SIMULATION AND
CHAOTIC BEHAVIOR
OF $\alpha$-STABLE
STOCHASTIC PROCESSES
Simulation and Chaotic Behavior of
$\alpha$–Stable Stochastic Processes

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Preface

Stochastic processes are recognized to play an important role in a wide range of problems encountered in mathematics, physics, chemistry, engineering, economics and finance. Recent developments show that in many practical applications leading to appropriate stochastic models a particular class of Lévy α-stable processes is involved. While the attempt at mathematical understanding of these processes leads to severe analytical difficulties, there exist very useful approximate numerical techniques.

This monograph is about Lévy α-stable processes, referred to as α-stable or stable processes. After preliminary remarks in Chapter 1, we demonstrate in Chapter 2 various properties of α-stable random variables and processes for \( \alpha \in (0, 2] \), i.e., including the Gaussian case \( \alpha = 2 \). It turns out that with the use of suitable statistical estimation techniques, computer simulation procedures, and numerical discretization methods described in Chapter 3, it is possible to construct approximations of stochastic integrals with stable measures as integrators. Their updated mathematical theory is systematically presented in Chapters 4 and 5. As a consequence we obtain an effective general method allowing us to construct approximate solutions to a wide class of stochastic differential equations involving such integrals.

Applications of computer graphics, Chapters 6 and 7, provide useful quantitative and visual information on those features of stable variates that distinguish them from their commonly used Gaussian counterparts. It is possible to demonstrate time evolution of densities with heavy tails of various stochastic processes, visualize the effect of jumps of trajectories, etc. We try to demonstrate that stable variates can be very useful in stochastic modeling of various problems arising in science and engineering, and often provide better description of real life phenomena than their Gaussian counterparts.

The final part of the book, Chapters 9 and 10, contains a theoretical study of the hierarchy of chaos (ergodicity, weak mixing, \( p \)-mixing, Kolmogorov property and exactness) for \( \alpha \)-stable stochastic processes. We are concerned with two basic questions: how are the ergodic and mixing properties of stable and more general infinitely divisible stationary processes related to the spectral representation introduced in Chapter 5, and how similar is the general symmetric \( \alpha \)-stable situation to the Gaussian?
The book can be followed without introductory studies by readers with basic knowledge of advanced probability and stochastic processes. It may be followed by less specialized readers as well. This is especially true of the computer simulations and graphic representations. Most of the results (including two original topics of this monograph: computer simulation of $\alpha$–stable processes and their chaotic behavior) are presented with detailed proofs. However, in some places, basic concepts from stochastic integration, stochastic differential equations, convergence of approximate methods, statistical estimation, numerical discretization, etc., are introduced without proofs.

The only technique that touches every area dealt with is the use of computers for simulation and visualization. In the Appendix we present the computer program STOCH–Lm.c which was employed to produce many graphical examples contained in this book. Running this program one can solve approximately stochastic differential equations with respect to the $\alpha$–stable Lévy motion.

The monograph should be of interest to mathematicians working in such fields as theory of stochastic processes, chaos, stochastic modeling, discrete and approximate methods in stochastic analysis, and applications of statistical methods. It can also be recommended as an auxiliary or basic reading for university graduate and postgraduate students interested in various mathematical and computer courses. It should be useful for those who are interested in numerical construction and computer simulation and visualization of solutions of various problems arising in stochastic analysis and stochastic modeling (also for physicists, chemists, economists, and many others).

The mathematical content, all figures and computer programs providing them, and the \LaTeX source code of the whole text, i.e. everything in the book, were produced by the authors themselves. There is no one to blame for possible occasional mistakes but the authors. Nevertheless, we are happy to acknowledge our indebtedness to many colleagues who have offered their help and a variety of constructive criticisms. Among others these include Professors S. Cambanis, M. Maejima, S. T. Rachev, J. B. Robertson, J. Rosinski and W. A. Woyczynski. We also thank our former and present students, in particular, A. Gross, P. Kokoszka, Z. Michna, K. Podgórski and A. Rejman, for their assistance, lively reactions during the lectures, and a fruitful collaboration.

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Chapter 1

Preliminary Remarks

1.1 Historical Overview

We start with the historical overview of a development of investigations and applications of \( \alpha \)-stable random variables and processes. The interested reader looking for documented sources on this topic is referred to Fienberg (1992) and to some books reviewed there, e.g., Stigler (1986).

