Sheldon Newhouse proved (among other things) that his notion of local entropy provides an upper bound of the defect of upper semicontinuity of the entropy function h. Then, using a result of Yomdin (Israel J. 1987), he provided an upper estimate (let us denote it by S) of local entropy, which turns zero in C^∞ systems, implying upper semicontinuity of h. Eight years later Jérome Buzzi argued in a similar way, replacing Newhouse's estimate by a different parameter. By definition, Buzzi's parameter is precisely the Misiurewicz's *topological conditional entropy* $\mathbf{h}^*$. At that time this equality escaped the author's attention, so, while he only claimed to have obtained a new proof of upper semi-continuity of h in C^∞ systems, Buzzi actually proved much more: all C^∞ maps on compact manifolds are asymptotically h-expansive. My more recent studies of so-called *entropy structure* reveal that Newhouse's upper bound S of the defect of h also equals $\mathbf{h}^*$, and hence his C^∞ result in Annals is also equivalent to asymptotic h-expansiveness. In the lecture I will try to present a "shortcut way" to prove asymptotic h-expansiveness of C^∞ maps using Yomdin's estimate and the theory of entropy structure.

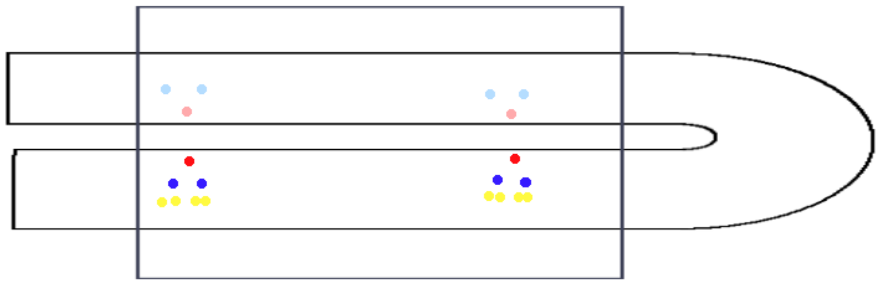
Let (X,T) be a topological dynamical system. The *topological entropy* is defined (after Bowen) as

$$h(T) = \lim_{\epsilon\to 0} \uparrow \lim_{n} \downarrow \frac{1}{n} \log r(n,\epsilon),$$

where $r(n,\epsilon)$ is the minimal number of (n,ϵ)-balls needed to cover X.

It measures the rate of the exponential growth of the number ϵ-distinguishable n-orbits as n-grows.

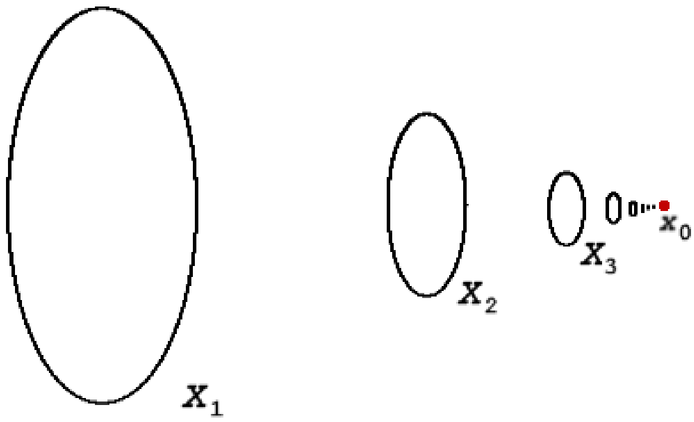
For example, a "k-fold horseshoe" in the system generates entropy at least $\log k$ (FIGURE 1).



