Unit 2

Logic, Sets and Subsets of the Set of Real Numbers

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1 Propositional calculus

Basic information and formulas

In mathematical logic, *a proposition* (also called *a sentence*) is a statement that is either true or false. We usually denote prepositions by letters $p, q, r, s, \ldots$. When the proposition is true, we say its *logical value* is 1 and if it's false — the logical value of that sentence is 0.

Logical operators (also called *logical connectives*) are used to perform various operations on the propositions and their logical values.

The simplest logical operator which only takes one proposition is *negation*. It is written using the symbol $\neg$ and it changes the logical value of the sentence to the opposite one.

p	$\neg p$
1	0
0	1

Most logical operators are *binary*, so they take two propositions and join them to form more complicated ones. One of them is called *conjunction*. It is written using the following symbol: $\wedge$. Result of conjunction is only true when both component sentences are true.

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Another kind of a connective, marked as $\vee$ is called *alternative*. It is true when at least one of the components is true.

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Next one is called *implication*, marked as an arrow to the right $\implies$.

p	q	$p \implies q$
1	1	1
1	0	0
0	1	1
0	0	1

Last one that we will define is *equivalence*, written using bi-directional arrow symbol $\iff$.

p	q	$p \iff q$
1	1	1
1	0	0
0	1	0
0	0	1

Using multiple binary operators we can write *compound* logical sentences. Such sentences can also be either true or false. There are some compound sentences which are always true, no matter what are the logical values of the components. They are called *tautologies*.

Exercises

1.1. Assess, if the following sentences are logical propositions

a) Let's go home!

b) 8 is a prime number.

c) x is a positive number.

d) Do you like ice cream?

e) Earth orbits the Sun.

f) $1 + 2 = 3$

1.2. Evaluate if the following logical propositions are true or false

a) One litre of water weights over 5kg.

b) The capital of the United Kingdom is London.

c) $\pi < 4$.

d) Triangle having side lengths of 3, 4 and 5 is a right triangle.

e) 1 is a prime number.

f) 2 is a prime number.

g) Every square is also a rectangle.

h) Every rectangle is also a square.

i) $(-1)^2 = 1$.

j) $\sqrt{1} = -1$.

1.3. Evaluate if the following compound logical propositions are true or false

a) $\neg(2 \cdot 2 = 2^2)$;

b) $\neg(3 \cdot 3 = \sqrt[3]{3})$;

c) $(\sqrt{2} > 1) \wedge \left(\frac{3}{6} = \frac{1}{2}\right)$;

d) $(-2 < -3) \wedge \left(3^2 \cdot 4^2 = (3 \cdot 4)^2\right)$;

e) $\left(\frac{8}{5} = 1\frac{3}{5}\right) \vee (4! = 20)$;

f) $(\sqrt[3]{1} = 3) \vee (3^3 = 9)$;

g) $\left(\frac{1}{7} > \frac{1}{11}\right) \implies (7 > 11)$;

h) $(3 + 4 = 5) \implies (3^2 + 4^2 = 5^2)$;

i) $((-1) = (-1)^{-1}) \iff (4^0 = 4)$;

j) $(1^{100} > 100^1) \iff (5^{-2} = -10)$.

1.4. Verify that the following sentences are tautologies

a) $\neg(\neg p) \iff p$;

b) $(p \wedge q) \implies p$;

c) $p \implies (p \vee q)$;

d) $\neg(p \wedge q) \iff (\neg p \vee \neg q)$;

e) $\neg(p \vee q) \iff (\neg p \wedge \neg q)$;

f) $(p \implies q) \iff (\neg p \vee q)$;

g) $(p \implies q) \iff (\neg q \implies \neg p)$;

h) $((p \implies q) \wedge (p \implies \neg q)) \implies \neg p$;

i) $((p \implies q) \wedge (q \implies p)) \iff (p \iff q)$;

j) $(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$;

k) $(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$.

2 Sets and operations on sets

A set is a primitive notion so it does not have a formal definition. Intuitively, we understand it as a collection of some mathematical objects (very often — numbers). We use capital letters: $A, B, C, \ldots$ to denote sets.