The early problem considered by pioneering statisticians of the 18th and 19th centuries was to find the best fit of an equation to a set of observed data points. After some false starts they hit upon the method of least squares. Legendre's work seemed to be the most influential at the time. Laplace elaborated upon it, and finally, in a discussion of the distribution of errors, Gauss emphasized the importance of the normal or Gaussian distribution. Laplace was a great enthusiast of generating functions and solved many complicated probability problems exploiting them. As the theory of Fourier series and integrals emerged in the early 1800’s, he and Poisson made the next natural step applying such representations of probability distributions as a new natural tool for analysis, thus introducing the powerful characteristic function method. Laplace seemed to be especially pleased noticing that the Gaussian density was its own Fourier transform. In the early 1850’s Cauchy, Laplace’s former student, became interested in the theory of errors and extended the analysis, generalizing the Gaussian formula to a new one

\[
f_N(x) = \frac{1}{\pi} \int_0^\infty \exp(-ct^N) \cos(tx) \, dt,
\]

expressed as a Fourier integral with \( t^N \) replacing \( t^2 \). He succeeded in evaluating the integral (in the non-Gaussian case) only for \( N = 1 \), thus obtaining the famous Cauchy law defined by the density

\[
f_1(x) = \frac{c}{\pi(c^2 + x^2)}.
\]

It was realized only much later (in 1919, thanks to Bernstein) that \( f_N \) is positive-definite, and hence is a probability density function only when \( 0 < N \leq 2 \).
Replacing $N$ by a real parameter $\alpha$ with values in $(0, 2]$, we find that the integral defining functions $f_\alpha = f_\alpha(x)$ is a source of remarkable surprises. After Cauchy there was a decline in mathematicians' interest in this subject until 1924, when the theory of stable distributions originated with Paul Lévy. When fashion sought the most general conditions for the validity of the Central Limit Theorem, Lévy found simple exceptions to it, namely the class of $\alpha$-stable distributions with index of stability $\alpha < 2$. The ambiguous name stable has been assigned to these distributions because, if $X_1$ and $X_2$ are random variables having such distribution, then $X$ defined by the linear combination $cX = c_1X_1 + c_2X_2$ has a similar distribution with the same index $\alpha$ for any positive real values of the constants $c_1$ and $c_2$ with $c^\alpha = c_1^\alpha + c_2^\alpha$. Lévy noted that the Gaussian case ($\alpha = 2$) is "singular" because for all $\alpha \in (0, 2)$ all nondegenerate densities $f_\alpha(x)$ have inverse power tails, i.e. $\int_{\{|x| > \lambda\}} f_\alpha(x) \, dx \approx C \cdot \lambda^{-\alpha}$ for large $\lambda$. Since these distributions have no second moments, the second moment existence condition for the CLT is violated, allowing for the possibility of unusual results.

Research concerning stable stochastic processes and models has been directed towards delineating the extent to which they share the features of the Gaussian models, and even more significantly, towards discovering their own distinguishing and often surprising features, e.g., Weron (1984). In the last ten years many important results characterizing different properties of these processes (and of other subclasses of processes with independent increments) have been obtained by several authors. Of particular importance are the results concerning representations involving stochastic integrals. A collection of papers edited by Cambanis, Samorodnitsky and Taqqu (1991) provides a review of the state of the art on the structure of stable processes as models for random phenomena.

Modern stochastic integration originated in the early work of Wiener. Stochastic integrals with respect to Brownian motion were defined by Itô (1944). Doob (1953) proposed a general integral with respect to $L^2$-martingales. On the basis of the Doob-Meyer decomposition theorem, Kunita and Watanabe (1967) further developed the theory of this integral. Meyer and Doleans-Dade (1970) extended the definition of the stochastic integral to all local martingales and subsequently to semimartingales. The natural role of semimartingales was made evident thanks to the contribution of Bichteler (1981) and Dellacherie (1980), who established that semimartingales are the most general class of integrators for which one can have a reasonable definition of stochastic integral against predictable integrands.

Our main tool of description of stochastic processes in which we are interested is a stochastic integral with respect to the $\alpha$-stable Lévy motion or, in a more general setting, with respect to $\alpha$-stable stochastic measures (see Samorodnitsky and Taqqu (1993)). The $\alpha$-stable Lévy motion together with the Poisson process and Brownian motion are the most important examples of Lévy processes, which form the first class of stochastic processes being studied in the modern spirit.

They still provide the most important examples of Markov processes as well as of semimartingales (see Protter (1990)).
Thus, on the one hand, the class of stochastic processes in which we are interested is much broader than the class of Gaussian processes and, on the other, it is contained in the class of infinitely divisible processes (e.g., Lévy processes), which itself is contained in the class of semimartingales.

1.2 Stochastic $\alpha$–Stable Modeling

In the past few years there has been an explosive growth in the study of physical and economic systems that can be successfully modeled with the use of stable distributions and processes. Especially infinite moments, elegant scaling properties and the inherent self-similarity property of stable distributions are appreciated by physicists. For a recent survey, see Janicki and Weron (1992). We believe that stable distributions and stable processes do provide useful models for many phenomena observed in diverse fields. The central–limit–type argument often used to justify the use of the Gaussian model in applications may also be applied to support the choice of the non–Gaussian stable model. That is, if the randomness observed is the result of summing many small effects, and those effects themselves follow a heavy–tailed distribution, then a non–Gaussian stable model may be appropriate. An important distinction between Gaussian and non–Gaussian stable distributions is that stable distributions are heavy–tailed, always with the infinite variance, and in some cases with the infinite first moment. Another distinction is that they admit asymmetry, or skewness, while a Gaussian distribution is necessarily symmetric about its mean. In certain applications then, where an asymmetric or heavy–tailed model is called for, a stable model may be a viable candidate. In any case, non–Gaussian stable distributions furnish tractable examples of non–Gaussian behavior and provide points of comparison with the Gaussian case, highlighting the special nature of Gaussian distributions and processes.

