Unit 6
Quadratic Function

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1 Standard and vertex form of a quadratic function

A quadratic function f is a function expressed by the following formula

$$f(x) = ax^2 + bx + c \tag{1}$$

where $a, b, c \in \mathbb{R}, a \neq 0$.

The graph of a quadratic function is a *parabola*.

Equation (1) presents what is called *the standard form* of how a quadratic function can be expressed. Another form of expressing a quadratic function is *the vertex form* presented below

$$f(x) = a(x - p)^2 + q \tag{2}$$

where $a, p, q \in \mathbb{R}, a \neq 0$. Relation between p and q in formula (2) and a, b and c in formula (1) is as follows

$$p = \frac{-b}{2a}, \tag{3}$$

$$q = \frac{-\Delta}{4a}. \tag{4}$$

Parameter Δ is called *a discriminant* and it can be calculated as follows

$$\Delta = b^2 - 4ac. \tag{5}$$

Parameters p and q have a natural interpretation of being x and y coordinates of the vertex (the tip) of the parabola.

Exercises

1.1. Draw graphs of the following functions

a) $f(x) = x^2$,
b) $f(x) = (x-2)^2 + 1$,
c) $f(x) = -(x+1)^2 - 3$,
d) $f(x) = 2x^2 - 1$,
e) $f(x) = \frac{1}{2}(x+3)^2 + 4$,
f) $f(x) = -\frac{2}{3}(x-1)^2$.

1.2. Bring the following functions to the vertex form

a) $f(x) = x^2 + 2x - 1$,
b) $f(x) = x^2 + 6x + 4$,
c) $f(x) = x^2 - 12x + 45$,
d) $f(x) = \frac{1}{2}x^2 + 4x + 1$,
e) $f(x) = \sqrt{2}x^2 + 4x$,
f) $f(x) = x^2 - 6\sqrt{3}x + \sqrt{2} + 27$.

1.3. Bring the following functions to the standard form

a) $f(x) = (x+1)^2$,
b) $f(x) = (x-3)^2 + 2$,
c) $f(x) = -(x-1)^2 - 5$,
d) $f(x) = 2(x+9)^2 - 2$,
e) $f(x) = \frac{1}{2}(x+4)^2 + 10$,
f) $f(x) = -\frac{2}{3}(x+12)^2$

2 Quadratic equations and inequalities

Solving a quadratic equation is finding solutions of the following equality

$$ax^2 + bx + c = 0 \tag{6}$$

Solving a quadratic equation is a process which need to be performed in order to find the roots (zeros) of a quadratic function.

The number of zeros that a quadratic function has is determined by the value of the discriminant Δ, introduced in the equation (5). Concerning that, the following options are possible

- if $\Delta > 0$, the function has two zeros and the associated quadratic equation has two solutions — x_1 and x_2:

$$x_1 = \frac{-b - \sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a}. \tag{7}$$

  In such case the formula of the function can be rewritten as

$$f(x) = a(x - x_1)(x - x_2). \tag{8}$$

- if $\Delta = 0$, the function has one zeros and the associated quadratic equation has one solution — x_0:

$$x_0 = -\frac{b}{2a}. \tag{9}$$

  In such case the formula of the function can be rewritten as

$$f(x) = a(x - x_0)^2. \tag{10}$$

- if $\Delta < 0$, the function does not have any zeros and the associated quadratic equation has no solutions.

Forms (8) and (10) of expressing a quadratic functions are called *factored forms.*

Exercises

2.1. Bring the following functions to the standard form

a) $f(x) = (x - 3)(x + 2)$,

b) $f(x) = (x - 1)(x + 1)$,

c) $f(x) = 8(x - 2)\left(x - \frac{1}{2}\right)$,

d) $f(x) = \frac{1}{3}(x - 3)(x + 3)$.

2.2. Bring the following functions to the vertex form

a) $f(x) = (x - 4)(x + 2)$,

b) $f(x) = -(x - 5)(x - 1)$,

c) $f(x) = (x - 3)^2$,

d) $f(x) = \frac{\sqrt{2}}{2}(x + \sqrt{2})(x + 3\sqrt{2})$.

2.3. If possible, bring the following functions to the factored form

a) $f(x) = x^2 - 6x + 8$,

b) $f(x) = 9x^2 - 3x - 2$,

c) $f(x) = x^2 - \frac{11}{2}x + 6$,

d) $f(x) = (x - 1)^2 - 4$,

e) $f(x) = (x-2)^2 + 3$,

f) $f(x) = 25 - x^2$.