Suppose for a moment that the system is expansive with the expansive constant ϵ_0. Then, in the definition of $h(T)$ we can replace ϵ by ϵ_0 and skip the first limit. This is because for every ϵ there is m such that every (m,ϵ_0) ball has radius smaller than ϵ (by compactness). Thus $h(\epsilon,T^m) \le mh(\epsilon_0,T)$ (and $h(\epsilon,T^m) = mh(\epsilon,T)$). In other words, the "resolution" ϵ_0 suffices to detect all "entropy generating dynamics". Expanive systems are known to have the following property: the entropy function $\mu \mapsto h_\mu(T)$ is upper semicontinuous, hence at least one measure μ_0 of maximal entropy $h_{\mu_0}(T) = h(T)$ exists.

Similar property have systems which are not necessarily expansive, but *asymptotically h-expansive* as defined by Misiurewicz. Let us explain this notion starting with examples.

EXAMPLE 1. Let the system consist of a sequence of expansive subsystems (X_n, T_n) with diameters of X_n decreasing to zero (hence their expansive constants ϵ_n also decrease to zero) and accumulating at a fixpoint (FIGURE 2). Assume that the entropies $h(T_n)$ also decrease to zero. The whole system is clearly not expansive. However, if we look at it with resolution ϵ (small) we miss only the dynamics near the fixpoint, which generates small entropy. Such system is *asymptotically h-expansive* and has u.s.c. entropy function.



EXAMPLE 2. In the previous example assume that $h(T_n)$ are bounded but do not converge to zero (for example all equal $\log 2$). Then, although the entropy $h(T)$ is finite the system is not *asymptotically h-expansive*. If we see it with any given resolution $\epsilon > 0$ we miss the essential dynamics near the fixpoint. Although we see dynamics that generates the maximal entropy, we miss equally important dynamics for many invariant measures. The entropy function has a "bad jump" from $\log 2$ to zero at the pointmass measure at the fixpoint, so it is not u.s.c.

Asymptotic h-expansiveness is defined via *topological tail entropy*, as follows:

$$h^*(T) = \lim_{\epsilon \to 0} \downarrow \lim_{\delta \to 0} \uparrow \lim_{n} \downarrow \frac{1}{n} \log r(n, \delta | \epsilon),$$

where $r(n, \delta|\epsilon)$ is the minimal number of (n, δ)-balls sufficient to cover every (n, ϵ)-ball. The system is asymptotically h-expansive if $h^*(T) = 0$. For small ϵ there are few δ-distinct n-orbits inside any (n, ϵ)-ball.

In EXAMPLE 2, the ϵ-ball around the fixpoint is in fact an (n, ϵ)-ball for any n. Yet with resolution $\delta \ll \epsilon$ we can distinguish lots of large entropy generating dynamics inside this ball.

Asymptotically h-expansive systems are important not only for the existence of measure of maximal entropy. They are characterized as systems "digitalizable without loss or gain of information". What does that mean?

A system (X, T) is "*digitalizable without loss of information*" if it admits an expansive (equivalently symbolic) extension, i.e., there is a subshift (Y, S) of which (X, T) is a topological factor: $\exists\, \pi : (Y, S) \to (X, T)$.

A system (X, T) is "*digitalizable without loss or gain of information*" if it admits a *principal* symbolic extension, i.e., an extension as above which preserves entropy of invariant measures: $h_\nu(S) = h_\mu(T)$ whenever $\pi^*(\nu) = \mu$.

• THEOREM [Boyle & D.]. A system is asymptotically h-expansive if and only if it admits a principal symbolic extension.

Now suppose X is an m-dimensional Riemannian manifold (with or without boundary) of class C^∞ and T is C^r smooth. What is known about the digitalizability properties of such systems?
There are three important facts and one open problem:

• THEOREM 1 (Newhouse 1989, Buzzi 1997, following Yomdin 1987): If T is C^∞ then (X,T) is asymptotically h-expansive. (Both authors claimed to have proved only that h is u.s.c., but in fact they proved as stated above. In case of Buzzi this is quite obvious, so the result is attributed to Buzzi. As for Newhouse, to see that his proof implies asymptotic h-expansiveness is far from obvious and follows from the much later theory of *entropy structure* [D]).

