Unit 3
Functions and Properties of Functions

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1 Functions

A function f is a relation between two sets that associates every element of a first set X with exactly one element of the second set. This is written as

$$f : X \to Y. \tag{1}$$

Set X is called *the domain* of the function and elements are *arguments*. Set Y is called *the codomain* or *the target set*. A set of all the elements of Y which have some element of X assigned to them is called *the image* of the function and elements of this set are *values*.

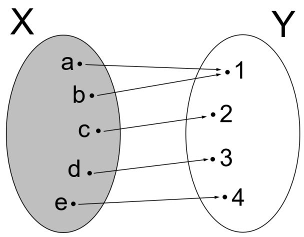
A function may be interpreted as a transformation of one number $x \in X$ into another one $y \in Y$ which is written as

$$f(x) = y. \tag{2}$$

A graph of a function f is a visualisation of all points (x, y) such that $y = f(x)$.

Exercises

1.1. Assess, if the following relations are functions

a) 

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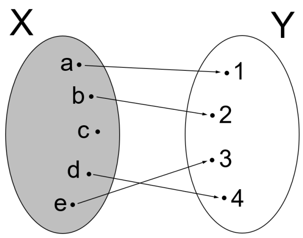
b) 

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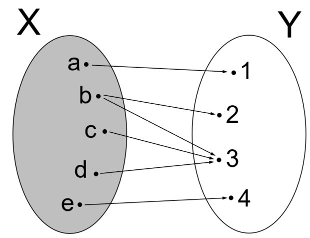
c) 

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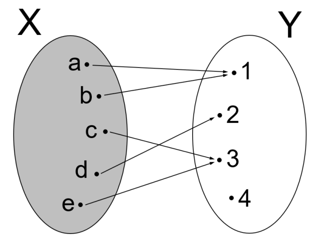
d) 

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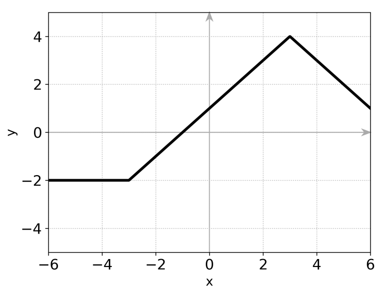
1.2. Assess, if the following graphs represent functions

a) 

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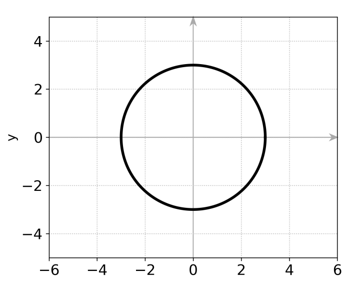
b) 

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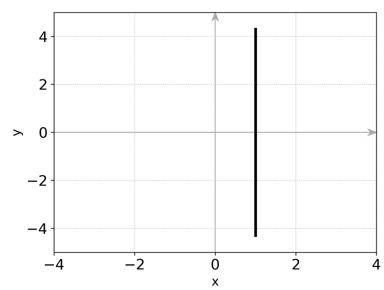
c) 

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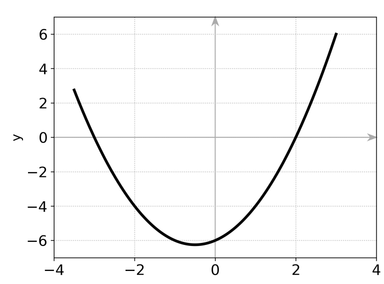
d) 

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1.3. Determine the domain of the following functions

a) $f(x) = x^2 + 2x - 3$;

b) $f(x) = \frac{1-x}{x^2+4}$;

c) $f(x) = \sqrt{x^2 + 1}$;

d) $f(x) = \frac{1}{x}$;

e) $f(x) = \frac{2x-1}{(6x-2)\left(\frac{1}{2}x-1\right)}$;

f) $f(x) = \frac{3}{(x-2)(x+1)(2x-8)}$;

g) $f(x) = \frac{x^2}{x^2-2x}$;

h) $f(x) = \frac{1}{x^2-4x+4}$;

i) $f(x) = \frac{2-x}{x^2-1}$;

j) $f(x) = \frac{x+1}{4x^2-25}$;

k) $f(x) = \sqrt{x - 1}$;

l) $f(x) = \sqrt{2x - 6}$;

m) $f(x) = 3\sqrt{5 - x}$;

n) $f(x) = \sqrt{\sqrt{3}x - 9}$;

o) $f(x) = \frac{2}{\sqrt{\frac{1}{2}x-6}}$;

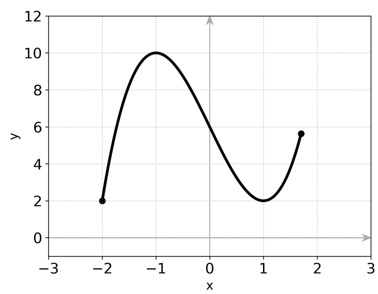
p) $f(x) = \frac{x+1}{\sqrt{6-2x}}$;

q) $f(x) = \frac{3x}{x^2-4} + \sqrt{x}$;

r) $f(x) = \frac{3-x}{x-2} + \frac{1}{x^2-9}$;

s) $f(x) = \frac{\sqrt{x}}{x^2+3} - \frac{1}{\sqrt{5-x}}$.

