Zero topological entropy and asymptotic pairs

Czech-Slovak-Spanish-Polish
Workshop on Discrete Dynamical Systems
in honor of Francisco Balibrea Gallego
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Tomasz Downarowicz

Institute of Mathematics and Computer Science
Wroclaw University of Technology
Poland

This is joint work with

Yves Lacroix

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- *forward mean proximal*, whenever

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} d(T^i x, T^i y) = 0.$$

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Convention: the system is μ-*something* if it satisfies “something” after removing μ-null set.

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- FMD $\implies$ MD and NAP
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Lack of implications:

- NAP $\not\Leftrightarrow$ MD
- μ-NAP $\not\Leftrightarrow$ μ-MD

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Theorem 4 [D-L]

Every topological dynamical system (X,T) of topological entropy zero has a topological extension (Y,S) which is NAP.

$$
\begin{array}{ccccc}
 & & \text{FMD} & & \\
 & \swarrow & & \searrow & \\
\text{MD} & \longrightarrow & \text{NBAP} & \longleftarrow & \text{NAP} \\
\downarrow & & & & \downarrow \\
\mu\text{-MD} & & & & \mu\text{-NAP} \\
 & \searrow & & \swarrow & \\
 & & h(\mu) = 0 & & \\
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Attention:
BNAP does not imply $h_{\text{top}} = 0$

(all the more μ-BNAP does not imply $h(\mu) = 0$)

Example:
Bilaterally deterministic systems with positive entropy
(a topological version)

(*if we have time*)

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REMARK: We cannot hope to have (Y, S) in form of a subshift.

Corollary

The following conditions are equivalent for a topological dynamical system (X, T):

- $h_{\text{top}}(T) = 0$,
- (X, T) is a topological factor of a NAP system [D-L, 2009],
- (X, T) is a topological factor of a subshift via a map that collapses asymptotic pairs [D-L, 2009],
- (X, T) is a topological factor of an FMD system [D-L, 2010],
- (X, T) is a topological factor of a subshift via a map that collapses forward mean proximal pairs [D-L, 2010].

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Two binary blocks A, B of the same length n are said to be *well separated* if they disagree at at least $\frac{n}{3}$ and agree at least $\frac{n}{3}$ positions.

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In other words $d_H(A, B) \in [\frac{1}{3}, \frac{2}{3}]$ or $(d_H(A, B) \geq \frac{1}{3}$ and $d_H(A, \bar{B}) \geq \frac{1}{3})$.

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Key combinatorial fact

Ther exists a positive number s such that for each $n \geq 2$ there exists a family of at least 2^{sn} pairwise well-separated binary blocks.

It the first part of the proof we build a symbolic extension which is not FMD yet, but which has entropy zero and collapses all forward proximal pairs. By taking a direct product, we can assume that (X, T) has an odometer factor. The odometer will be used to cut every $x \in X$ into blocks of equal lengths.

To each $x \in X$ we will associate its "preimages" $y \in Y$.

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- We choose n_1 such that the number of all n_1-blocks appearing in X is smaller than 2^{sn_1}. This allows to injectively associate to every n_1-block B of X a binary block $\phi(B)$ from a pairwise well separated family.
- Using the odometer, we represent every $x \in X$ as a concatenation of n_1-blocks. We let y be a "candidate for a preimage" of x if it has, above each n_1-block B of x, either $\phi(B)$ or its negation $\bar{\phi}(B)$.

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- We allow y be a "candidate" for a preimage of x if above each n_k-block B of x if it has either $\phi(B)$ (in fact depending also on the preceding block) or its negation $\bar{\phi}(B)$ (the preimages decrease).

In the end, above "almost every" x there are only two elements: some y and its negation $\bar{y}$.

This is not true for those elements x in which the division into n_k-blocks has a "cut of infinite order". Such x has four preimages $y|z$, $\bar{y}|z$, $y|\bar{z}$ and $\bar{y}|\bar{z}$. Note that we have here two asymptotic pairs! But such asympotic pairs are collapsed by the factor map.

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- If y, y' map to x, x' which map to different elements of the odometer then, since the odometer is equicontinuous, the pair y, y' is distal, so cannot be forward mean proximal.
- Suppose y, y' map to different points x, x' mapping to the same element of the odometer. Then x, x' have the same division into n_k-blocks. There is a coordinate n where x differs from x'. Then, for every k, n is covered by two different n_k-blocks, say, in x it is B and in x' it is B'. This implies that the FOLLOWING n_k-blocks in y and y' are well separated. It is now easy to see that y and y' are not forward mean proximal (the density of differences is at least $\frac{1}{6}$).

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- By induction, we have a sequence of symbolic extensions $Y_k \mapsto Y_{k-1}$, where each map collapses all forward mean proximal pairs.
- We let Y be the inverse limit of this sequence. It is obvious that Y is an extension of X. By an elementary argument, Y has no forward mean proximal pairs, i.e., it is FMD.

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Thank you, that's all.