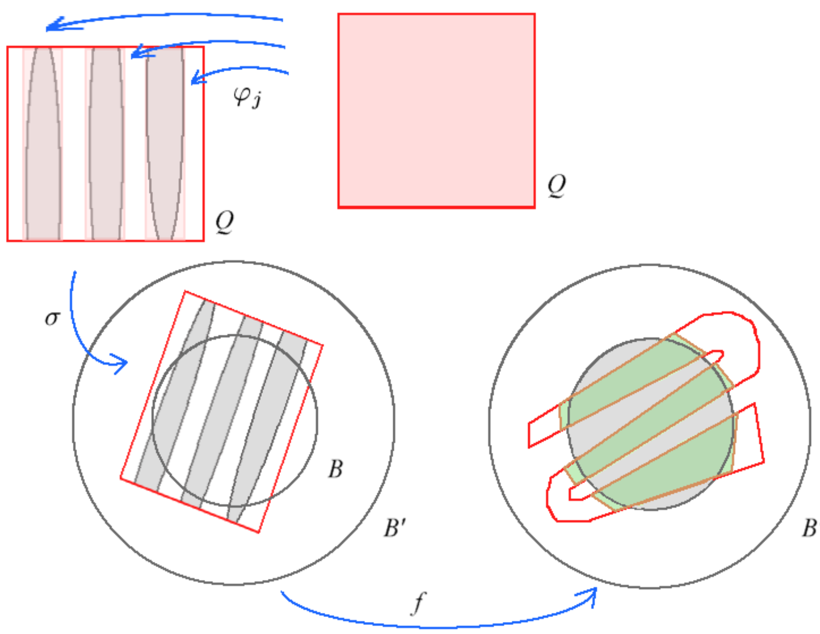
• THEOREM 2. (Newhouse & D.) There are C^1 maps T for which (X,T) has NO expansive extension (even with much larger entropy).

• THEOREM 3. (Newhouse & D.) There are C^r maps T for which (X,T) has no principal expansive extensions (every expansive extension has much larger topological entropy).

QUESTION: Let $1 < r < \infty$. Does every C^r map admit a symbolic extension? (Conjecture [Newhouse & D.]: Yes)

We will now sketch the idea behind the proof of THEOREM 1. The key tool is provided by Yomdin's *volume growth*:

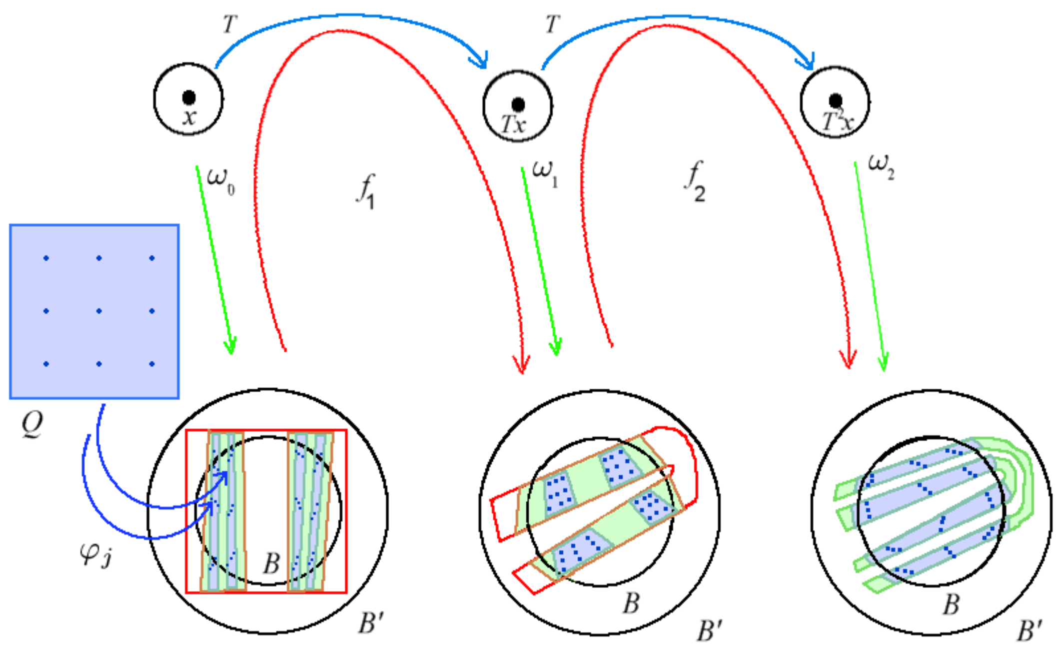
[Y] THEOREM 2.1. Let B and B' be the balls of radius 1 and 2 around zero in R^m, respectively. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^r-map with $\|d^s f(x)\| \le M$, $x \in B'$, $s = 1, \ldots, r$. Let $\sigma \in C^r$ $\sigma : Q \to B'$, where $Q = [-1, 1]$ satisfy $\|d^s \sigma(x)\| \le 1$, $x \in Q$, $s = 1, \ldots, r$. Then there exist at most $\kappa = a(r, m)(\log M)^{b(r,m)} \cdot M^{2m/r}$ diffeomorphisms $\varphi_j : Q \to Q$ $(j = 1, \ldots, \kappa)$ such that
1. the images of φ_j cover $S = (f \circ \sigma)^{-1}(B)$,
2. the image of $f \circ \sigma \circ \varphi_j$ is contained in B' $(j = 1, \ldots, \kappa)$
3. $\|d^s(f \circ \sigma \circ \varphi_j)(x)\| \le 1$, $(x \in Q, \quad s = 1, \ldots, r, \quad j = 1, \ldots, \kappa)$.
(FIGURE 3)