In order to appreciate the basic difference between a Gaussian distribution and a distribution with a long tail, Montroll and Shlesinger (1983b) proposed to compare the distribution of heights with the distribution of annual incomes for American adult males. An average individual who seeks a friend twice his height would fail. On the other hand, one who has an average income will have no trouble to discover a richer person, who, with a little diligence, may locate a third person with twice his income, etc. The income distribution in its upper range has a Pareto inverse power tail; however, most of the income distributions follow a log–normal curve, but the last few percent have a stable tail with exponent $\alpha = 1.6$ (cf., Badgar (1980)), i.e., the mean is finite but the variance of the corresponding $1.6$–stable distribution diverges.
1.3 Statistical versus Stochastic Modeling

The notions of a statistical model and a stochastic model may be understood differently and may be ambiguous in some situations. In order to clear up what we mean by these terms we formulate a few remarks on this subject. One of a number possible descriptions of statistical model is the following (see Clogg (1992)).

What statistical methodology refers to in most areas today is virtually synonymous with statistical modeling. A statistical model can be thought of as an equation, or set of equations, that (a) link "inputs" to "outputs"..., (b) have both fixed and stochastic components. (c) include either a linear or a nonlinear decomposition between the two types of components, and (d) purport to explain, summarize or predict levels of or variability in the "outputs".

A very interesting discussion about how to model the progression of cancer, the AIDS epidemic, and other real life phenomena, is contained in the chapter "Model building: Speculative data analysis" of Thompson and Tapia (1990). The main idea is to derive a stochastic process that describes as closely as possible an investigated problem. Starting from an appropriate system of axioms one has to arrive at a formula (a stochastic model) defining this process, construct it explicitly in some way, and verify its correctness and usefulness. Appealing to one of the problems they are interested in, Thompson and Tapia (1990) say "If we wish to understand the mechanism of cancer progression, we need to conjecture a model and then test it against a data base."

Quite often a stochastic model is a synonym of a stochastic differential equation or system of stochastic differential equations. Thanks to Itô's theory of stochastic integration with respect to Brownian motion, it is commonly understood that any continuous diffusion process \( \{X(t) : t \geq 0\} \) with given drift and diffusion coefficients can be obtained as a solution of the stochastic differential equation

\[
X(t) = X_0 + \int_0^t a(s, X(s)) \, ds + \int_0^t c(s, X(s)) \, dB(s), \quad t \geq 0,
\]

where \( \{B(t) : t \in [0, \infty)\} \) stands for a Brownian motion process and \( X_0 \) is a given Gaussian random variable. The theory of such stochastic differential equations is well developed (see, e.g., Arnold (1974)) and they are widely applied in stochastic modeling.

However, it is not so commonly understood that a vast class of diffusion processes \( \{X(t) : t \geq 0\} \) with given drift and diffusion coefficients can be described by the stochastic differential equation

\[
X(t) = X_0 + \int_0^t a(s, X(s)) \, ds + \int_0^t c(s, X(s)) \, dL_\alpha(s), \quad t \geq 0,
\]
where \( \{L_\alpha(t) : t \in [0, \infty)\} \) stands for a stable Lévy motion process and \( X_0 \) is a given \( \alpha \)-stable random variable. Note that, in general, diffusion processes \( \{X(t) : t \geq 0\} \) defined above do not belong to the class of \( \alpha \)-stable processes. On the other hand, they belong to the import class of diffusions with jumps.

As sources of information on modern aspects of stochastic analysis (e.g., on various properties of stochastic integrals and existence of solutions of stochastic differential equations driven by stochastic measures of different kinds) we recommend, among others, Protter (1990) and Kwapien and Woyczyński (1992).

An application of stochastic differential equations to statistical or stochastic model building is not an easy task. So it seems that the use of suitable statistical estimation techniques, computer simulation procedures, and numerical discretization methods should prove to be a powerful tool.

In most of non-trivial cases ... the "closed form" solution is itself so complicated that it is good for little other than as a device for pointwise numerical evaluation. The simulation route should generally be the method of approach for non-trivial time-based modeling. ... Unfortunately, at the present time, the use of the modern digital computer for simulation based modeling and computation is an insignificant fraction of total computer usage.

(Thompson and Tapia (1990), p. 232–233.)

We agree with this opinion and add that, unfortunately, as far as we know, practical approximate methods for solving stochastic differential equations involving stochastic integrals with stable integrands or integrators are only now beginning to be developed.

In our exposition we emphasize the methods exploiting computer graphics. Let us cite Thompson's opinion:

I feel that the graphics-oriented density estimation enthusiasts fall fairly clearly into the exploratory data analysis camp, which tends to replace statistical theory by the empiricism of a human observer. Exploratory data analysis, including nonparametric density estimation, should be a first step down a road to understanding the mechanism of data generating systems. The computer is a mighty tool in assisting us with graphical displays, but it can help us even more fundamentally in revising the way we seek out the basic underlying mechanisms of real world phenomena via stochastic modeling.

