

# Flow and elasticity in human eyes and the measurement of IOP – by eye shape ?!

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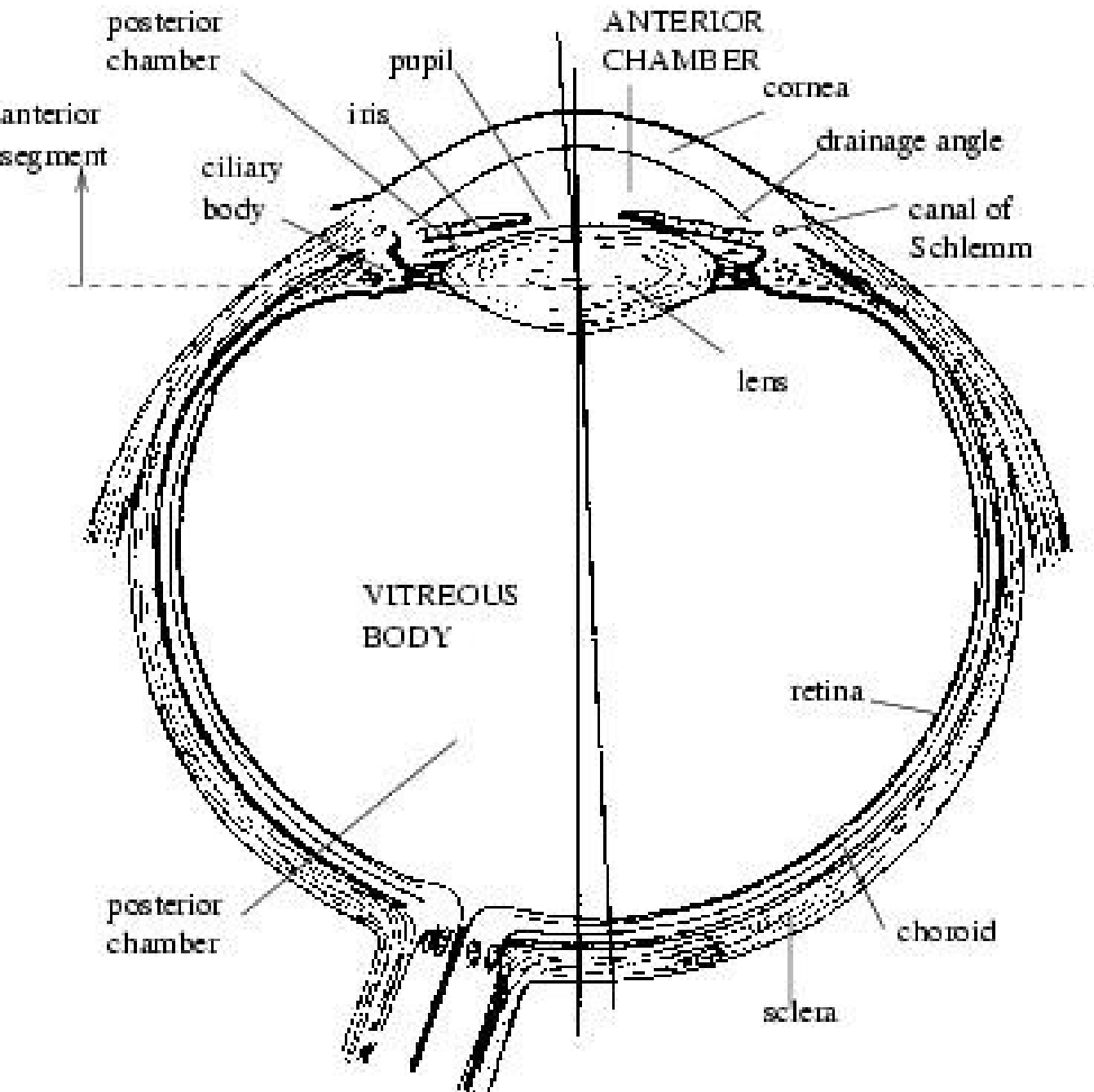
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Bedlowo, Poland October 2010

# Philosophy and plan.....

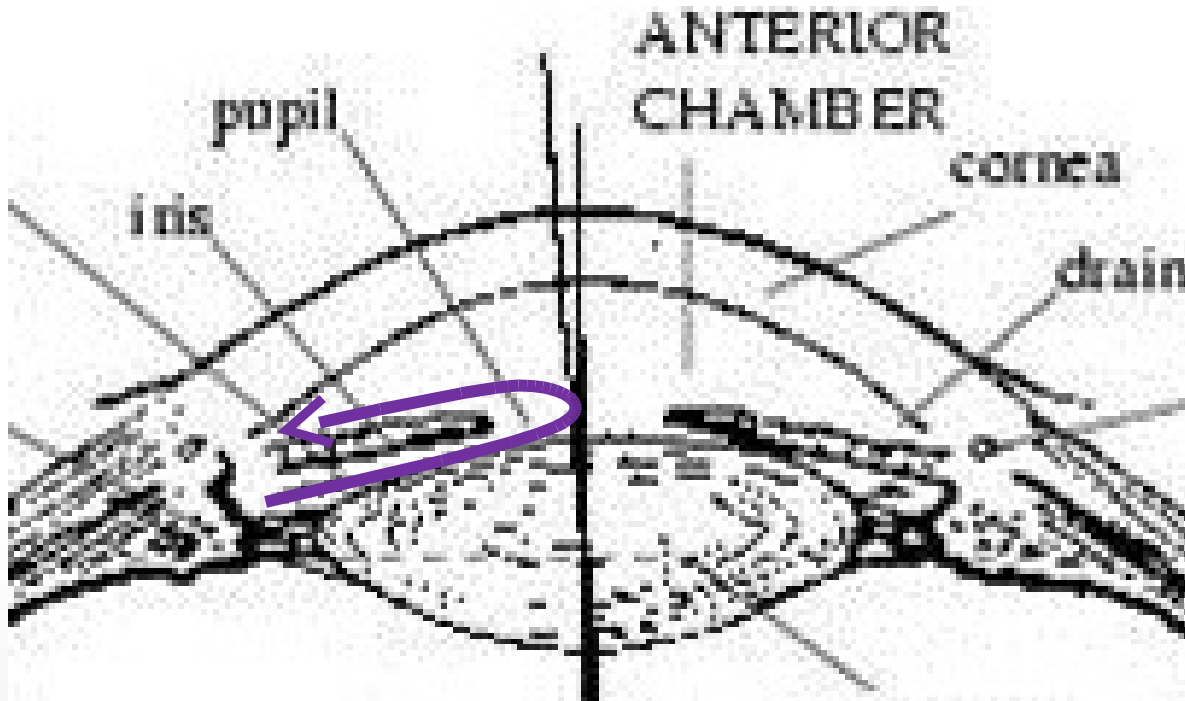
- This talk is about problems from *Mathematical Medicine*: a “popular area” but should be done not only in a University Department but also with proper medics
- *Go to operations! Smell the blood! Dish the gore!* etc. etc.
- I’ll tell you about a range of eye-related problems and their solutions
- The specific problem for this week is of a very interesting type that may involve

- *Consideration of measured eye shape data*
- *Elasticity theory*
- *Numerical calculations*
- *..... And many other things too – we don’t know yet!*