Sketch of proof of asymptotic h-expansiveness of C^∞ maps (after Buzzi). Consider an n-orbit $x, Tx, \dots, T^n x$ in X and the fixed "atlas" of maps from X into $\mathbb{R}^m$ such that each 2ϵ-ball B_i' around $T^i x$ is fully contained in one chart (this is possible for sufficiently small ϵ). By appropriate composition with linear maps we can assume that the atlas maps each B_i' by some map ω_i roughly onto B' (and the ϵ-ball B_i roughly onto B). Then on B' we have maps $f_i = \omega_i \circ f \circ \omega_{i-1}^{-1}$ which satisfy roughly the same bounds on the derivatives as f. The (n,ϵ)-ball $\tilde{B}$ around x_0 corresponds (via the map ω_0) to the intersection

$$V = \bigcap_{k=1}^{n} (f_k \circ \cdots \circ f_1)^{-1}(B).$$

Fix some δ. In order to estimate the cardinality of an (n,δ)-spanning set in $\tilde{B}$ it suffices to estimate do the same in V (the change of δ in this passage is inessential, because we will give the same estimate for every small δ). Here we use repeatedly Yomdin's Theorem: start with a δ-spanning set F in Q (of cardinality c). Eventually there will be at most κ^n contacting images of Q that cover V, and the union of images of F (of cardinality $c\kappa^n$ will form an (n,δ)-spanning set (FIGURE 4).



We have obtained

$$\frac{1}{n}\log r(n,\delta|\epsilon) \le \frac{\log c}{n} + \log a(r,m) + b(r,m)(\log\log M) + \frac{2m}{r}\log M,$$

hence $h^*(T) \leq \log a(r,m) + b(r,m)(\log\log M) + \frac{2m}{r}\log M$.
The same applies to T^n (now M becomes M^n, while the other constants remain), hence

$$h^*(T^n) \leq \log a(r,m) + b(r,m)(\log n + \log\log M) + \frac{2mn}{r}\log M.$$

Because $h^*(T^n) = nh^*(T)$ we divide by n and pass with n to infinity, so, eventually

$$h^*(T) \leq \frac{2m}{r}\log M.$$

For $r = \infty$ this is zero and the system is asymptotically h-expansive. $\square$