(Thompson and Tapia (1990), p. xiv.)

In our approach we feel strongly inspired by the work of S. M. Ulam, who was one of the first enthusiasts of application of computers not only to scientific calculations or to the construction of mathematical models of physical phenomena but even to the investigation of new universal laws of nature; consult e.g., A. R. Bednarek and F. Ulam (1990).
1.4 Hierarchy of Chaos

The past few years have witnessed an explosive growth in interest in physical, chemical, engineering and economic systems that could be studied using stochastic and chaotic methods, see Berliner (1992) and Chatterjee and Yilmaz (1992). "Stochastic" and "chaotic" refer to nature's two paths to unpredictability, or uncertainty. To scientists and engineers the surprise was that chaos (making a very small change in the universe can lead to a very large change at some later time) is unrelated to randomness. Things are unpredictable if you look at the individual events; however, one can say a lot about averaged-out quantities. This is where the stochastic stuff comes in.

Our aim in the second part of the monograph is the theoretical investigation and computer illustration of the hierarchy of chaos for stochastic processes (Gaussian and non-Gaussian stable) with applications to stochastic modeling.

The ergodic theory is one of the very few parts of mathematics that has undergone substantial changes in recent decades. Previously, the ergodic theory had been solving rather general and qualitative problems but now it has become a powerful tool for studying statistical and chaotic properties of dynamical systems. This, in turn, makes ergodic theory quite interesting not only for mathematicians but also for physicists, biologists, chemists and many others.

Indeed, the ergodic theory of stochastic processes is a part of general ergodic theory that is now intensively being developed. Until recently, characterizations of ergodic properties were known only for Markov or Gaussian processes. However, due to modern results on representations of different classes of stochastic processes in terms of stochastic integrals with respect to stochastic measures, it is now possible to characterize ergodic properties for a much wider spectrum of stochastic processes and establish the whole hierarchy of chaos (ergodicity, weak mixing, $p$-mixing, exactness and K-property). A crucial role in our theoretical study is played by the dynamical functional, which describes dynamical properties of stochastic processes and facilitates the investigation of ergodic properties of stochastic processes having a spectral representation.

One of our ideas is to study and characterize ergodic properties of different classes of stochastic processes by means of purely theoretical as well as computer methods. We attempt to demonstrate usefulness of constructive numerical methods of approximation and computer simulations of such processes in investigation of their ergodic properties, providing some new quantitative information on their dynamical properties.

1.5 Computer Simulations and Visualizations

Using our own software packages of computer programs written in languages Turbo C or Turbo Pascal for IBM PC, we want to show that proper use of computer graphics can provide useful and sometimes surprisingly interesting information enabling us to better understand phenomena that are of complicated, chaotic or stochastic nature.
To our knowledge, up to now the numerical analysis of stochastic differential systems driven by Brownian motion has essentially focused on such problems as mean-square approximation, pathwise approximation or approximation of expectations of the solution, etc. (see, e.g., Pardoux and Talay (1985), Talay (1983) and (1986), Yamada (1976)). There are some results on convergence of approximate solutions. The results on the rate of convergence of approximations of stochastic integrals driven by Brownian motion can be found in Rootzén (1980).

Our aim is to adapt some of these constructive computer techniques based on discretization of the time parameter $t$ to the case of stochastic integrals and stochastic differential equations driven by stable Lévy motion. We describe some results on the convergence of approximate numerical solutions. There are still many open questions concerning this problem.

The research in the theory of convergence of constructive approximations of stochastic integrals and stochastic differential equations driven by infinitely divisible measures or by semimartingales is in progress. There is a growing literature on the stability of stochastic integrals and stochastic differential equations with jumps (we refer the reader to Kasahara, Yamada (1991), Jakubowski, Mémin and Pages (1989), Slomiński (1989), Kurtz and Protter (1991) and (1992)). We have to rely on some results in this more general setting, though they are not so easily applicable in practice.

Our idea is to represent the discrete time processes approximating stochastic processes with continuous time parameter $t$ by appropriately constructed finite sets of random samples in order to obtain kernel estimators of densities of these processes on finite sets of values of $t$ and to get some useful quantitative information on their behavior.


### 1.6 Stochastic Processes

We restricted ourselves precisely to problems strictly connected with the theory of $\alpha$-stable stochastic processes and with computer techniques of their approximate construction and simulation. We tried to make the book as much self-contained as possible. Our main goal was to convince the reader that our constructive approach provides powerful tools for modeling and solving approximately a wide variety of problems from the physical, biological and social sciences, so we provide detailed descriptions of necessary algorithms and a lot of examples of graphical presentation and visualization of stochastic processes of different kinds. Concentrating our attention on computer simulations and ergodic properties of $\alpha$-stable processes, we tried to make our exposition as simple as possible, but the interested reader, on the basis of the very quickly growing literature concerning modern stochastic integration and its various applications, has an opportunity to go further in the investigation of the problems which are discussed here.
We assume that the reader is familiar with some of the well known textbooks on probability theory, such as Breiman (1968), Feller (1966) and (1971), Shiryaev (1984) or many others. As the main source of information on \( \alpha \)-stable processes we consider the book by Samorodnitsky and Taqqu (1993) (see also a survey paper of Weron (1984)). Basic facts concerning some aspects of stochastic analysis and especially the theory of stochastic integrals and stochastic differential equations of different types can be found, for example, in Arnold (1974), Elliot (1982), Ikeda and Watanabe (1981), Jacod (1979), Kallianpur (1980), Kwapieni and Woyczyński (1992), Liptser and Shiryaev (1977) and (1978), McKean (1969), Métivier (1982), Protter (1990) or Revuz and Yor (1991).
Chapter 2