# THE MAIN BITS OF A HUMAN EYE

Diameter – about 1  
inch

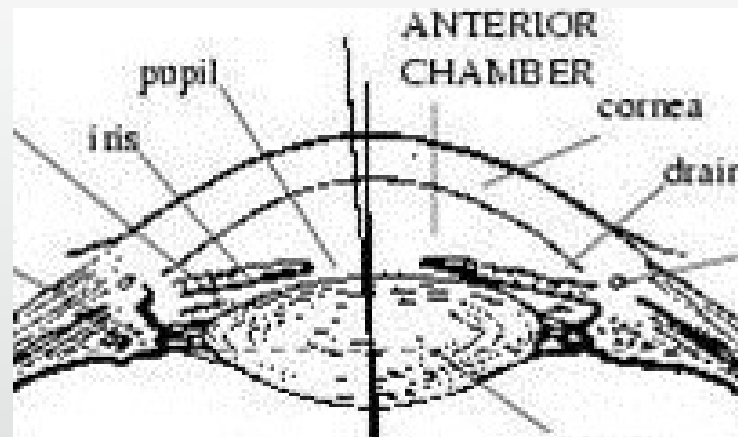


THE MAIN  
BITS OF A  
HUMAN  
EYE - FLOW

*But there can be a lot more to it than that – as we'll see later!*

# What's the deal with IOP?

- The first thing that you have to understand is the significance of IOP (Intra Ocular Pressure)
- To keep its shape the human eye is internally pressurized – a little bit
- The pressurization is passive in the posterior segment, but a very important issue in the anterior chamber (AC)



# The main issues with IOP:

- What is normal IOP?
- What happens if the IOP is not normal?
- How is IOP maintained?
- What can lead to abnormal IOP?
- How is IOP measured?

# The main issues with IOP:

- What is normal IOP?
  - Normal IOP is about 12-22 mm Hg
  - IOP varies during the day and with age and sex
  - Yes – mm Hg is not SI, but that's what they use
    - so get over it!
  - $760 \text{ mm Hg} = 101325 \text{ Pa} = 1 \text{ atm}$
  - $1 \text{ mm Hg} = 133.322 \text{ Pa}$
  - Therefore IOP of 15 mm Hg means that the eye is pressurised to about 1.02 atm

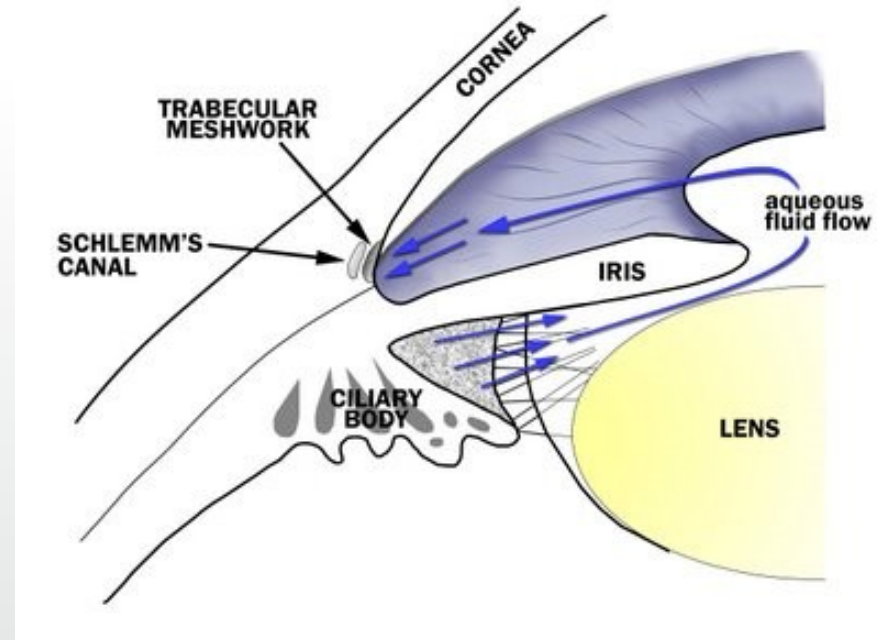
# The main issues with IOP:

- What happens if the IOP is not normal?
  - Many bad things happen if IOP is not normal
  - If IOP is too high (ocular hypertension) then the optic nerve can be damaged by defenestration of the lamina cribrosa
  - If IOP is too high the blood supply to the retina can be cut off – leading to blindness in a few minutes
  - If IOP is too low (hypotony) then the retina, lens and cornea can all become distorted. In severe cases the cornea can collapse like a deflated balloon



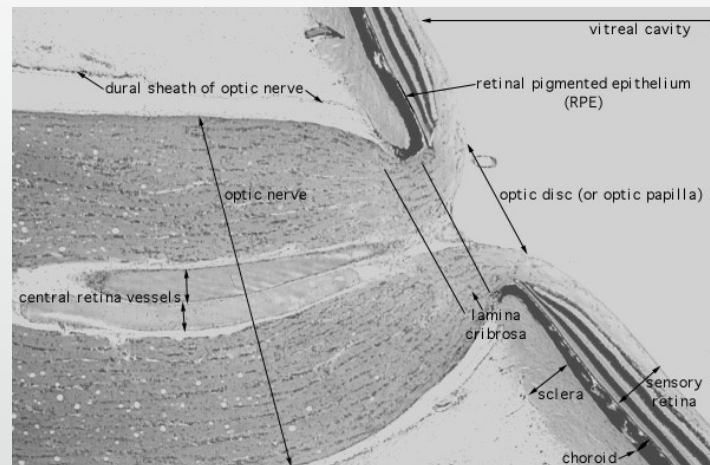
# The main issues with IOP:

- How is IOP maintained?
  - Ciliary body ultrafiltrates blood to make AH (Aqueous humour) ( $\sim 2 \mu$  litres/min)
  - Ciliary body can regulate itself – to a degree
  - AH flows out of the drainage angle, through the Trabecular meshwork, into the Canal of Schlemm and away into the lymph system
  - “Uveoscleral” outflow can also take place from the “back door” of the ciliary body
  - This can also be regulated - to a degree



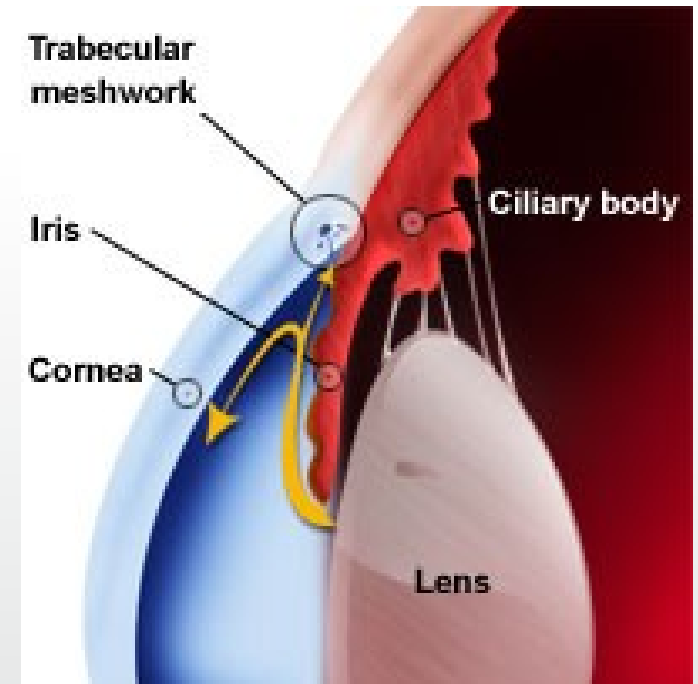
# The main issues with IOP:

- What can lead to abnormal IOP (i)?
  - The most common cause of high IOP is glaucoma
  - Glaucoma is an optic nerve neuropathy – NOT just high IOP
  - - but patients with high IOP may not have glaucoma
  - - and patients with glaucoma may not have high IOP



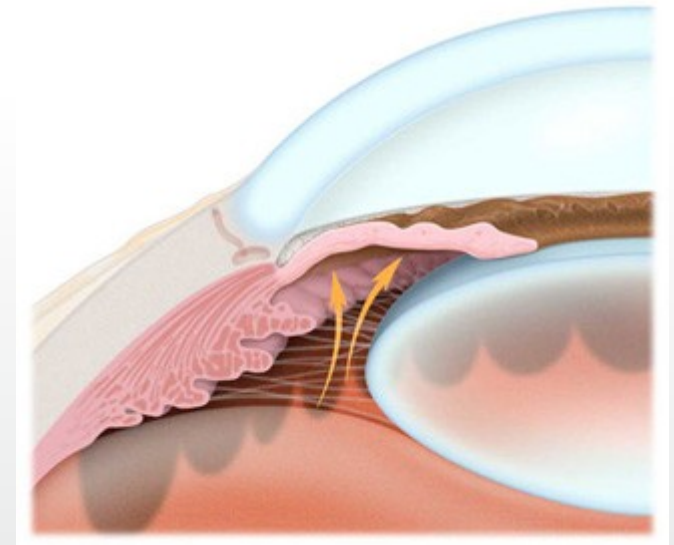
## The main issues with IOP:

- What can lead to abnormal IOP (ii)?
  - Open angle glaucoma:
    - TM and/or Schlemm canal may become blocked
    - Ciliary body cannot slow down enough to stop producing aqueous humour
    - IOP slowly rises.....



## The main issues with IOP:

- What can lead to abnormal IOP (iii)?
  - Closed angle glaucoma:
    - Lens touches iris and stops flow
    - Ciliary body makes AH but it has nowhere to go
    - IOP rises very fast – rapid pain and blindness.....
  - (*also buphthalmos, neovascular glaucoma – other rarer glaucomas*)



# The main issues with IOP:

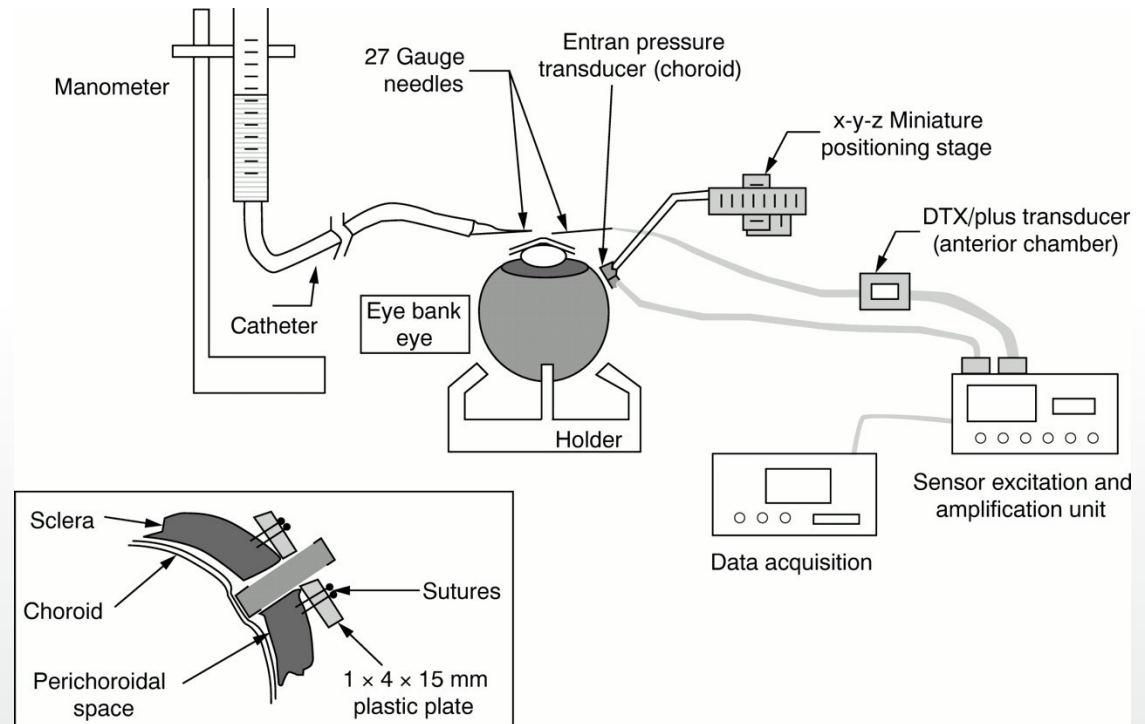
- How is IOP measured?
  - When patients have glaucoma etc. It is essential to be able to measure the IOP accurately. The science of doing this is TONOMETRY. Methods of tonometry include:

- Direct pressure measurement
- Applanation tonometry
- Rebound tonometry
- Other methods (OCT, Transpalpebral, Pneumatometry, Impression, Ultrasound tonometry)?

- *EYE SHAPE TONOMETRY?*

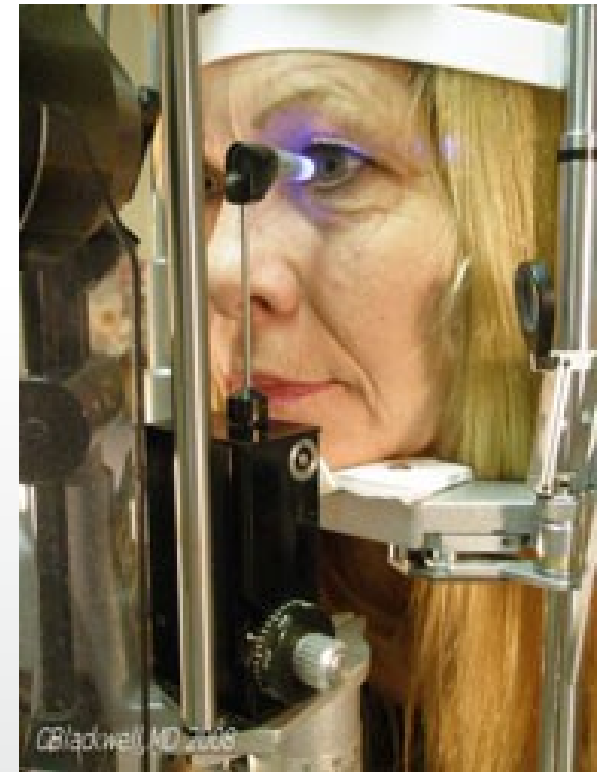
# Direct pressure:

- Wire the eye up to a manometer
- Requires an operation, very invasive
- Real-time measurements, very accurate
- May be done to lab animals
- Sometimes done to humans
  
- Not at all practical for everyday use



# Applanation tonometry:

- Topically anaesthetize the eye
- Use e.g. Goldmann tonometer
- Using slit lamp to judge, press against cornea till known circle (diameter 3.06 mm) is flattened (“applanated”)
- Use Imbert-Fick rule: each 0.1 gm force required to do this corresponds to 1 mm Hg IOP
- Pretty accurate
- Standard procedure
- Involves contact – prions?
- Very practical for everyday use