If A is a set and x is one of the objects of this set, this is denoted as $x \in A$ and is read as x *is an element of* A, x *belongs to* A, or x *is in* B. If we want to say that x is not an element of A — we denote it as $x \notin A$.

A set is fully determined by the elements it has. There are two main ways to specify members of the set:

- *enumeration* — listing all elements of the set in curly brackets, e.g. $A = \{1, 2, 3, 4, 5\}$. It may contain the *ellipsis*, if the number of elements is big and the pattern is clear e.g. $B = \{2, 4, 6, \ldots, 22, 24\}$;

- *set-builder notation* — defining a rule which elements must satisfy to be in the set, e.g. if $A = \{x : x^2 = 4\}$, then $A = \{-2, 2\}$;

A set that does not have any elements is called *an empty set*. Is is denoted with a symbol $\emptyset$.

There is a few sets of numbers which are considered most important, as all the numbers that belong to those sets share some specific features. These sets are

- set of natural numbers — $\mathbb{N}$
$$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}; \tag{1}$$
number zero (0) is sometimes considered an element of $\mathbb{N}$;

- set of integer numbers — $\mathbb{Z}$
$$\mathbb{Z} = \{x : x = 0 \vee x \in \mathbb{N} \vee -x \in \mathbb{N}\} = \{0, 1, -1, 2, -2, 3, -3 \ldots\}; \tag{2}$$

- set of rational numbers — $\mathbb{Q}$
$$\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge q \neq 0\right\} = \tag{3}$$
$$= \left\{0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, 3, -3, \frac{3}{2}, -\frac{3}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, 4, -4, \ldots\right\}; \tag{4}$$
set of rational numbers contains all integer numbers and all fractions;

- set of real numbers — $\mathbb{R}$; set of real numbers contains all numbers on the number line.

The above sets — $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ — are *subsets* of the set of real numbers. Set A is a subset of set B, which we write as $A \subseteq B$ if all elements of set A are also elements of set B or, in other words, if the fact that an element belongs to set A implies that it also belongs to set B
$$x \in A \implies x \in B. \tag{5}$$

Intervals are specific subsets of real numbers, containing all numbers lying between two specific ones a and $b, a < b$. We define

- *an open* interval
$$(a, b) = \{x : x > a \wedge x < b\}; \tag{6}$$

- *a left-closed* or *a right-open* interval
$$[a, b) = \{x : x \geqslant a \wedge x < b\}; \tag{7}$$

- *a right-closed* or *a left-open* interval

$$(a, b] = \{x : x > a \wedge x \leqslant b\}; \tag{8}$$

- *a closed* interval

$$[a, b] = \{x : x \geqslant a \wedge x \leqslant b\}. \tag{9}$$

Unbounded intervals are also frequently used. For any real number a we define

- *a left-unbounded, right-open* interval

$$(-\infty, a) = \{x : x < a\}; \tag{10}$$

- *a left-unbounded, right-closed* interval

$$(-\infty, a] = \{x : x \leqslant a\}; \tag{11}$$

- *a right-unbounded, left-open* interval

$$(a, +\infty) = \{x : x > a\}; \tag{12}$$

- *a right-unbounded, left-closed* interval

$$[a, +\infty) = \{x : x \geqslant a\}. \tag{13}$$

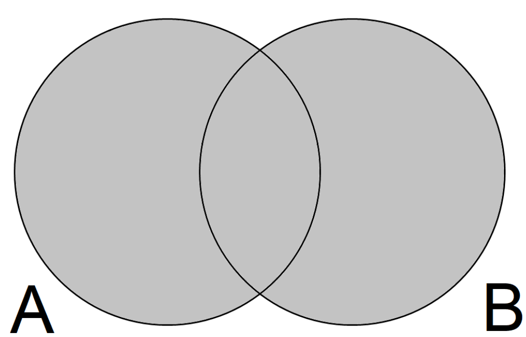
It is possible to do operations on sets. One of the most basic operations for sets is *a set sum*, denoted as $\cup$. For any sets A and B we define

$$A \cup B = \{x : x \in A \vee x \in B\} \tag{14}$$

or equivalently

$$x \in (A \cup B) \iff x \in A \vee x \in B. \tag{15}$$

An element belongs to the sum of sets A and B if it belongs to any of the two sets — A or B (or possibly — to both).