Brownian Motion, Poisson Process, $\alpha$–Stable Lévy Motion

2.1 Introduction

In this chapter we recall briefly the main, most important properties of the Brownian motion and Poisson processes, list some properties of $\alpha$–stable random variables, and introduce an $\alpha$–stable Lévy motion, placing it somewhere in between Brownian motion and Poisson processes, among a vast class of infinitely divisible processes. These three classes of "elementary" processes will serve as principal tools for constructing stochastic measures, allowing us to describe vast classes of stochastic processes with the use of stochastic integrals of different kinds.

We emphasize the importance of constructive methods of description of stochastic processes. We believe that a few graphs of trajectories of the Brownian motion, the Poisson process and the $\alpha$–stable Lévy motion will convince the reader that computer simulation methods are quite powerful and provide some useful information on behavior of stochastic processes. We apply such methods in the investigation of more complicated problems presented in the next chapters of the book.

2.2 Brownian Motion

We find it interesting to start with the presentation of some methods of construction of Brownian motion processes and computer graphs of their trajectories, which these methods provide (see Figures 2.2.1 – 2.2.3). We present briefly three different methods: the first is based on the Lévy–Ciesielski Representation, the second (from our point of view the most important) is based on the summation of independent increments, and the third exploits random walk process. All of them provide "essentially the same result", thus from the point of view of practical computations and applications, justifying their correctness.
Taking as a given a complete probability space $(\Omega, \mathcal{F}, P)$ (together with a filtration $\{\mathcal{F}_t\}$), let us recall the definition of the standard, one-dimensional Brownian motion.

**Definition 2.2.1** The **standard, one-dimensional Brownian motion** is a process $\{B(t) : t \geq 0\}$ (or in full notation: $\{B(t, \omega) : t \geq 0, \omega \in \Omega\}$), satisfying the following conditions

1. $P\{\omega : B(0, \omega) = 0\} = 1$;

2. $\{B(t, \omega)\}$ has independent increments, i.e. for any sequence $0 = t_0 < t_1 < \ldots < t_n$, the random variables $B(t_j) - B(t_{j-1})$, $j = 1, 2, \ldots, n$, are independent (in other words, for any $0 \leq s < t$, the increment $B(t) - B(s)$ is independent of $\mathcal{F}_s$):

3. for any $0 \leq s < t$, the random variable $B(t) - B(s)$ is normally distributed with mean $0$ and variance $t - s$, i.e.

   $$P\{a \leq B(t) - B(s) \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{x^2}{2(t-s)}} \, dx;$$

4. $P\{\omega : B(\cdot, \omega) \text{ is a continuous function}\} = 1$.

Notice that

$$E \left[ \exp \left( i \sum_{j=1}^{n} a_j (B(t_j) - B(t_{j-1})) \right) \right] = \prod_{j=1}^{n} \exp \left( -\frac{1}{2} a_j^2 (t_j - t_{j-1}) \right), \quad (2.2.1)$$

for any sequence $0 = t_0 < t_1 < \ldots < t_n < \infty$ and any real numbers $a_j$, $j = 1, 2, \ldots, n$, ($i = \sqrt{-1}$).

It can be derived easily from the definition that the marginal distribution of $B(t, \omega)$ for Borel sets $A$ in $\mathbb{R}$ can be given by the formula

$$P \{(B(t_1), \ldots, B(t_n)) \in A\} = \int_A \left( \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} e^{-\frac{(x_j - t_{j-1})^2}{2(t_j - t_{j-1})}} \right) \, dx_1 \ldots dx_n,$$

for any sequence $0 = t_0 < t_1 < \ldots < t_n$ and $x_0 = 0$.

In other words, if we define the Gaussian kernel

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}, \quad t > 0, \quad x, y \in \mathbb{R},$$

then the cumulative distribution function for $(B(t_1), B(t_2), \ldots, B(t_n))$ has the form
\[ F_{(t_1, t_2, \ldots, t_n)}(x_1, x_2, \ldots, x_n) \]
\[ = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} p_{t_1}(0, y_1)p_{t_2 - t_1}(y_1, y_2) \cdots p_{t_n - t_{n-1}}(y_{n-1}, y_n) \, dy_n \cdots dy_2 \, dy_1, \]
for \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\).

A stochastic process is very often thought of as a consistent family of marginal distributions. Thus we can regard a Brownian motion as specified by the above marginal distributions. Of course, by the Kolmogorov Extension Theorem, such a family of marginal distributions specifies a stochastic process which has a version \(\{B(t, \omega) : t \geq 0\}\) as above.

