“JEWETT-KRIEGER” TYPE THEOREMS FOR NON-UNIQUELY ERGODIC AND NON-INVERTIBLE SYSTEMS

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Let (X, Σ, μ) be a standard probability space and let T be a measurable measure-preserving transformation from X into itself, i.e., such that $\mu(A) = \mu(T^{-1}(A))$ for every $A \in \Sigma$. Then (X, Σ, μ, T) is called a *measure-theoretic dynamical system.* A measure-theoretic dynamical system is *ergodic* if all T-invariant sets (i.e., $A \in \Sigma$ satisfying $T(A) \subset A$) have either measure 1 or 0. Two measure-theoretic dynamical systems (X, Σ, μ, T) and (X', Σ', μ', T') are said to be *isomorphic* if there exists a measurable bijection $\psi : X_0 \to X_0'$, where $X_0 \in \Sigma$, $X_0' \in \Sigma'$, $\mu(X_0) = \mu'(X_0') = 1$, which sends the measure μ to μ' (i.e., $\mu(A) = \mu'(A')$ whenever $A' = \psi(A)$, $A \in \Sigma$), and which is *equivariant*, i.e., $\psi \circ T = T' \circ \psi$ μ-almost everywhere. A system isomorphic to an ergodic one is ergodic.

By an *assignment* we will mean a function Ψ defined on an abstract metrizable Choquet simplex $\mathcal{P}$, whose “values” are measure-theoretic dynamical systems, i.e., for $p \in \mathcal{P}$, $\Psi(p)$ has the form $(X_p, \Sigma_p, \mu_p, T_p)$. Two assignments, Ψ on a simplex $\mathcal{P}$, and Ψ' on a simplex $\mathcal{P}'$, are said to be *equivalent* if there exists an affine homeomorphism of Choquet simplexes $\pi : \mathcal{P} \to \mathcal{P}'$ such that for every $p \in \mathcal{P}$ the systems $\Psi(p)$ and $\Psi'(p')$, where $p' = \pi(p)$, are isomorphic.

By a *topological dynamical system* we shall mean a pair (X, T), where X is a compact metric space and T is a continuous map of X into itself. A topological dynamical system (X, T) is minimal if for every $x \in X$ the orbit $\{T^n(x) : n \in \mathbb{N}\}$ is dense in X. In the context of a topological dynamical system (X, T), by a *measure* we will always mean a probability measure on the Borel sigma-field Σ. By $\mathcal{P}_T(X)$ we will denote the collection of all T-invariant measures on X, i.e., measures μ preserved by T, in other words such that (X, Σ, μ, T) becomes a measure-theoretic dynamical system. It is well known that $\mathcal{P}_T(X)$ is a nonempty compact, for the weak* topology of measures, metrizable Choquet simplex whose extreme points are precisely the ergodic invariant measures. A topological dynamical system (X, T) determines a natural assignment on the simplex $\mathcal{P}_T(X)$ by the rule: $\mu \mapsto (X, \Sigma, \mu, T)$.

This note contributes to the investigation of the following abstract problem: *Characterize the assignments equivalent to the natural assignments arising from minimal topological dynamical systems*? The renowned Jewett-Krieger theorem solves the problem for trivial (one-point) simplexes and invertible maps; every assignment of an invertible ergodic measure-theoretic dynamical system can be equivalently realized by a minimal (strictly ergodic) invertible topological system.

For nontrivial simplexes or noninvertible maps there is no known answer in full generality. Though, there are some obvious restrictions on the available assignments:

(R1) Ψ assigns ergodic systems to all extreme points;
(R2) Ψ is affine, i.e., the system (X, Σ, μ, T) assigned to a convex combination $\alpha p_1 + (1-\alpha)p_2$ $(0 < \alpha < 1)$ is non-ergodic and admits a decomposition $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ with both μ_1 and μ_2 preserved by the transformation T, and such that (X, Σ, μ_1, T) and (X, Σ, μ_2, T) are isomorphic to the systems $\Psi(p_1)$ and $\Psi(p_2)$ assigned to p_1 and p_2, respectively;
(R3) the assigned measure-theoretic dynamical systems must be nonatomic;

The restictions (R1) and (R2) apply to assignments arising from any (not necessarily minimal) topological dynamical systems and follow from the basics of ergodic theory. Condition (R3) in minimal systems is immediate; an atomic invariant measure is supported by finitely many periodic points, so, by minimality, the whole system would be one periodic orbit. Such systems have trivial simplex of invariant measures.