# Rebound tonometry:

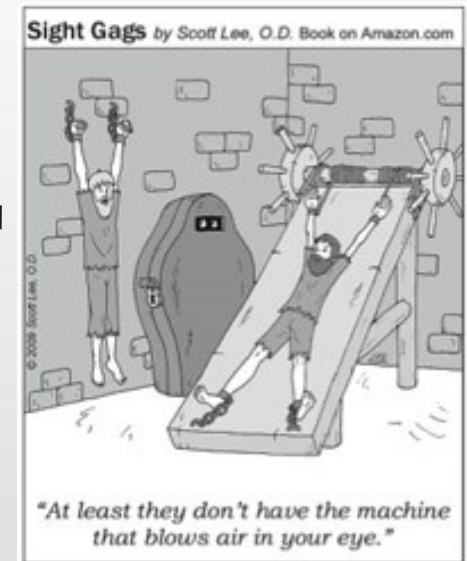
- Done for example using the “tonopen”
- Fire a (very small!) rounded spike into the cornea
- Electronically measure how fast it retracts after it hits the cornea
- Use this to “calculate” (?) the IOP
- Largely used on animals up to now
- Produces “no sensation at all”
- Starting to be used on humans





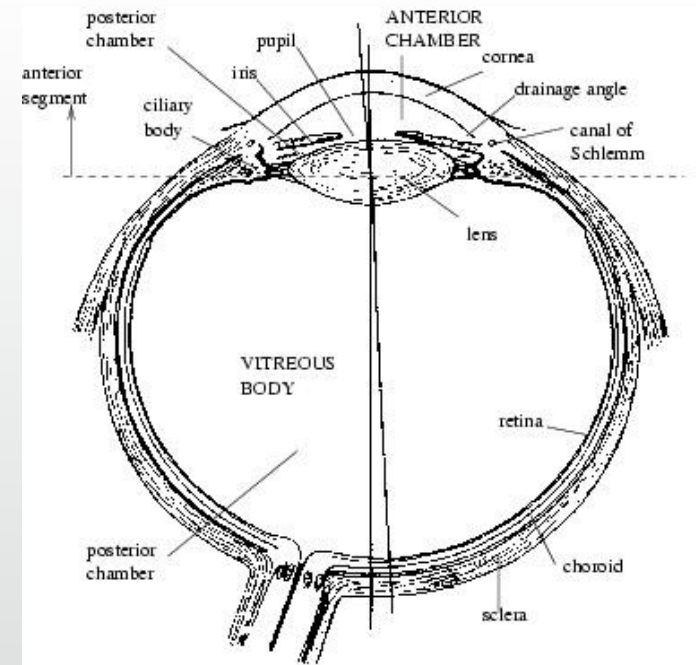
## Other tonometry methods:

- Many different ideas have been trialled, e.g.
- Ultrasonic – use sound to measure pressure
- Transpalpebral – measure through closed eyelids
- “Air puff” – measure reaction of cornea to air puff
- Methods have to be quick, easy, non-invasive
- .....and acceptable to the patient!
- Accuracy is an issue with all of these methods
- **ONE FINAL IDEA: USE THE SHAPE OF THE EYE?!!**



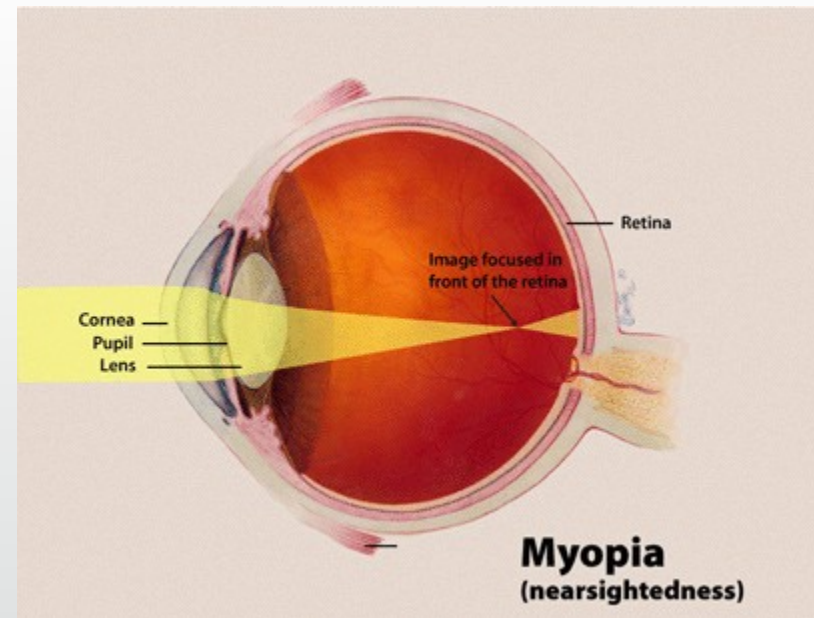
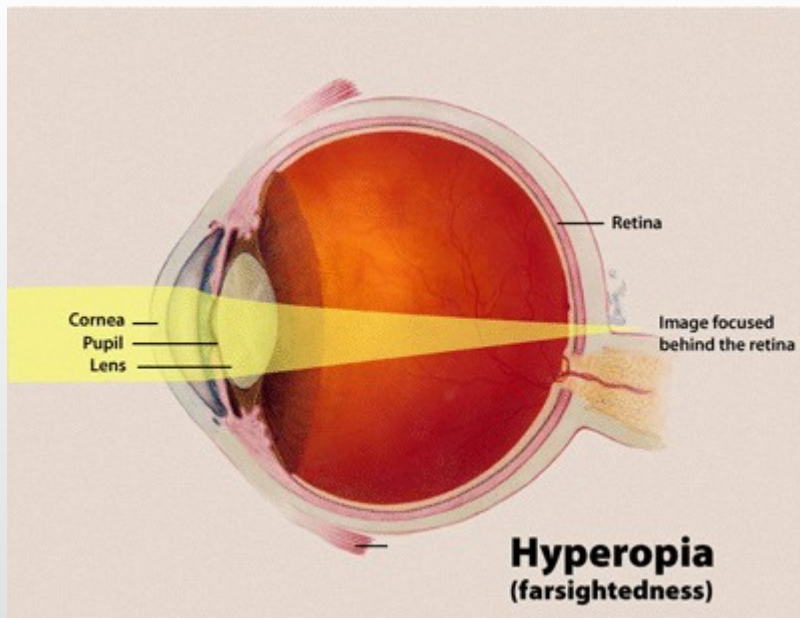
# Funny-shaped eyes:

- You already know that eyes are not perfectly spherical and have a complicated structure
- However there are many ways that this non-spherical shape can be even more abnormal, including
  - Myopia
  - Hyperopia
  - Keratoconus



# Funny-shaped eyes: myopia and hyperopia

- Myopia – short sight – the eyeball is too long
- Hyperopia – long sight – the eye is too short
- (Astigmatism – the cornea is not uniformly shaped)



# Keratoconus

- A degenerative disorder
- Changes in the structure of the cornea cause it to thin in places
- The resulting conical shape causes all sorts of visual difficulties including multiple “ghost” images
- Usually treated using drugs, contact lenses or surgery



# Flow in the eye, other complications and mathematical modelling

- So far, things have been quite simple
- However the whole science of tonometry is hugely complicated by a vast range of factors including
  - Gas bubbles in the eye
  - Bands around the eye
  - Flow in the eye
  - Basic accuracy of tonometry calculations

# Gas bubbles in the eye

- A common treatment for RD (retinal detachment) involves
  - Remove the vitreous humour (“vitrectomy”)
  - Replace some of it with saline (or even maybe mineral oil)
  - Inject a gas bubble into the eye to “tamponade” the retina to the choroid
- Big problem: what happens if you fly soon after this surgery (e.g. from Guernsey)?
- Clearly the IOP may rise rapidly as cabin pressure drops – so how safe is it to fly?