**Transition probabilities.** Let us also recall that transition probabilities are defined by
\[ p_t(x, A) = \frac{1}{\sqrt{2\pi t}} \int_A e^{-\frac{(u-x)^2}{2t}} \, du \]
and satisfy the Chapman–Kolmogorov Equation, i.e.
\[ p_{t+s}(x, A) = \int_{\mathbb{R}} p_t(x, dy) \, p_s(y, A). \]

**Semigroup representation.** The semigroup representation has the form
\[ (P_t f)(x) = \int_{\mathbb{R}} f(y) \, p_t(x, dy), \quad \text{for} \quad t > 0, \quad \text{with} \quad P_0 = I. \]
Here \(\{P_t : t \geq 0\}\) denotes a strongly continuous contraction semigroup on the Banach space of bounded uniformly continuous functions on \(\mathbb{R}\). The infinitesimal generator of \(\{P_t : t \geq 0\}\) is given by
\[ \lim_{t \to 0} \frac{(P_t f)(x) - f(x)}{t} = \frac{1}{2} \, f''(x). \]

Now we would like to present a few possible methods of constructing Brownian motion processes.

**Interpolation technique of approximate construction of Brownian motion on [0, 1].** Observe that with fixed \(0 \leq t_1 < t_2 < \infty\), the random variable \(B(s)\) for \(s = (t_1 + t_2)/2\) is normal with mean \(w = (z_1 + z_2)/2\) and variance \(\sigma^2 = \tau/2\) with \(\tau = (t_2 - t_1)/2\), under conditions: \(z_1 = B(t_1), z_2 = B(t_2)\). Indeed, using the formulas
\[ P[B(t_1) \in A_x, B(s) \in A_y, B(t_2) \in A_z] = \int_{A_x} \int_{A_y} \int_{A_z} p_{t_1}(0, x)p_{t_2}(x, y)p_{t_2}(y, z) \, dx \, dy \, dz \]
\[ = \int_{A_x} \int_{A_y} \int_{A_z} p_{t_1}(0, x)p_{t_2 - t_1}(x, z) \left( \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left[ -\frac{(y - w)^2}{2\sigma^2} \right] \right) \, dx \, dy \, dz \]
describing the joint density of $B(t_1), B(s) - B(t_1), B(t_2) - B(s)$, and

$$P[B(t_1) \in A_x, B(t_2) \in A_z] = \int_{A_x} \int_{A_z} p_{t_1}(0, x) p_{t_2-t_1}(x, z) \, dx \, dz,$$

we obtain

$$P[B((t_1 + t_2)/2) \in A_y | B(t_1) = z_1, B(t_2) = z_2] = \frac{1}{\sigma \sqrt{2\pi}} \int_{A_y} e^{-y(w)^2/(2\sigma^2)} \, dy.$$ 

This suggests that we can construct the Brownian motion on some finite interval of $t$, say, on the interval $[0, 1]$.

Assume that the probability space $(\Omega, \mathcal{F}, P)$ is rich enough to carry a countable collection $\{\xi_k : k = 0, 1, \ldots\}$ of independent, standard normal random variables (i.e. $\xi_k \sim \mathcal{N}(0, 1)$ for $k = 0, 1, \ldots$). Using linear interpolation and a recursion formula we define the sequence of processes $B^{(n)} = \{B^{(n)}(t) : t \in [0, 1]\}$, as follows.

Starting with the processes

$$B^{(0)}(t) = t\xi_0, \quad t \in [0, 1],$$

we can assume that for any fixed $n \geq 1$ we are given the real-valued stochastic processes $\{B^{(n-1)}(t) : t \in [0, 1]\}$ which is piecewise-linear on subintervals $[k/2^{n-1}, (k+1)/2^{n-1}]$ of the interval $[0, 1]$ for $k = 0, 1, \ldots, 2^{n-1} - 1$.

In order to construct the process $\{B^{(n)}(t) : t \in [0, 1]\}$ we put

$$B^{(n)}(k/2^{n-1}) = B^{(n-1)}(k/2^{n-1}), \quad k = 0, 1, \ldots, 2^{n-1},$$

and, for $k = 0, 1, \ldots, 2^{n-1} - 1$,

$$B^{(n)}((2k+1)/2^n) = \frac{1}{2} \left\{ B^{(n-1)}((k/2^{n-1}) + B^{(n-1)}((k+1)/2^{n-1}) \right\}$$

$$+ 2^{-(n+1)/2} \xi_{2^{n-1} + k}.$$

Linear interpolation on subintervals $[k/2^n, (k+1)/2^n]$ of $[0, 1]$ completes the construction of $B^{(n)}$ on $[0, 1]$.

