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ON VARIOUS TYPES OF RECURRENCE

A *dynamical system* in ergodic theory is (X, Σ, μ, T), where (X, Σ, μ) is a probability measure space and $T : X \to X$ is measurable preserving μ by preimage, i.e.,

$$\forall\, A \in \Sigma \quad \mu(T^{-1}(A)) = \mu(A).$$

Example: A topological dynamical system is a pair (X, T), where X is a compact Hausdorff space and $T : X \to X$ is a continuous mapping. Basic fixpoint theorem (e.g. Bogoliubov-Krylov) implies that:

There exists a regular Borel probability measure μ on X invariant under T.

(From now on by an *invariant measure* we will mean a regular Borel probability measure invariant under T.)

Then (X, Σ_B, μ, T) becomes a dynamical system in terms of ergodic theory. There may be more than one invariant measure on X!

A set $Y \subset X$ is called *invariant* if $T(Y) \subset Y$. A closed invariant subset Y can be regarded as a *subsystem* $(Y, T|_Y)$.

Examples:
1. The topological support of an invariant measure is a closed invariant set.
2. For any point $x \in X$ the orbit-closure of x

$$O_x = \overline{\{x, Tx, T^2x, \dots\}}$$

is a closed invariant set.

A system is called *minimal* if there are no proper closed invariant subsets in X. Equivalently, when $X = O_x$ for every $x \in X$. It is a standard fact (using Zorn's Lemma) that:

Every compact system contains an invariant set which is minimal.

In a minimal system every invariant measure has *full support*, i.e., its topological support is the whole space.

A point in a topological dynamical system is *recurrent* if it returns to every its open neighborhood:

$$\forall \text{ open } U \ni x \quad \exists\, n > 0 \quad T^n x \in U.$$

In a minimal system every point is recurrent (for otherwise O_{Tx} would be a proper closed invariant set). A point x whose orbit-closure is minimal is called *uniformly recurrent*. It is not true that a system in which every point is recurrent is minimal or that it is a union of minimal sets.

Suppose x is recurrent or uniformly recurrent. We are interested in the properties of the set of times of recurrence

$$N(x, U) = \{n \in \mathbb{N} : T^n x \in U\}.$$

Question 1: Does this set have any interesting algebraic properties?

Question 2: What if this set has additional density properties?

Definition 1 A set $S \subset \mathbb{N}$ is called *syndetic* if it has "bounded gaps", i.e., there exists $k_0 \in \mathbb{N}$ such that

$$\forall\, n \in \mathbb{N} \qquad S \cap \{n, n+1, \ldots, n+k_0-1\} \neq \emptyset.$$

Definition 2 A set $S \subset \mathbb{N}$ has *positive upper Banach density* if

$$\limsup_{k\to\infty} \sup_{n\in\mathbb{N}} \frac{\#(S \cap \{n, n+1, \ldots, n+k-1\})}{k} > 0.$$

Definition 3 A set $S \subset \mathbb{N}$ is called an IP-set if there exists an increasing sequence $(p_1, p_2, p_3, \ldots)$ of positive integers such that any finite sum $p_{i_1} + p_{i_2} + \cdots + p_{i_k}$ belongs to S.

Every syndetic set has positive upper Banach density (at least $\frac{1}{k_0}$), but not vice-versa.

Theorem 1: If $x \in X$ is recurrent then for every open $U \ni x$ the set $N(x, U)$ is an IP-set. Conversely, if S is an IP-set, then there is a compact dynamical system (X, T), a recurrent point x and an open $U \ni x$ such that $N(x, U) \subset S$.

Theorem 2: A point $x \in X$ is uniformly recurrent if and only if for every open $U \ni x$ the set $N(x, U)$ is syndetic.

Definition 4 A dynamical system X is *measure saturated* if for every open set $U \in X$ there exists an invariant measure μ such that $\mu(U) > 0$.

For example, any minimal system is measure saturated. There are however many not minimal measure saturated systems.

Definition 5 A point $x \in X$ is *essentially recurrent* if the orbit closure of x is measure saturated.

Theorem 3: If $x \in X$ is essentially recurrent if and only if for every open $U \ni x$ the set $N(x, U)$ has positive upper Banach density. In particular, every essentially recurrent point is indeed recurrent.

Proofs (sketchy)

Thm 2

$\Longrightarrow$. Suppose x is uniformly recurrent (its orbit closure is minimal), yet $N(x,U)$ is not syndetic, i.e., for every k there is n_k such that

$$\{T^{n_k}x, T^{n_k+1}x, \ldots, T^{n_k+k}x\} \cap U = \emptyset.$$

Then let y be any accumulation point of the sequence $T^{n_k}x$. The entire orbit of y is contained in the complement of U, thus its orbit closure is a proper invariant set of the orbit closure of x, hence the latter is not minimal, a contradiction.

$\Longleftarrow$. Suppose the orbit closure O_x of x is not minimal. Let M be a minimal subset in O_x. Clearly, $x \notin M$. By compactness and T_2, there is an open set U containing x disjoint from another open set V containing M. Then $N(x,U)$ is not syndetic, since the orbit of x spends in V arbitrarily long intervals of the time.

Thm 1.

$\Longrightarrow$. Fix $U \ni x$, where x is recurrent. Let p_1 be such that $T^{p_1}x \in U$. The same holds for y in some $U_1 \subset U$. Let $p_2 > p_1$ be such that $T^{p_2}x \in U_1$. Then $T^{p_1+p_2}x \in U$. And so on...

$\Longleftarrow$. Let S be an IP-set. We can assume that S is the set of finite sums of a rapidly growing sequence (p_i). Consider the "full shift on two symbols" system $(\{0,1\}^{\mathbb{N}}, \sigma)$, where $\{0,1\}^{\mathbb{N}}$ is the compact space of all binary sequances $x = (x_n)$ and σ is the shift map $\sigma(x)_n = x_{n+1}$. In this space the characteristic function of S is apoint x. Clearly, $S = N(x, U)$, where U is the set of all binary sequances starting with "1". By the IP-property it is seen that x is recurrent (if (p_i) grows fast enough).

Thm 3. (From a joint paper with Vitaly Bergelson)
$\Longrightarrow$. Let x be essentially recurrent and pick an open set $U \ni x$. There is an invariant measure μ supported by the orbit closure of x with $\mu(U) > 0$. By the ergodic theorem, there is a point x' in the orbit closure of x such that $N(x', U)$ has positive density. Because there are times n when $T^n x$ is very close to x', it is easily seen that $N(x, U)$ has positive upper Banach density.

$\Longleftarrow$. Let x be such that $N(x, U)$ has positive upper Banach density for every open set $U \ni x$. Fix some open U and then let $V \ni x$ be open and with closure contained in U. The set $N(x, V)$ has positive Banach density , say 2ϵ. Let n_k be the starting times of the intervals of time of length k in which the frequency of $N(x, V)$ is at least ϵ. Consider the probability measures

$$\frac{1}{k} \sum_{i=0}^{k-1} \delta_{T^{n_k+i}x}.$$

where δ_z denotes the point mass at z. Every such measure assigns to $\overline{V}$ a value larger than ϵ. These measures have an accumulation point μ which is an invariant measure, and it assigns to $\overline{V}$ a value at least ϵ (it is important that $\overline{V}$ is closed). But then $\mu(U) > 0$, as we needed.