Note that to understand how this all works you need to know about uveoscleral outflow, ciliary body production, drainage via TM, Schlemm canal, IOP, cornea etc. etc. – so it’s a good problem to start with

# Gas bubbles in the eye

- No time to give full details but essentially it comes down to some ODEs

$$\left[ \frac{1}{K(p_A(t) - p_a(t))} + \frac{C}{p_A^2(t)} \right] \frac{dp_A}{dt} = F_i(t) - F_o(t) + \left( \frac{1}{K(p_A(t) - p_a(t))} \right) \frac{dp_a}{dt}$$

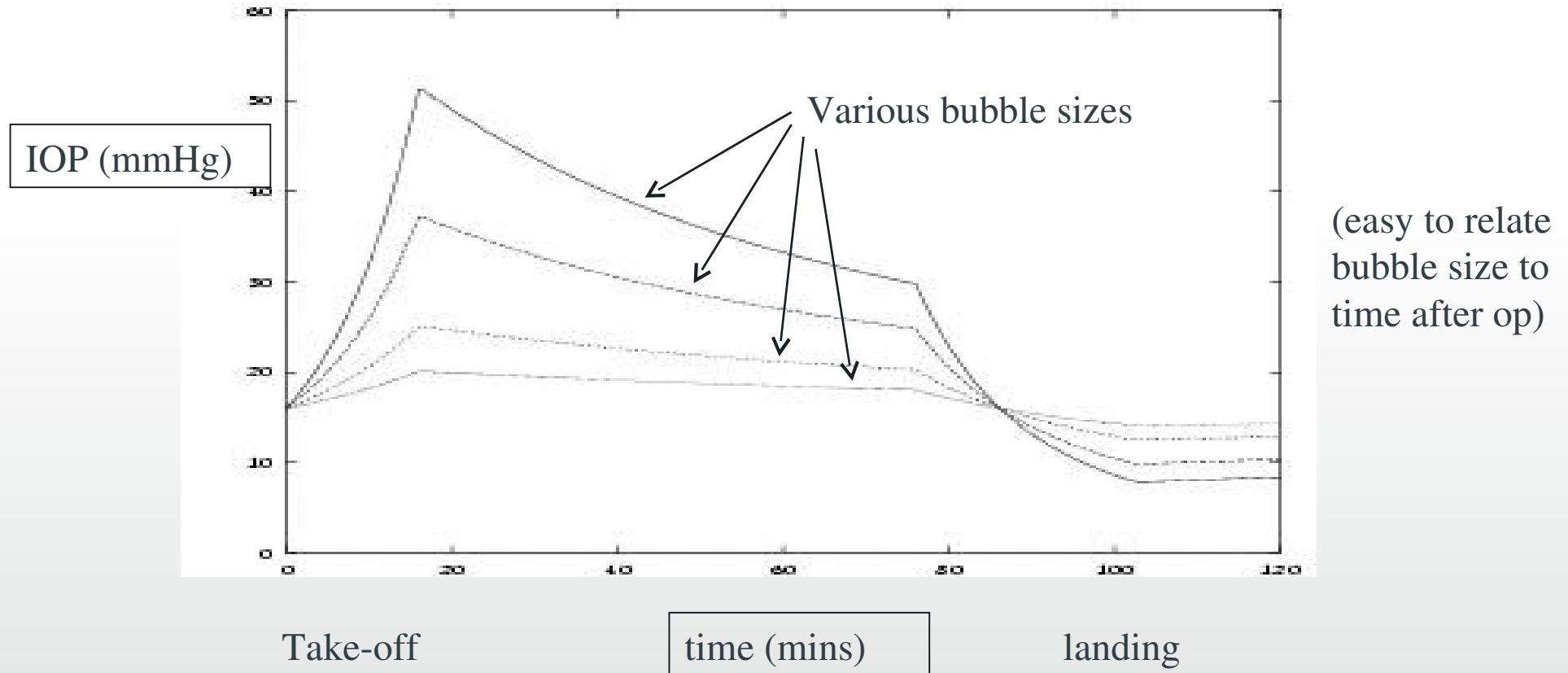
$$F_i(t) = \alpha - H(\ell_1)H(\beta - \ell_1)\ell_1 - \beta H(\ell_1 - \beta) \leftarrow \text{inflow}$$

$$F_o(t) = H(\ell_2)\ell_2 \leftarrow \text{outflow}$$

$$\ell_1 = C_p(p_A(t) - p_a(t) - p_i)$$

$$\ell_2 = C_t(p_A(t) - p_a(t) - p_c)$$

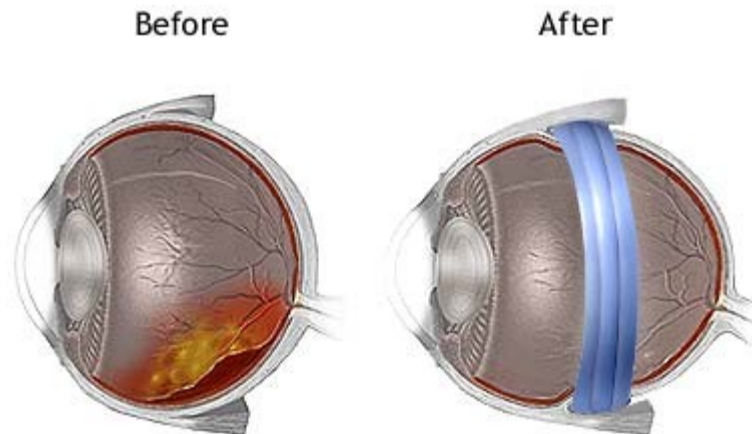
# First problem - gas bubbles in the eye





# Bands around the eye

- Another common treatment for RD (retinal detachment) involves fitting a “scleral buckle”
  - Tamponades the retinal detachment
  - Normally stays fitted for life
  - Makes IOP measurements very awkward