1.4. Based on the graphs, determine the image of the functions

a) 

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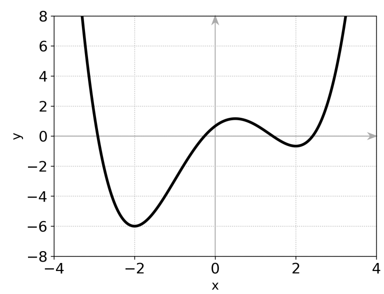
c) 

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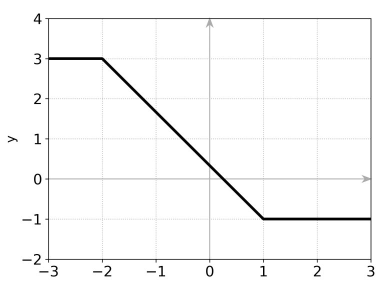
b) 

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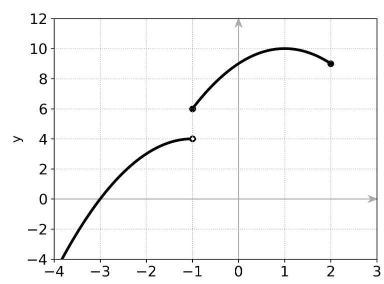
d) 

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1.5. Determine the image of the following functions on the given domains

a) $f(x) = 2x - 1, X = \{1, 2, 4, 6, 10\}$;
b) $f(x) = -x^2 + 1, X = \{-\sqrt{2}, -1, \sqrt{2}, \sqrt{3}\}$;
c) $f(x) = \frac{3}{x-2}, X = \{-1, 0, 1, 3, 5\}$;
d) $f(x) = 2^{-x}, X = \{-2, -1, 0, 1, 2\}$.

1.6. For the following functions and their domains check which points of set A belong to the image of the function

a) $A = \{4, 7, 8\}, f(x) = 3x + 1, X = \{1, 2, 3\}$;
b) $A = \left\{-1, 0, \frac{1}{3}, 4\right\}, f(x) = x^2, X = \mathbb{Q}$;
c) $A = \left\{0, \frac{1}{2}, \right\}, f(x) = \frac{6}{x+1}, X = \mathbb{Q} \setminus \{-1\}$.

1.7. Verify if the given point P, Q and R belong to the given function with a specified domain

a) $P = (3, 6), Q = (2, -5), R = (2\sqrt{3}, 9), f(x) = x^2 - 3, X = \mathbb{R}$;
b) $P = (2, 2), Q = (1, -1), R = \left(\frac{1}{4}, 2\right), f(x) = 3 - 4x, X = \mathbb{Z}$;
c) $P = (-3, 1), Q = \left(\frac{9}{4}, \frac{5}{2}\right), R = (5, 3), f(x) = \sqrt{x + 4}, X = \mathbb{N}$.

1.8. Draw the graphs of the following functions on specified domains

a) $f(x) = 2x - 1, X = \mathbb{R}$;
b) $f(x) = -x + 2, X = \mathbb{N}$;
c) $f(x) = \sqrt{x - 3}, X = [3, \infty)$;
d) $f(x) = \frac{2}{x}, X = \left\{\frac{1}{4}, \frac{1}{2}, 1, 2, 4\right\}$

2 Zero of a function

A zero, also called *a root* of a function f is an argument x_0 such that the function attains the value of 0 at x_0

$$f(x_0) = 0. \tag{3}$$

Exercises

2.1. Find the zeros of the following functions on a given domain

a) $f(x) = 2x - 8, X = (-3, 4]$;
b) $f(x) = -x + 3, X = \{0, 2, 4, 6\}$;
c) $f(x) = \frac{1}{2}x^2 - \frac{25}{2}, X = \mathbb{Q}$;
d) $f(x) = (x - 4)(x + 6), X = \{x : x^2 = 16\}$.

