Unit 4

Linear Function

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1 Linear function and its properties

A *linear function* f a function expressed by the following formula

$$f(x) = ax + b. \tag{1}$$

where a and b are constants, $a, b \in \mathbb{R}$. Number a is called a *slope* of a function and number b is called an *intercept*.

Slope of the function determines of it is increasing, decreasing or constant

- if $a > 0$ then the function is increasing,
- if $a < 0$ then the function is decreasing,
- if $a = 0$ then the function is constant.

The graph of a linear function is a straight line. Hence - two points on the plane are sufficient to find out the formula of the full formula of a linear function.

If we have two functions $f(x) = a_1x + b_1$ and $g(x) = a_2x + b_2$ then the lines of the graphs of those functions

- are parallel if $a_1 = a_2$,
- are perpendicular if $a_1 \cdot a_2 = -1$.

Exercises

1.1. Assess, if point A belongs to the graph of the function f

a) $A = (2, 5), f(x) = 3x - 1;$

b) $A = (4, 1), f(x) = -x + 4;$

c) $A = (3, 3), f(x) = \frac{2}{3}x + 1;$

d) $A = (2\sqrt{2}, 3), f(x) = \sqrt{2}x - 2.$

1.2. Complete the formula of a function (find a or b) based on the point A which belongs to its graph. Draw the graphs of those functions.

a) $A = (-3, -8), f(x) = ax - 2;$

b) $A = (4, 12), f(x) = 3x + b;$

c) $A = (8, -1), f(x) = ax + 7;$

d) $A = (-2, 1), f(x) = \frac{3}{2}x + b.$

1.3. Find the formula of a function based on points A and B which belongs to its graph

a) $A = (3, 4), B = (5, 6);$

b) $A = (4, -3), B = (8, -6);$

c) $A = (\sqrt{3}, 3 + \sqrt{5}),$
$B = (3, 3\sqrt{3} + \sqrt{5});$

d) $A = (0, -1), B = (\frac{1}{4}, 0).$

1.4. Knowing that the graph of a linear function f is parallel to graph of a linear function g and that the graph of f includes point A, find the formula for the function f:

a) $A = (-1, -2), g(x) = 3x - 2;$

b) $A = (4, 5), g(x) = \frac{1}{2}x - 1;$

c) $A = (3, 10), g(x) = 7x + 8;$

d) $A = (-1, 4), g(x) = 1.$

1.5. Knowing that the graph of a linear function f is perpendicular to graph of a linear function g and that the graph of f includes point A, find the formula for the function f:

a) $A = (4, 2), g(x) = -x + 2;$

b) $A = (2, 5), g(x) = 2x + 3;$

c) $A = (-1, 1), g(x) = -\frac{1}{3}x + 10;$

d) $A = (6, 3\sqrt{3}), g(x) = -\sqrt{3}x + 1.$

1.6. Determine for which values of a parameter m the following functions represent given monotonicity

a) $f(x) = (2m-3)x + 4$, increasing;
b) $f(x) = -(4m+1)x + 9m$, decreasing;
c) $f(x) = \frac{1}{m-1}x + \frac{m}{3}$, increasing;
d) $f(x) = \sqrt{3m-9}x - 3$, constant.

1.7. Determine for which values of parameter m the following linear functions f and g are parallel

a) $f(x) = 3x + 2m, \quad g(x) = (2m+1)x + 2;$
b) $f(x) = (9m-7)x + 3m, \quad g(x) = (-3m+5)x - (13m-2);$
c) $f(x) = (m^2-9)x + 2m, \quad g(x) = 12m;$
d) $f(x) = \frac{1}{m-3}x + 1, \quad g(x) = \frac{1}{2m+1} + m.$

1.8. Determine for which values of parameter m the following linear functions f and g are perpendicular

a) $f(x) = -(2m-1)x + 3, \quad g(x) = \frac{1}{3}x + 1;$
b) $f(x) = mx + 4, \quad g(x) = -mx + 3m;$
c) $f(x) = mx, \quad g(x) = 3;$
d) $f(x) = \frac{1}{m-1}x + 1, \quad g(x) = (2m+10)x - 24m.$

2 Linear equations and inequalities

Solving a linear equation is finding the solution of the following equality

$$ax + b = 0 \tag{2}$$

which is given by

$$x = -\frac{b}{a}, \quad a \neq 0. \tag{3}$$

Solving a linear equation is a process which need to be performed in order to find the root (zero) of a linear function.