..... we will return to ways of modelling this

# Flow in the anterior chamber

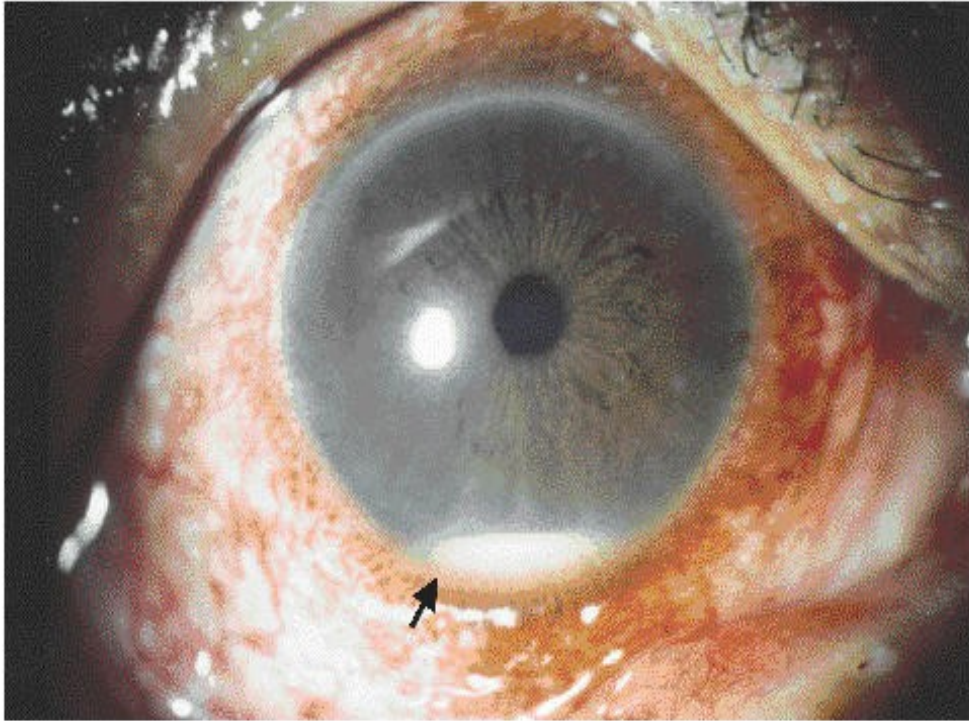
- We know that there is flow in the anterior chamber. Essentially:
  - AH Flow produced by ciliary body
  - AH flows behind iris and through pupil aperture
  - AH “circulates” in the anterior chamber before leaving at the limbus
  - (AH passes through trabecular meshwork, into Schlemm’s canal and away...)
- OBSERVATION: many surgeons have speculated that the “circulation” may be caused by buoyancy – but many do not believe this and there is no theory
- Task – to provide a simple model for this that allows predictions to be made
- To be able to explain hyphemas, hypopyons, Krukenburg spindles and to understand how the flow may affect IOP measurement

# RED BLOOD CELLS (HYPERHEMA)



*....a squash ball in the eye?*

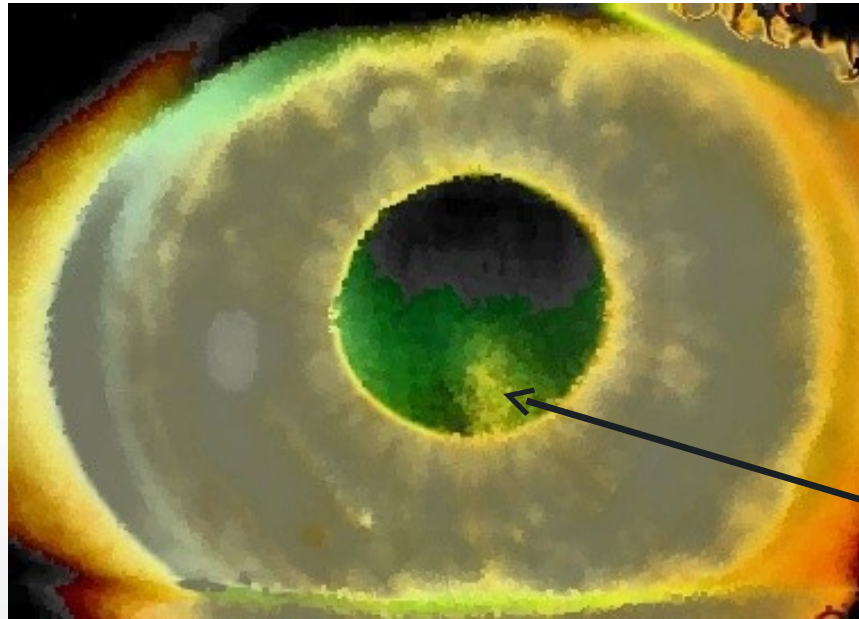
# WHITE CELLS (HYPOPYON)



*keratic precipitates*

*....an infection somewhere around the eye?*

# PIGMENT (KRUKENBERG SPINDLE)



*RPDS – blond haired  
Scandinavian Males  
< 25 yrs old*

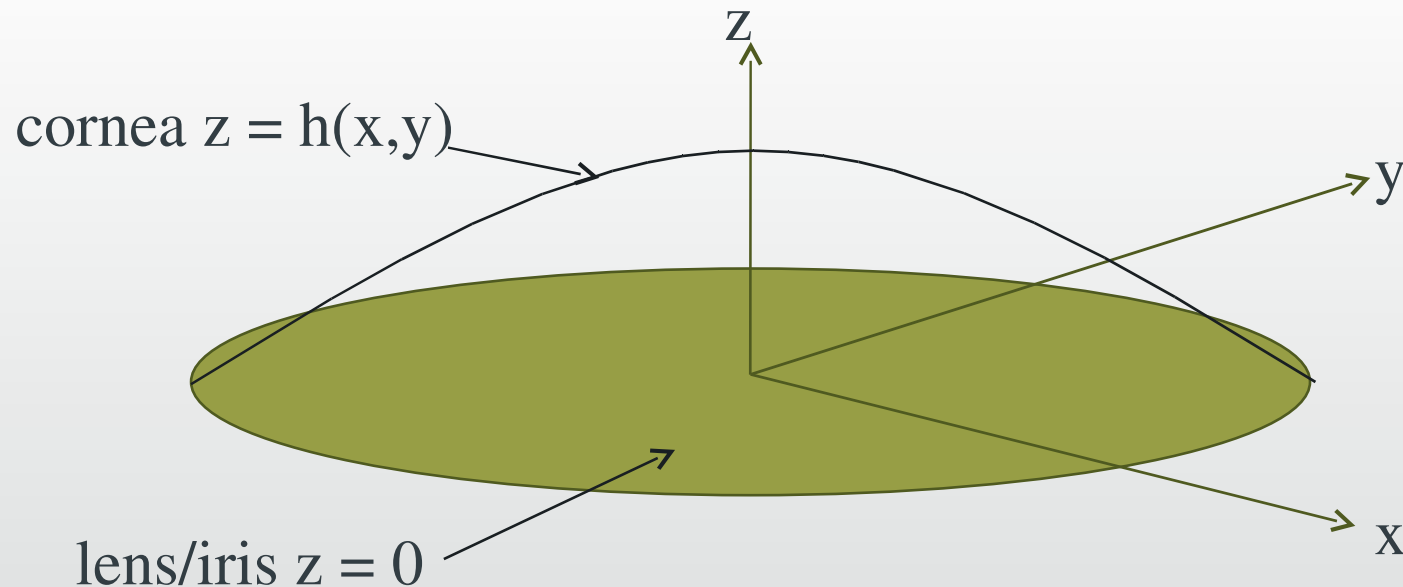
pigment cells

We have been able to analyze all of these flows, essentially by using lubrication theory (anterior chamber is long and thin)

– some examples (not much time for explanations):

# Flow when no detached membrane is present

- The basic flow was established by Canning, Dewynne, Fitt & Greaney (2002)
- Flow is buoyancy-driven and circulation is inevitable
- Schematic of anterior chamber with flow:



# Flow equations

- We solve the Navier-Stokes and advection-diffusion heat equations

$$\rho_0(\underline{q}_t + (\underline{q} \cdot \nabla) \underline{q}) = -\nabla p + \rho_0 \nu \nabla^2 \underline{q} + \rho_0 (1 - \alpha(T - T_0)) \underline{g}$$

$$\nabla \cdot \underline{q} = 0$$

$$T_t + (\underline{q} \cdot \nabla) T = \frac{k}{\rho_0 c_p} \nabla^2 T + \frac{1}{\rho_0 c_p} \Phi$$