2.2. Find the zeros of the following functions on their natural domains

a) $f(x) = \frac{x+4}{x^2-16}$;
b) $f(x) = \frac{\frac{1}{2}x-1}{(x-2)(x+3)}$;
c) $f(x) = \frac{(x-3)(4x+12)}{x^2-9}$;
d) $f(x) = \frac{7-x}{x^2+1}$;
e) $f(x) = \frac{x^2-4}{\sqrt{x-3}}$;
f) $f(x) = \frac{x+5}{\sqrt{4-x}}$;
g) $f(x) = \frac{x^2+2x}{\sqrt{x+2}}$;
h) $f(x) = \frac{(x+1)(x+2)(x+3)}{\sqrt{-x+2}}$.

3 Injection

An injective function or, in short, *an injection* is a function that maps distinct elements of its domain to distinct elements of its codomain. Formally

$$\text{for all } x_1, x_2 \in X \quad f(x_1) = f(x_2) \implies x_1 = x_2. \tag{4}$$

Exercises

3.1. Draw the graphs of the following functions and assess if they are injective

a) $f(x) = \frac{1}{2}x + 1$;
b) $f(x) = 2$;
c) $f(x) = -x^2 + 1$;
d) $f(x) = \frac{1}{x}$;
e) $f(x) = \sqrt{x}$.

3.2. Prove that the following functions are injective

a) $f(x) = 3x - 1$;
b) $f(x) = \frac{2}{3}x + 4$;
c) $f(x) = \frac{2}{x}$;
d) $f(x) = \frac{x-1}{x+1}$;
e) $f(x) = 3\sqrt{x}$;
f) $f(x) = \sqrt{2x - 1}$.

4 Monotonicity

A monotonic function f is a function which is either *increasing*, *decreasing* or *constant*.

A function is increasing on a given subset $A \subseteq X$ if for smaller arguments it attains smaller values and for bigger arguments it attains bigger values. Formally

$$\text{for all } x_1, x_2 \in A \quad x_1 < x_2 \implies f(x_1) < f(x_2). \tag{5}$$

or equivalently

$$\text{for all } x_1, x_2 \in A \quad x_1 - x_2 < 0 \implies f(x_1) - f(x_2) < 0 \tag{6}$$

A function is decreasing on a given subset $A \subseteq X$ if for smaller arguments it attains bigger values and for bigger arguments it attains smaller values. Formally

$$\text{for all } x_1, x_2 \in A \quad x_1 < x_2 \implies f(x_1) > f(x_2) \tag{7}$$

or equivalently

$$\text{for all } x_1, x_2 \in A \quad x_1 - x_2 < 0 \implies f(x_1) - f(x_2) > 0. \tag{8}$$

A function is constant on a given subset $A \subseteq X$ if for all arguments it attains the same value. Formally

$$\text{for all } x_1, x_2 \in A \quad x_1 \neq x_2 \implies f(x_1) = f(x_2) \tag{9}$$

or equivalently

$$\text{for all } x_1, x_2 \in A \quad x_1 - x_2 \neq 0 \implies f(x_1) - f(x_2) = 0. \tag{10}$$

Exercises

4.1. Draw the graphs of the following functions and determine their monotonicity

a) $f(x) = \frac{1}{2}x + 1$;

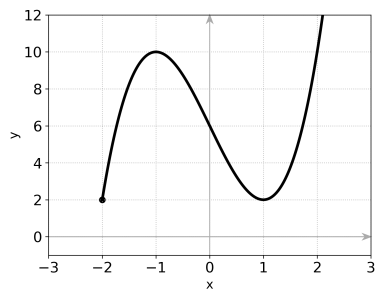
b) $f(x) = 2$;

c) $f(x) = -x^2 + 1$;

d) $f(x) = \frac{1}{x}$;

e) $f(x) = \sqrt{x}$.

4.2. Based on the graphs, determine the intervals of monotonicity of the functions

a) 

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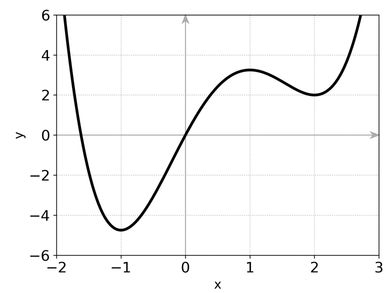
c) 

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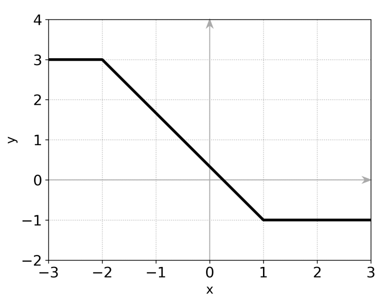
b) 