$A \cup B$

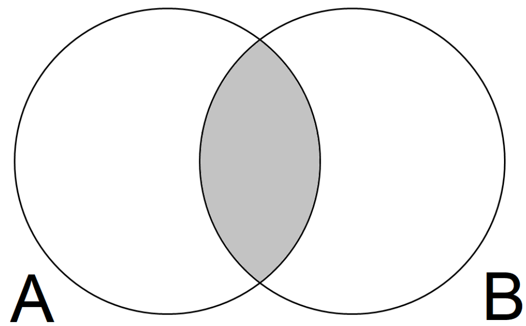
A set product is a set operation denoted as $\cap$. For any sets A and B

$$A \cap B = \{x : x \in A \wedge x \in B\} \tag{16}$$

or equivalently

$$x \in (A \cap B) \iff x \in A \wedge x \in B. \tag{17}$$

An element belongs to the product of sets A and B if it belongs to both sets A and B at the same time.



$A \cap B$

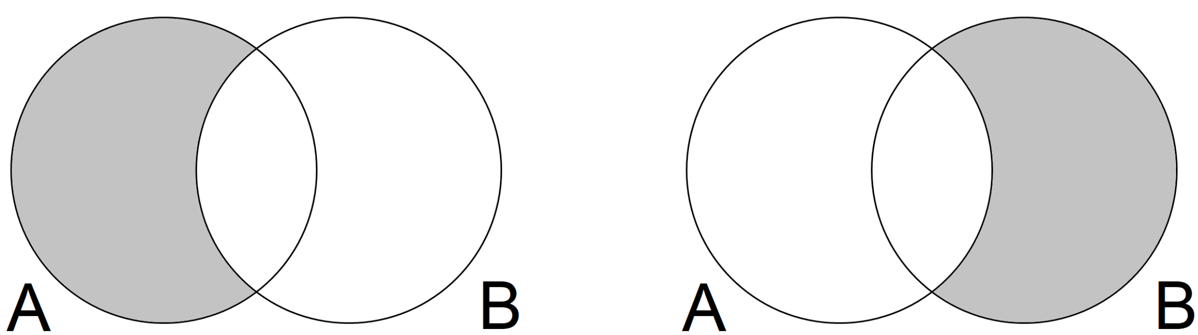
A set difference is a set operation denoted as $\setminus$. For any sets A and B

$$A \setminus B = \{x : x \in A \wedge \neg(x \in B)\} = \{x : x \in A \wedge x \notin B\} \tag{18}$$

or equivalently

$$x \in (A \setminus B) \iff x \in A \wedge \neg(x \in B) \iff x \in A \wedge x \notin B. \tag{19}$$

An element belongs to the difference of sets A and B if it belongs to the set A but not to the set B. Note that $A \cup B = B \cup A$, $A \cap B = B \cap A$ but $A \setminus B \neq B \setminus A$.



(a) $A \setminus B$ (b) $B \setminus A$

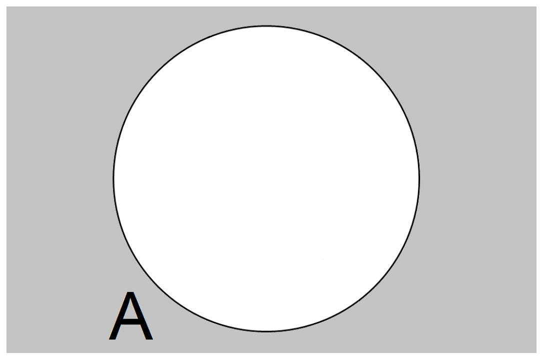
Figure 1

The last set operation that we will describe is *a set compliment*. For any set A it is denoted as A^c and defined as follows

$$A^c = \{x : \neg(x \in A)\} = \{x : x \notin A\} \tag{20}$$

or equivalently

$$x \in A^c \iff \neg(x \in A) \iff x \notin A. \tag{21}$$



A^c

Exercises

2.1. Knowing that

- $A = \{0, 2, 4, 6, 8\}$;
- $B = \{2, 3, 5, 7\}$;
- $C = \{1, 2, 3, 4\}$;
- $D = \{9\}$

assess, if the following sentences are are true or false

a) $2 \in A$;
b) $1 \in B$;
c) $4 \notin C$;
d) $0 \notin A$;
e) $(6 \in A) \wedge (9 \in D)$;
f) $(8 \notin A) \vee (0 \in C)$.

