Unit 1
Basic Operations in the Set of Real Numbers

Jarosław Gruszka

1 Powers and roots

Basic information and formulas

We call the operation written as x^a *rising* x *to the power* a.
For $x \in \mathbb{R}$ and $a \in \mathbb{N}$, we define x^a to be

$$x^a = \underbrace{x \cdot x \cdot \ldots \cdot x}_{a \text{ times}}. \tag{1}$$

We separately define that for all $x \in \mathbb{R}, x \neq 0$

$$x^0 = 1. \tag{2}$$

For $x \in \mathbb{R}, x \neq 0$ and $a \in \mathbb{N}, a < 0$, we define x^a to be

$$x^a = \left(\frac{1}{x}\right)^{-a}. \tag{3}$$

We call the operation of $\sqrt[a]{x}$ *taking the* a*-th root of* x.
For $x \in \mathbb{R}$ and $a \in \mathbb{R}, a \neq 0$, we define taking the root as an operation inverse to rising to the power

$$\sqrt[a]{x} = y \quad \text{if and only if} \quad y^a = x. \tag{4}$$

For $x \in \mathbb{R}$, $a \in \mathbb{R}$ and $b \in \mathbb{R}, b \neq 0$, we define $x^{\frac{a}{b}}$ to be

$$x^{\frac{a}{b}} = \sqrt[b]{x^a}. \tag{5}$$

The following equalities are true

$$x^a \cdot x^b = x^{a+b} \qquad \text{for all } x, a, b \in \mathbb{R}, \tag{6}$$

$$\frac{x^a}{x^b} = x^{a-b} \qquad \text{for all } x, a, b \in \mathbb{R}, x \neq 0, \tag{7}$$

$$(x^a)^b = x^{a\cdot b} \qquad \text{for all } x, a, b \in \mathbb{R}, \tag{8}$$

$$(x \cdot y)^a = x^a \cdot y^a \qquad \text{for all } x, y, a \in \mathbb{R}, \tag{9}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \qquad \text{for all } x, y, a, b \in \mathbb{R}, y \neq 0. \tag{10}$$

Exercises

1.1. Compute

a) 3^4;
b) $(-4)^2$;
c) $\left(-1\frac{1}{4}\right)^3$;
d) $\sqrt{3}^3$;
e) 4^{-1};
f) 2^{-3};
g) $\left(\frac{2}{3}\right)^{-2}$;
h) $(2\frac{1}{3})^{-3}$;
i) $8^{\frac{1}{3}}$;
j) $\sqrt{2}^{\frac{1}{2}}$;
k) $((-10)^2)^{\frac{1}{3}}$.

1.2. Compute

a) $10^3 \cdot 10^4$;
b) $(-5)^2 \cdot (-5)$;
c) $\left(\frac{2}{5}\right)^3 \cdot \left(1\frac{1}{4}\right)^3$;
d) $4^{-2} \cdot \left(\frac{1}{16}\right)^{-2}$;
e) $2^{-8} \cdot 2^5$;
f) $125^{\frac{2}{3}}$;
g) $\left(\frac{4}{7}\right)^{-5} \cdot \left(\frac{4}{7}\right)^4$;
h) $\frac{(3^{-2})^{-3}}{3^8}$;
i) $\frac{0^6+6^3}{(6^2)^2} \cdot \left(\frac{1}{36}\right)^{-1}$;
j) $5^{-1} \cdot 25^{\frac{2}{3}}$;
k) $\left(\frac{(4\sqrt{2})^2-\left(\frac{3^5}{3^2}+9^0\right)}{7^{12}\cdot\left(\frac{1}{7}\right)^{12}}\right)^{-\frac{2}{2^2}}$;
l) $\sqrt{8} - \sqrt{32} + \sqrt{128}$.