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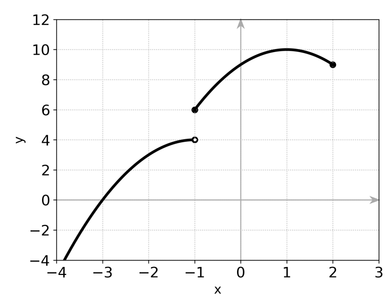
d) 

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4.3. Prove that the following functions are monotonic (in a specified way) on a given interval

a) increasing, $f(x) = 3x - 1, A = \mathbb{R}$;

b) decreasing, $f(x) = -\frac{2}{3}x + 4, A = \mathbb{R}$;

c) decreasing, $f(x) = \frac{2}{x}, A = (0, \infty)$;

d) increasing, $f(x) = \frac{x-1}{x+1}, A = (-\infty, -1)$;

e) constant, $f(x) = 3\sqrt{x}, A = \mathbb{R}$;

f) increasing, $f(x) = \sqrt{2x - 1}, A = \left[\frac{1}{2}, \infty\right)$.

5 Parity

To determine function's *parity* is to tell whether a function is *even* or *odd*.

A function is even when its plot is symmetric with respect to the y-axis. Formally

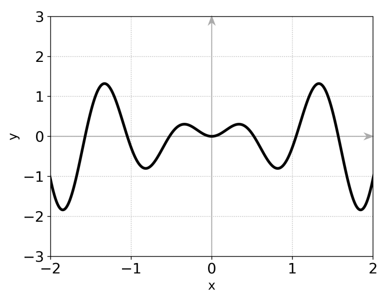
$$\text{for any } x \in X \quad f(x) = f(-x). \tag{11}$$

A function is odd when its plot is symmetric with respect to the origin of the x-y-plane. Formally

$$\text{for any } x \in X \quad f(x) = -f(-x). \tag{12}$$

Exercises

5.1. Based on the graphs, determine parity of the functions

a) 

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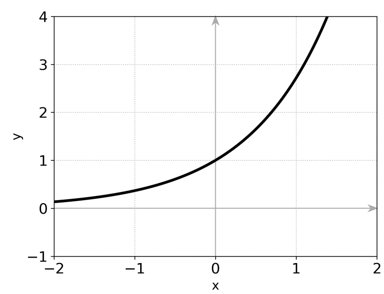
d) 

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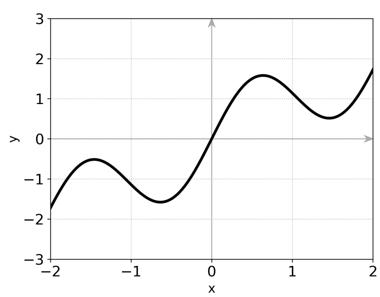
b) 

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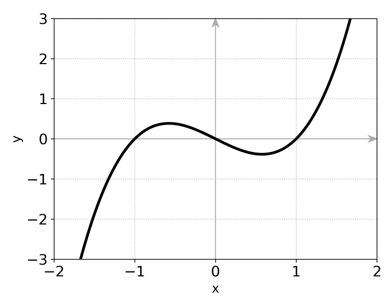
e) 

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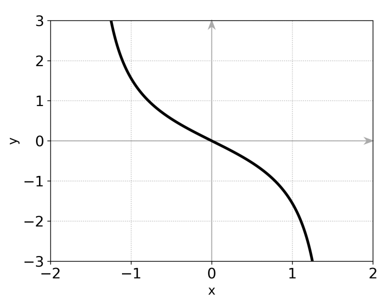
c) 

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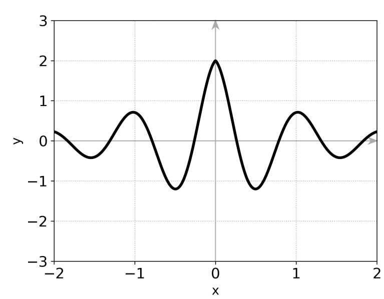
f) 

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5.2. Draw the graphs of the following functions and determine their parity

a) $f(x) = 3x$;
b) $f(x) = -x^2 + 4$;
c) $f(x) = x^3$;
d) $f(x) = \frac{2}{x}$;
e) $f(x) = 0$;

5.3. Prove specific parity of the following functions

a) even, $f(x) = \frac{x^2}{x^2+1}$;
b) odd, $f(x) = -\frac{2}{3x}$;
c) odd, $f(x) = x^3 - 3x$;
d) even, $f(x) = \frac{1}{x^4}$;
e) even, $f(x) = 2^x + 2^{-x}$;
f) odd, $f(x) = \frac{x}{x^2-1}$.