It is obvious that the above list of restrictions is incomplete. There must be some kind of "measurability" or even "continuity" of the assignment involved, but due to lack of a natural topology or measurable structure in the "world of classes" of measure-theoretic dynamical systems modulo isomorphisms, they seem extremely difficult to capture. A manifestation of the existence of such type of restriction is seen in the following condition, valid without assuming minimality:

(R4) The entropy function $p \mapsto h(\Psi(p)) := h_{\mu_p}(T_p)$ must be a nondecreasing limit of upper-semicontinuous functions (see [D-S]).

In this note we exploit the following approach: an assignment determined by a non-minimal topological dynamical system should possess all the (undiscovered) "measurability" and "continuity" properties. Does minimality impose any further restrictions other than (R3)? In other words, if Ψ is an assignment determined by an arbitrary topological dynamical system (Y, S) having no periodic points (this is equivalent to (R3) for such assignments), does there exist a minimal topological dynamical system (X, T) whose assignment is equivalent to Ψ?

We will answer this question affirmatively in case Y is zero-dimensional. The result then autoamltcally extends to systems with so-called *small boundary property*, i.e., existence of arbitrarily fine finite open covers by sets with boundaries having measure zero for all invariant measures. Every system with small boundary property has a Borel*-isomorphic zero-dimensional extension. The construction is standard.

The small boundary property has been exploited in the works of E. Lindenstrauss. Lack of periodic orbits plus any of the properties listed below suffices for (Y, S) to have small boundary property:

- Y is zero-dimensional (includes all subshifts over finite or countable alphabets);
- (Y, S) has finitely or countably many ergodic measures;
- S is invertible and Y finite-dimensional [Kulesza];
- (Y, S) is invertible, has finite topological entropy and a nonperiodic minimal topological factor [Lindenstrauss].

Finally, we notice that zero-dimensionality is needed only for the existence of closed-and-open markers, which immediately implies that the theorem also extends to the case where (Y, S) admits an infinite factor with the small boundary property.

In fact (with this assumption) we will prove more: we will conjugate the systems (Y, S) and (X, T) in a very strong sense, implying equivalence of their assignments. We need a definition for that.

Let (X, T) be a topological dynamical system. A Borel subset $X_0 \subset X$ is called a *full set* if it is T-invariant and has measure 1 for every invariant measure $\mu \in \mathcal{P}_T(X)$.

Definition. By a *Borel* isomorphism* between two topological dynamical systems (X, T) and (X', T') we shall understand an equivariant Borel-measurable bijection $\phi : X_0 \to X_0'$ between full sets $X_0 \subset X$ and $X_0' \subset X'$, such that the adjacent map $\phi^* : \mathcal{P}_T(X) \to \mathcal{P}_{T'}(X')$ defined by the rule $\phi^*(\mu)(A) = \mu(\phi^{-1}(A))$ $(A \subset X)$ is a homeomorphism with respect to the weak* topologies.

If ϕ is a Borel* isomorphism, then the pair ϕ and ϕ^* establish an equivalence between the assignments determined by (X, T) and (X', T'); ϕ^* plays the role of an affine homeomorphism between the simplexes, while, for each pair of measures μ, $\mu' = \phi^*(\mu)$, ϕ provides the isomorphism between (X, Σ, μ, T) and (X', Σ', μ', T'). Notice that any equivariant Borel-measurable bijection ϕ between full sets provides an affine bijection ϕ^* between the simplexes of invariant measures. By compactness of these simplexes, in order to verify ϕ^* as a homeomorphism (and thus a Borel* isomorphism) it suffices to check its weak* continuity.

We will exploit the notion of *n-markers*. The following Lemma is a non-invertible version of Krieger's marker lemma (see [B] for the invertible case) with absence of periodic points.

Definition. A subset F of a topological dynamical system (Y, S) is called an *n-marker* if

(a) the sets $T^{-i}(F)$ $(0 \le i \le n)$ are pairwise disjoint;

(b) the sets $T^{-i}(F)$ $(0 \le i \le m)$ cover Y for some $m \in \mathbb{N}$.

The system (X, T) is said to have the *marker property* if there exist closed-and-open n-markers for all $n \in \mathbb{N}$. Every zero-dimensional system has the marker property.

We are in a position to state the main theorem:

Theorem 1. *If (Y, S) has the marker property then it is Borel*-isomorphic to a minimal topological dynamical system (X, T). In particular the assignment determined by (Y, S) is equivalent to the assignment determined by the minimal system (X, T).*