- In the thin –layer limit where the ratio  $\varepsilon$  of the height  $h_0$  of the anterior chamber to its width is small – the leading order equations

become

$$-\frac{p_x}{\rho_0} + \nu u_{zz} + g(1 - \alpha(T - T_0)) = 0$$

$$-\frac{p_y}{\rho_0} + \nu v_{zz} = 0$$

$$p_z = 0$$

$$u_x + v_y + w_z = 0$$

$$T_{zz} = 0$$

# Simple 2-D flow

- It is easy to show that buoyancy dominates all other flow mechanisms
  - e.g. Ciliary body production
  - e.g. Phakodogenesis
  - e.g. Gravity when lying down face-up
- In its simplest form the flow is 2-D – equations are

$$-\frac{p_x}{\rho_0} + \nu u_{zz} + g(1 - \alpha(T - T_0)) = 0$$

$$p_z = 0$$

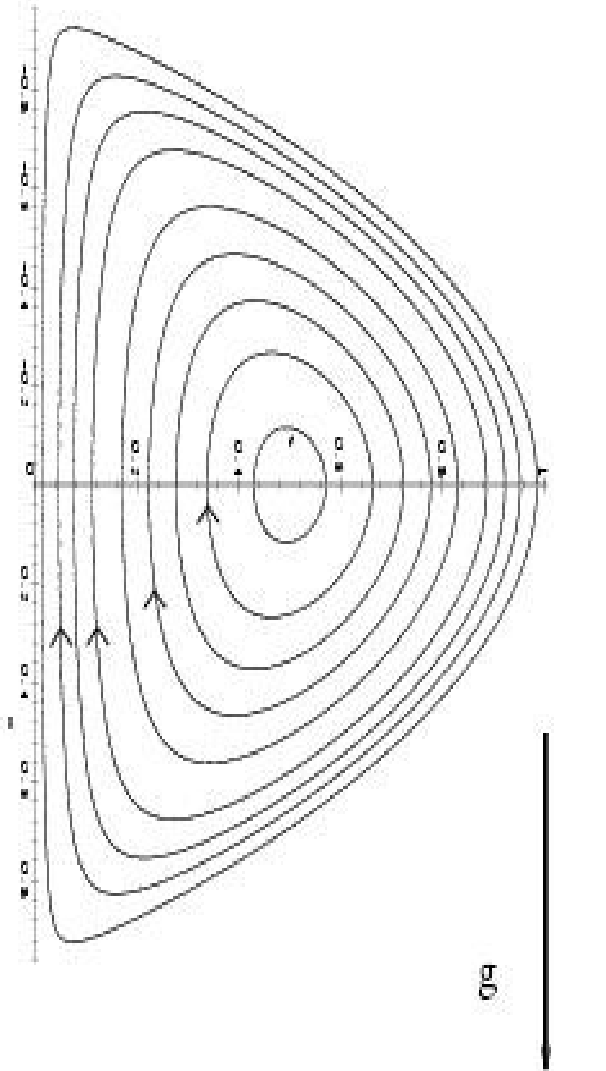
$$u_x + w_z = 0$$

$$T_{zz} = 0$$

- With the obvious boundary conditions



# BASIC BUOYANCY-DRIVEN FLOW



Flow driven by buoyancy in anterior chamber (external temperature gradient)

$$u = \frac{g\rho_0}{\mu} \left[ \frac{zh}{2} - \frac{z^2}{2} + \frac{\alpha z}{6h} (T_1 - T_0)(2h - z)(z - h) \right] + \frac{1}{2\mu} p_x (z^2 - zh)$$

$$v = \frac{1}{2\mu} p_y (z^2 - zh)$$

$$w = \text{HORRID}$$

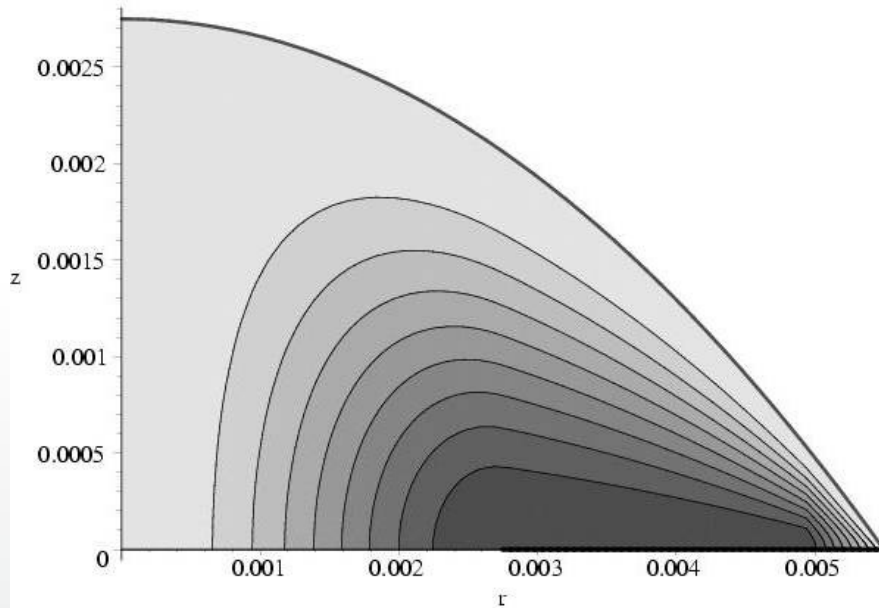
$$p = p_a + (x + a)g\rho_0 \left[ 1 - \frac{\alpha(T_1 - T_0)}{2} \right] + P(x, y)$$

$$(h^3 P_x)_x + (h^3 P_y)_y = -12\mu w_0$$

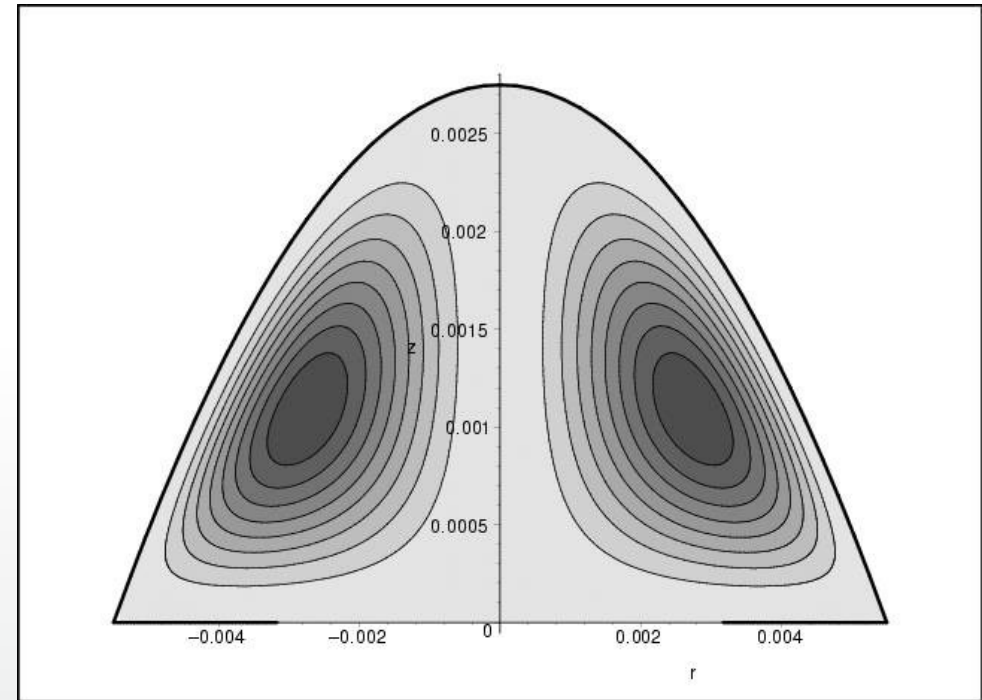
$$T = T_1 + \left( \frac{z}{h} \right) (T_0 - T_1)$$