**Lévy–Ciesielski construction.** The algorithm exploiting the Gaussian property of the Brownian motion described above is derived from the Lévy Ciesielski series representation of this process. In order to obtain the theorem on the convergence of the sequence $\{B^{(0)}(t), B^{(1)}(t), \ldots\}$ to the Brownian motion on $[0, 1]$, we recall this theoretical construction giving another description of these processes. We define the system $\{H^{(n)}_k\}$ of Haar functions, putting

$$H^{(0)}_0(t) = 1 \quad \text{for} \quad t \in [0, 1];$$

$$H^{(n)}_k(t) = \begin{cases} 2^{(n-1)/2}, & \text{if} \quad t \in [2k/2^n, (2k+1)/2^n), \\ -2^{(n-1)/2}, & \text{if} \quad t \in [(2k+1)/2^n, (2k+2)/2^n), \\ 0, & \text{elsewhere in} \quad [0, 1]. \end{cases}$$

for $n = 1, 2, \ldots, k = 0, 1, \ldots, 2^{n-1} - 1.$
We define also the Schauder functions by
\[ S_k^{(n)}(t) = \int_0^t H_k^{(n)}(u) \, du, \quad \text{for } t \in [0, 1], \]
where \( n = 0, 1, \ldots, \ k = 0, 1, \ldots, 2^{n-1} - 1. \)

It is clear, by induction with respect to \( n, \) that
\[ B^{(n)}(t, \omega) = \sum_{m=0}^n \sum_{k=0}^{2^{n-1}-1} \xi_{2^{n-1}+k}^{(m)}(\omega) S_k^{(m)}(t), \quad \text{for } t \in [0, 1], \] (2.2.2)
for \( n = 0, 1, \ldots. \)

**Theorem 2.2.1** As \( n \to \infty, \) the sequence \( \{B^{(n)}(t, \omega)\}_{n=0}^{\infty} \) given by (2.2.2) converges uniformly with respect to \( t \) on \([0, 1]\) to a continuous function \( B(\cdot, \omega) \) for a.a. \( \omega \in \Omega \) and the process \( \{B(t, \omega) : t \in [0, 1], \omega \in \Omega\} \) is the Brownian motion on \([0, 1].\)

**Proof.** Define \( b_n = \max_{0 \leq k \leq 2^{n-1}-1} \left| \xi_{2^{n-1}+k} \right|. \) For \( x > 0 \) we have
\[
P[|\xi_{2^{n-1}+k}| > x] = P[|\xi_0| > x] = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2} \, du
\leq \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{u}{x} e^{-u^2/2} \, du
= \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x},
\]
which, for \( n \geq 1, \) gives
\[
P[b_n > n] = P \left[ \bigcup_{k=0}^{2^{n-1}-1} \{ |\xi_{2^{n-1}+k}| > n \} \right] \leq 2^{n-1} P[|\xi_0| > n] \leq \sqrt{\frac{2}{\pi}} 2^{n-1} e^{-n^2/2} / n.
\]
Observe that \( \sum_{n=1}^\infty 2^{n-1} e^{-n^2/2}/n < \infty, \) thus the Borel-Cantelli lemma implies that there is a subset \( \hat{\Omega} \) of \( \Omega \) such that \( P(\hat{\Omega}) = 1 \) and for each \( \omega \in \hat{\Omega} \) there is an integer \( n(\omega) \) satisfying \( b_n(\omega) \leq n \) for all \( n \geq n(\omega). \) Then
\[
\sum_{n=n(\omega)}^\infty \sum_{k=0}^{2^{n-1}-1} |\xi_{2^{n-1}+k} S_k^{(n)}(t)| \leq \sum_{n=n(\omega)}^\infty n2^{-(n+1)/2} < \infty,
\]
so for \( \omega \in \hat{\Omega} \) the sequence \( \{B^{(n)}(t, \omega)\} \) converges uniformly in \( t \) to a limit \( B(t, \omega). \)

The continuity of \( \{B(t, \omega) : t \in [0, 1]\} \) follows from the uniformity of the convergence.

The Haar system \( \{H_k^{(n)}\} \) forms a complete orthonormal basis in the Hilbert space \( L^2[0, 1], \) so applying the Parseval Equality to functions \( f(u) = I_{[0, t]}(u) \) and \( g(u) = I_{[0, s]}(u) \) gives
\[
\sum_{n=0}^\infty \sum_{k=0}^{2^{n-1}-1} S_k^{(n)}(t) S_k^{(n)}(s) = s \wedge t, \quad \text{for } s, t \in [0, 1]. \] (2.2.3)
In order to prove that the process \( \{B(t, \omega) : t \in [0, 1] \} \) is the Brownian motion process, i.e. that the increments \( \{B(t_j) - B(t_{j-1})\} \), for \( 0 = t_0 < t_1 < \ldots < t_n \leq 1 \), are independent, normally distributed, with mean zero and variance \( t_j - t_{j-1} \), it suffices to prove (2.2.1).