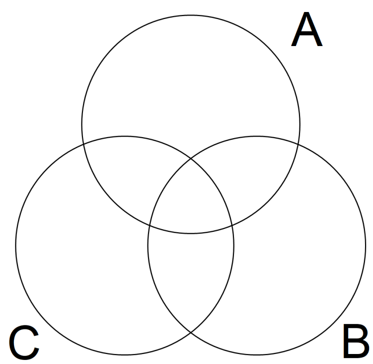
2.2. List all elements of the following sets

a) $A = \{x : 3x = 12\}$
b) $B = \{x \in \mathbb{N} : x < 7\}$;
c) $C = \{x \in \mathbb{Z} : x > -3 \wedge x \leqslant 4\}$;

d) $D = \{x \in \mathbb{N} : x > 10 \land x < 30 \land x \text{ is a prime number}\}$;

e) $E = \left\{ \frac{p}{q} : p \in \{-1, 1\}, q \in \{2, 3, 4\} \right\}$.

2.3. Using a set structure presented below



block in the following sets

a) $B \cup C$;

b) $A \setminus C$;

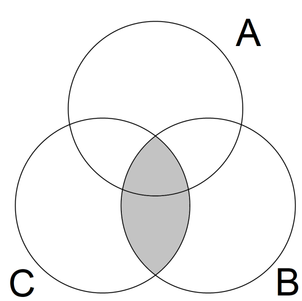
c) $(A \cup B) \cap C$;

d) $A \cup (B \cap C)$;

e) $(A \cup B) \setminus (A \cap B)$;

f) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$.

2.4. Describe the following sets by set operations

a) 

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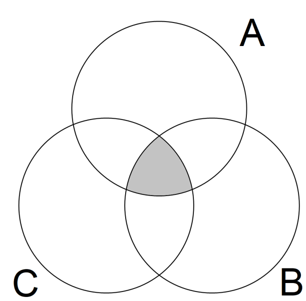
c) 

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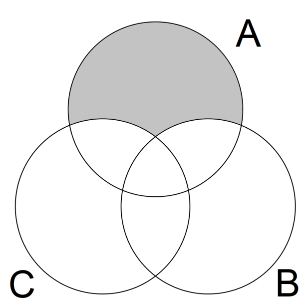
b) 

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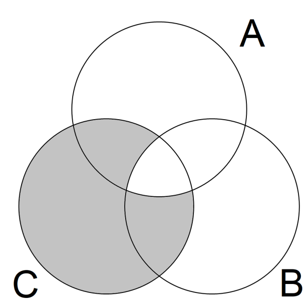
d) 

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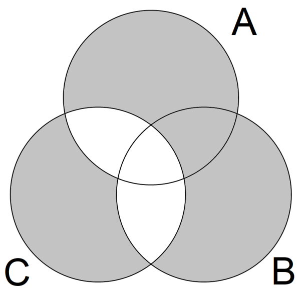
e) 

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2.5. For sets A, B, C and D defined as in task **2.1**, find sets

a) $A \cup B$;

b) $B \cap C$;

c) $B \cap D$;

d) $A \setminus C$;

e) $C \setminus A$;

f) $(A \cup C) \cap B$;

g) $(A \cap B) \cup D$.

2.6. Knowing that

- $A = (-\infty, -5]$;
- $B = [-5, 4)$;
- $C = (3, 7]$;
- $D = (8, 12]$

find sets

a) $A \cup B$;

b) $A \cap B$;

c) $B \cap C$;

d) $(A \cup B) \setminus C$;

e) $B \cap C$;

f) $C \setminus D$.