2.4. Solve the following quadratic equations

a) $x^2 - 4 = 0$,

b) $2x^2 + 1 = 0$,

c) $4x^2 - 5 = 0$,

d) $3x^3 - 9x = 0$,

e) $x^2 + 12x = 0$,

f) $(2x+4)^2 - 16 = 0$,

g) $4(x+1)^2 - 25 = 0$,

h) $81 - (3x-12)^2 = 0$,

i) $x^2 - 10x + 21 = 0$,

j) $x^2 + 12x + 34 = 0$,

k) $4x^2 + 12x + 9 = 0$,

l) $x^2 - 4x - 5 = 0$,

m) $3x^2 + 7x - 20 = 0$,

n) $-x^2 + 5x - 15 = 0$,

o) $(x-4)(x+5) = 7x - 29$,

p) $-(x-2)^2 - 5 = 0$,

q) $(x+2)\left(x+\frac{1}{2}\right) = 0$

r) $2(x-1)(x+4) = 0$,

s) $3(x-2) + (x+1)(x-2) = 0$.

2.5. Solve the following equations by bringing them to the form of quadratic equations

a) $x^4 - 25x^2 + 144 = 0$,

b) $x^4 + x^2 - 6 = 0$,

c) $\sqrt{x+14} + x + 2 = 0$,

d) $\sqrt{x - 1 + \sqrt{x+2}} = 3$,

e) $\sqrt{15-x} + \sqrt{3-x} = 6$.

2.6. Determine for which values of the m parameter following equations have two different solutions

a) $(m+2)x^2 - 2x + m + 2 = 0$,

b) $mx^2 - (m+1)x - 2m + 3 = 0$.

2.7. Solve the following quadratic inequalities

a) $x^2 + 5 > 0$,

b) $x^2 - 9 \leqslant 0$,

c) $-x^2 + 16 > 0$,

d) $x^2 - 2x \geqslant 0$,

e) $5x - x^2 < 0$,

f) $x^2 - 8x + 16 \leqslant 0$,

g) $-(x+5)^2 < 0$,

h) $3(x-2)(x+6) > 0$,

i) $-(x+5)(x-3) \geqslant 0$,

j) $x^2 + 2x - 3 < 0$,

k) $15 - 2x - x^2 > 0$,

l) $-x^2 - 2x - 18 \geqslant 0$.

3 Vieta's formulas

For any quadratic function of the form as in (1), if $\Delta > 0$, the following equalities hold, called the *Vieta's formulas*

$$x_1 + x_2 = -\frac{b}{a}, \tag{11}$$

$$x_1 \cdot x_2 = \frac{c}{a}. \tag{12}$$

In case $\Delta = 0$, those are reduced to

$$2x_0 = -\frac{b}{a}, \tag{13}$$

$$(x_0)^2 = \frac{c}{a}. \tag{14}$$

Exercises

3.1. Without calculating zeros of the following functions, find the sum and the product of their roots

a) $f(x) = x^2 - 8x + 15$,

b) $f(x) = 2x^2 + 12x - 14$,

c) $f(x) = \frac{1}{2}x^2 - x - 4$,

d) $f(x) = \frac{\sqrt{2}}{2}x^2 - 4\sqrt{2}$.

3.2. Assuming that in formula the (1) $a = 1$, calculate b and c knowing that

a) $x_1 = 2$ and $x_2 = 7$,

b) $x_1 = 3$ and $x_1 \cdot x_2 = 12$,

c) $x_1 = 5$ and $x_1 + x_2 = 2$,

d) $x_1 + x_2 = 4$ and $x_1 \cdot x_2 = -5$.

3.3. Transform the following expressions so that they can be calculated only using the values of the Vieta's formulas

a) $\frac{1}{x_1} + \frac{1}{x_2}$,

b) $(x_1)^2 + (x_2)^2$,

c) $\frac{1}{(x_1)^2} + \frac{1}{(x_2)^2}$,

d) $\frac{x_1}{x_2} + \frac{x_2}{x_1}$,

e) $(x_1 - x_2)^2$,

f) $|x_1 - x_2|$.

3.4. Determine for which values of the m parameter following equations have two solutions of the opposite sign

a) $x^2 - 2(m-1)x + 2m + 1 = 0$, b) $2x^2 - 3(m-1)x + 1 - m^2 = 0$.

3.5. Determine for which values of the m parameter following equations have two distinct, positive solutions

a) $x^2 - mx + \frac{1}{4}m(m-1) = 0$, b) $x^2 - 2(m-2)x + m^2 - 2m - 3 = 0$.

3.6. Determine for which values of the m parameter following equations have two distinct solutions of the same sign

a) $x^2 + (m-3)x + \frac{1}{4}m(m+2) = 0$, b) $2x^2 + (m-9)x + m^2 + 3m + 4 = 0$.

3.7. Determine for which values of the m parameter the sum of solutions of the following equations equals to given number S

a) $x^2 + 2(m-1)x + m^2 - 4 = 0$, $S = 12$,

b) $x^2 - mx - m(m-1) = 0$, $S = 1$.

3.8. Determine for which values of the m parameter the inverse of the sum of solutions of an equation $2x + m(1 - x^2) = 2 + 2x^2$ is positive.

3.9. Determine for which values of the m parameter the sum of solutions of an equation $x^2 - 2m(x-1) - 1 = 0$ is equal to the sum of squares of those solutions.

3.10. Determine for which values of the m parameter the sum of inverses of solutions of an equation $x^2 + mx - 16 = 0$ is equal to -4.