Exercises

2.1. Solve the following linear equations

a) $3x - 18 = 0$;

b) $3x + 21 = 5x + 23$;

c) $2(3 - x) = x$

d) $x(x - 4) = (x - 2)(x - 3)$;

e) $(x + 3)^2 = 2(3x + 5) + x^2$;

f) $\frac{x}{6-x} = 2$;

g) $\frac{x^2-4}{x-2} = x$;

h) $5x^2 + 2(5x + 2) = 4 + 5(x^2 + 2x)$.

2.2. Find the values of a parameter m such that the solution of the given equation meets given criteria

a) $(m + 2)x - 4 = 0$, solution is 1;

b) $(2m + 5)x - m - 10 = 0$, solution is 3;

c) $2(2x - m) = 2x + m$, solution is bigger than $\frac{3}{2}$;

d) $9x + 5m = -mx$, solution is smaller or equal than -2;

e) $x - 2m = 0$, solution is between 1 and 10 (inclusive).

2.3. Solve the following inequalities

a) $3x - 12 < 0$;

b) $4x \geqslant 2(5 + x)$;

c) $2(x + 3) - (1 + 3x) > 4 - x$;

d) $\left(x + \frac{1}{2}\right)^2 \geqslant (1 + x)^2 - \frac{7}{4}$;

e) $(x - 5)(x + 5) \leqslant (x - 1)(x + 1)$.

3 Linear systems of equations

Linear system of two equations is a system of the form

$$\begin{cases} ax + by &= p \\ cx + dy &= q \end{cases} \tag{4}$$

To solve such system, various methods can be used, including:

- elimination method (also known as the method of opposite coefficients);
- substitution method;
- graphical method;
- determinants method.

To use the *determinants method*, one should calculate three *determinants*:

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb; \tag{5}$$

$$A_x = \begin{vmatrix} p & b \\ q & d \end{vmatrix} = pd - qb; \tag{6}$$

$$A_y = \begin{vmatrix} a & p \\ c & q \end{vmatrix} = aq - cp. \tag{7}$$

Solution is given by

$$x = \frac{A_x}{A}, \quad y = \frac{A_y}{A} \tag{8}$$

provided $A \neq 0$. If $A = 0$, the system is either *indeterminate* which means there are infinitely many solutions or it is *inconsistent* (also called *contradictory*) which means no solutions exist.

Exercises

3.1. Solve each of the following systems of linear equations using the method of elimination, the method of substitution, the graphical method and the determinants method.

a) $\begin{cases} x + 2y = 8 \\ 2x + y = 7 \end{cases}$

b) $\begin{cases} 3x + y = 1 \\ 2x - 2y = -10 \end{cases}$

c) $\begin{cases} \dfrac{x+2y}{3} + x = 8 \\ \dfrac{2x+y}{4} + y = 5 \end{cases}$

d) $\begin{cases} (x+2)^2 - x^2 + 2(y-3) = 2 \\ (y+1)(y-1) + x - 2y = y^2 \end{cases}$

3.2. Find the values of a parameter m such the given system of equations has a unique solution, is indeterminate or contradictory

a) $\begin{cases} 2x + y = m \\ x + 2y = 1 \end{cases}$

b) $\begin{cases} x - my = 1 \\ mx - y = 1 \end{cases}$

c) $\begin{cases} 2x + 3y = 3 \\ 4x + my = 2m \end{cases}$

d) $\begin{cases} x + my = 3 \\ mx + 4y = 2m \end{cases}$

3.3. Block in regions of the plane to which solutions of the following systems of inequalities belong

a) $\begin{cases} -x + 2y \leqslant 0 \\ x + 2y \geqslant 0 \end{cases}$

b) $\begin{cases} x - y + 1 > 0 \\ x + 2y - 2 < 0 \\ x - 4y + 2 < 0 \end{cases}$

c) $\begin{cases} y > 0 \\ y < 3 \\ y < 3x \\ y > 3x - 6 \end{cases}$

d) $\begin{cases} 2x - y + 2 \geqslant 0 \\ -2x - y + 2 \geqslant 0 \\ 2x - y - 2 \leqslant 0 \\ -2x - y - 2 \leqslant 0 \end{cases}$