*N.B. PICS YESTERDAY.....*

## SOME OTHER RELATED FLOWS

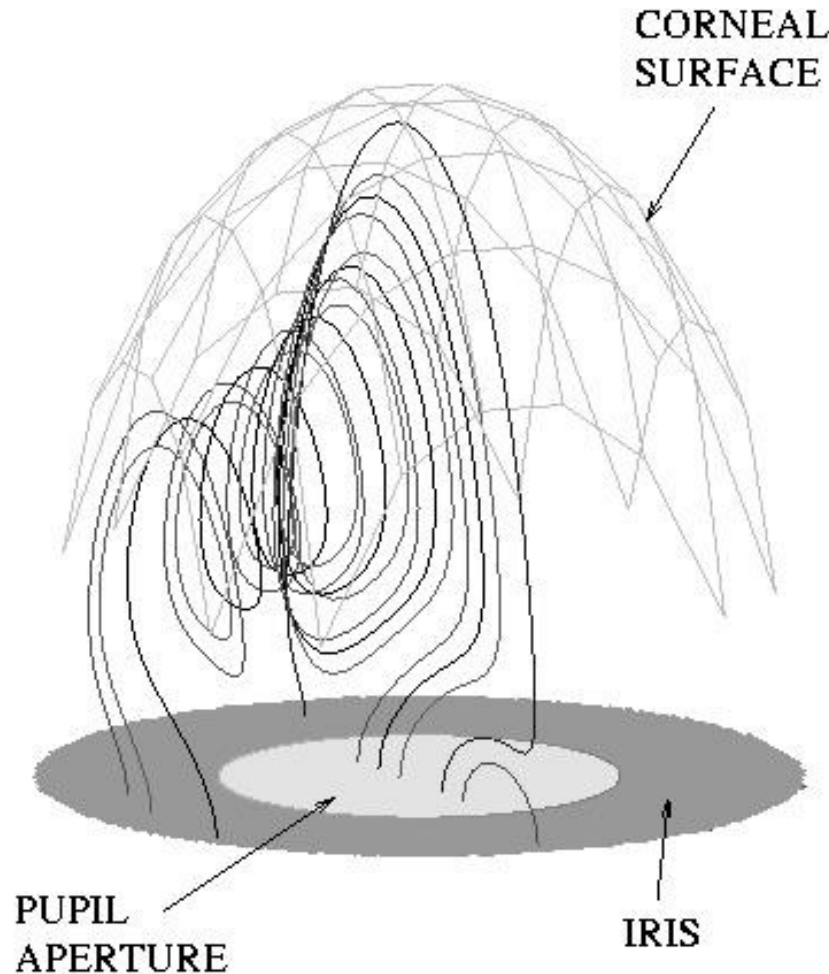


RH half of anterior chamber:  
secretory flow with no gravity  
from behind lens to drainage angle  
via trabecular meshwork



Gravity/buoyancy driven flow  
(asleep, face-up) – weak torodial  
flow

# SECRETORY FLOW BALANCES GRAVITY

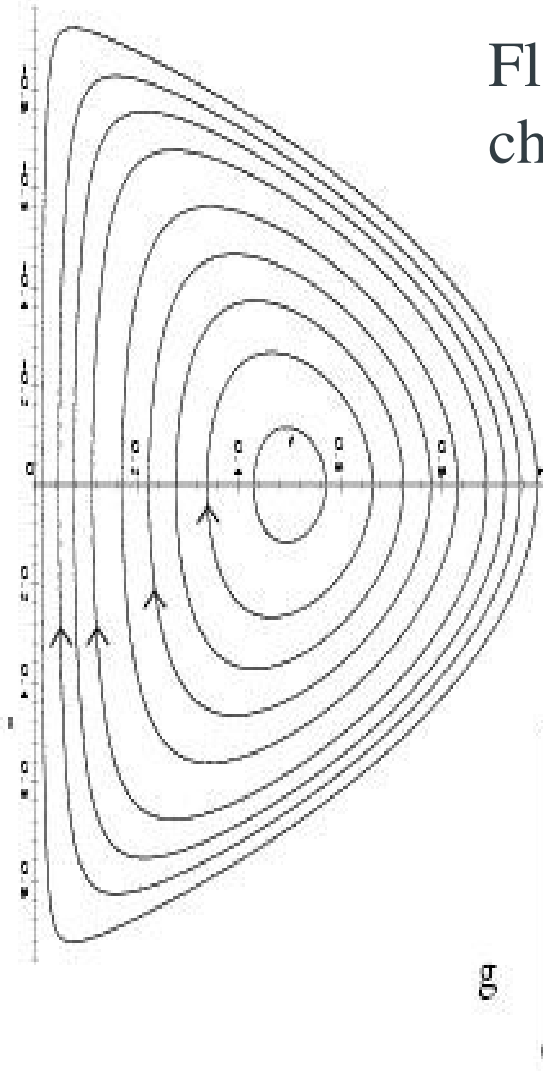


A truly 3-D flow whose velocity can be written down in closed form – something of a rarity

Both form and topology of streamlines are very complicated: fluid particles “spun out” to drainage angle from deep within centre of anterior chamber

## ONE FINAL NOTE

Flow driven by buoyancy in anterior chamber (external temperature gradient)



*The max flow speed is*

$$|u_{\max}| = \frac{g \alpha h_0^2 \sqrt{3} (T_1 - T_0)}{216 \nu}$$

*So put a cold patch on the eye to increase circulation (for drug delivery or “other purposes”) – but may also change IOP*

# Basic accuracy of tonometry calculations

- Tonometry calculations are based on old and inaccurate ideas
- For example – applanation tonometry uses the “Imbert-Fick” principle:
  - “When a disc of radius 3.06mm is applanated each 0.1 g force exerted by the tonometer is equivalent to 1 mm HG of IOP”
- This is very dubious
- Rebound tonometry is based on “look-up tables” and air-puff tonometry is based on some very simplistic calculations
  - SO THERE IS MUCH ROOM HERE FOR ERROR

# The current problem – some random thoughts

- The “inflated eye” is clearly an elasticity problem in one respect
- There is a “well-known” relationship between the diameters and pressures of an inflated eye – the Friedenwald law
- One could also assume that the eye was a linearly elastic body
  - There is even a “Love stress function” for axisymmetric deformations
- One could do FE calculations – but the structure is complicated and it is not clear how one could relate the calculated shape to the IOP
- A simpler possibility is to assume that the eye is a “membrane”
  - FAIRLY SIMPLE EQUATIONS

# Membrane equations

$$(a \sin \theta N_\theta)_\theta - a \cos \theta N_\phi + p_\theta a^2 \sin \theta = 0$$

$$N_\theta + N_\phi = ap_r$$

$$a(N_\theta - \nu N_\phi) = E\alpha(v_\theta + w)$$

$$a \sin \theta (N_\phi - \nu N_\theta) = E\alpha(\nu \cos \theta + w \sin \theta)$$

- The  $N$ 's are the stresses
- The displacements are  $v$  and  $w$
- The radius is  $a$ , the thickness is  $\alpha$  and  $E$  is Young's modulus
- The applied pressure are  $p_r$  and  $p_\theta$