2 Binomial expansion formulas

Basic information and formulas

For $a \in \mathbb{R}$ and $b \in \mathbb{R}$ the following equalities are true

$$(a+b)^2 = a^2 + 2ab + b^2, \tag{11}$$
$$(a-b)^2 = a^2 - 2ab + b^2, \tag{12}$$
$$(a+b)(a-b) = a^2 - b^2. \tag{13}$$

Corresponding formulas also exist for higher exponents

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \tag{14}$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3, \tag{15}$$
$$(a+b)(a^2 - ab + b^3) = a^3 + b^3, \tag{16}$$
$$(a-b)(a^2 + ab + b^3) = a^3 - b^3. \tag{17}$$

More generally, for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and for any $n \in \mathbb{N}$ it is true that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k. \tag{18}$$

We call the operation of $\binom{n}{k}$ a *binomial coefficient* and it is defined for any $n, k \in \mathbb{N}$ as.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{19}$$

where $n!$ is called *factorial* and it is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n. \tag{20}$$

Also, by definition, we accept that

$$0! = 1. \tag{21}$$

For binomial coefficient, the following recursive formula is true

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{22}$$

Alternatively, the value of the binomial coefficient can be read from the *Pascal triangle*, below

$n=0$							1							
$n=1$						1		1						
$n=2$					1		2		1					
$n=3$				1		3		3		1				
$n=4$			1		4		6		4		1			
$n=5$		1		5		10		10		5		1		
$n=6$	1		6		15		20		15		6		1	

Exercises

2.1. Expand algebraic expressions

a) $(2a+3b)^2$;

b) $(4m-7n)^2$;

c) $(5g-9h)(5g+9h)$;

d) $(pq^2+p^2q)^2$;

e) $\left(\frac{x}{y^2}-\frac{y}{x^2}\right)^2, \quad x,y \neq 0$;

f) $(z^{\frac{n}{2}}-1)(z^{\frac{n}{2}}+1)$;

g) $\left(\frac{a}{3}+\frac{b}{3}\right)^3$;

h) $(m^n-n^m)^3$.

2.2. Reduce algebraic expressions to the multiplicative form

a) $\frac{9}{16}x^2-\frac{1}{121}$;

b) $64a^2b^2-1$;

c) $25-(x+y)^2$;

d) $(a+2b)^2-(3c+4d)^2$;

e) $4x^2+4x+1$;

f) $9a^2-6a+1$;

g) $9-16x^2-40xy-25y^2$;

h) $a^3+18a^2+108a+216$;

i) $64-96x+48x^2-8x^3$.

2.3. Prove that

a) $5^{12}-1$ is divisible by 31;

b) $3^{18}-2^{18}$ is divisible by 19.

2.4. Remove irrationality from the denominator

a) $\frac{1}{2\sqrt{3}}$;

b) $\frac{3}{5\sqrt{2}}$;

c) $\frac{1}{2-\sqrt{3}}$;

d) $\frac{7}{\sqrt{2}-1}$;

e) $\frac{2}{\sqrt{3}+1}$;

f) $\frac{2}{1+\sqrt{5}}$;

g) $\frac{1-2\sqrt{7}}{4-\sqrt{7}}$;

h) $\frac{3}{\sqrt{5}-\sqrt{2}}$;

i) $\frac{9}{\sqrt{2}+\sqrt{3}}$;

j) $\frac{1}{\sqrt{\sqrt{10}+1}}$;

k) $\frac{1}{\sqrt[3]{2}-1}$;

l) $\frac{1}{\sqrt[3]{5}+2}$.

2.5. Compute

a) $5!$

b) $\frac{11!}{10!}$;

c) $\frac{10!}{7!}$;

d) $\binom{5}{3}$;

e) $\binom{8}{6}$;

f) $\binom{4}{3}+\binom{4}{2}$;

g) $\frac{\binom{6}{3}}{\binom{5}{1}}$;

h) $\binom{7}{2}\cdot\binom{5}{2}$.

2.6. Expand

a) $(2a+1)^5$;

b) $(x^2+y)^7$.

2.7. Determine the form of a component

a) containing a^4b^7 in the expansion of $(a+b)^{11}$;

b) containing p^5q in the expansion of $(2p+3q)^8$;

c) containing x^8y^{13} in the expansion of $(x^2y^2+xy^3)^5$;

d) without any z in the expansion of $\left(z^4+\frac{1}{z^4}\right)^6$.