Using the independence and standard normality of random variables \( \{\xi_j\} \), we have from (2.2.2)

\[
E\left[\exp \left\{ -i \sum_{j=1}^{n} (a_{j+1} - a_j) B^{(M)}(t_j) \right\} \right]
\]

\[
= E \left[ \exp \left\{ -i \sum_{m=0}^{M} \sum_{k=0}^{2^m-1} \xi_{2^m-1+k} \sum_{j=1}^{n} (a_{j+1} - a_j) S_k^{(m)}(t_j) \right\} \right]
\]

\[
= \prod_{m=0}^{M} \prod_{k=0}^{2^m-1} E \left[ \exp \left\{ -i \xi_{2^m-1+k} \sum_{j=1}^{n} (a_{j+1} - a_j) S_k^{(m)}(t_j) \right\} \right]
\]

\[
= \prod_{m=0}^{M} \prod_{k=0}^{2^m-1} \exp \left[ \frac{1}{2} \left\{ \sum_{j=1}^{n} (a_{j+1} - a_j) S_k^{(m)}(t_j) \right\}^2 \right]
\]

\[
= \exp \left[ \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} (a_{j+1} - a_j) (a_{i+1} - a_i) \sum_{m=0}^{M} \sum_{k=0}^{2^m-1} S_k^{(m)}(t_j) S_k^{(m)}(t_i) \right]
\]

for any \( 0 = t_0 < t_1 < \ldots < t_n \leq 1 \) and any real parameters \( a_j, j = 1, 2, \ldots, n+1 \) with \( a_{n+1} = 0 \). Letting \( M \to \infty \) and using (2.2.3), we obtain

\[
E\left[\exp \left\{ i \sum_{j=1}^{n} a_j (B(t_j) - B(t_{j-1})) \right\} \right]
\]

\[
= E \left[ \exp \left\{ -i \sum_{j=1}^{n} (a_{j+1} - a_j) B(t_j) \right\} \right]
\]

\[
= \exp \left\{ - \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} (a_{j+1} - a_j) (a_{i+1} - a_i) t_j - \frac{1}{2} \sum_{j=1}^{n} (a_{j+1} - a_j)^2 t_j \right\}
\]

\[
= \exp \left\{ - \sum_{j=1}^{n-1} (a_{j+1} - a_j) (-a_{j+1}) t_j - \frac{1}{2} \sum_{j=1}^{n} (a_{j+1} - a_j)^2 t_j \right\}
\]

\[
= \exp \left\{ \frac{1}{2} \sum_{j=1}^{n-1} (a_{j+1}^2 - a_j^2) t_j - \frac{1}{2} a_n^2 t_n \right\}
\]

\[
= \prod_{j=1}^{n} \exp \left\{ - \frac{1}{2} \sum_{j=1}^{n-1} a_j^2 (t_j - t_{j-1}) \right\}
\]

This ends the proof. \( \square \)
The figure below (Fig. 2.2.1) contains the result of computer realization of this method.

![Figure 2.2.1. Trajectories of Brownian motion (Lévy–Ciesielski representation).](image)

The technique used to produce all figures in this chapter will be elaborated and applied later on several times to visualize various types of stochastic processes and will be explained there in detail.

**Construction of Brownian motion on \([0, \infty)\).** The construction of the Brownian motion on \([0, 1]\) described above and a simple patching-together technique provide a Brownian motion process defined on \([0, \infty)\).

Starting with a sequence \((\Omega_n, \mathcal{F}_n, P_n), n = 1, 2, \ldots\), of probability spaces together with the Brownian motion \(\{X^{(n)}(t) : t \in [0, 1]\}\) on each space, define \(\Omega = \Omega_1 \times \Omega_2 \times \ldots\), \(\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \ldots\), \(P = P_1 \times P_2 \times \ldots\), and finally

\[
B(t, \omega) = \sum_{j=1}^{n-1} X^{(j)}(1, \omega) + X^{(n)}(t-n+1, \omega), \quad n-1 \leq t < n, \quad \text{with } n = 1, 2, \ldots,
\]

The process \(\{B(t) : t \in [0, \infty)\}\) is clearly continuous and has independent increments, which are symmetric Gaussian random variables with proper variances.

**Approximation of Brownian motion by summation of increments.** In order to obtain an approximate construction of the Brownian motion on a given interval \([0, T]\) one can proceed as follows.

Introduce a mesh \(\{t_i = i\tau : i = 0, 1, \ldots, I\}\) on \([0, T]\) with fixed natural number \(I\) and \(\tau = T/I\). For a given finite sequence \(\{\zeta_i\}, i = 1, 2, \ldots, I\), of independent
Gaussian variables with mean 0 and variance \( \tau \) (i.e. \( \zeta_i \sim \mathcal{N}(0, \tau) \)), put

\[
B'(0) = 0 \quad \text{a.e.,}
\]

for \( i = 1, 2, \ldots, I \); compute

\[
B'(t) = B'(t_{i-1}) + (t - t_{i-1})\zeta_i, \quad \text{for } t \in (t_{i-1}, t_i].
\]

The process \( \{B'(t) : t \in [0, T]\} \) converges to the Brownian motion process on \([0, T] \), when \( \tau \to 0 \).

The result of application of this method is presented in Fig. 2.2.2 below, where \( T = 4 \) and \( \tau = 0.002 \) (\( I = 2000 \)).

![Figure 2.2.2. Trajectories of Brownian motion (summation of increments).](image)

**Random walk approximating Brownian motion.** Now we are going to state and prove a theorem on convergence of constructions described above, but in a slightly more general framework.

Let us consider a sequence \( \{\xi_j\}_{j=1}^{\infty} \) of independent, identically distributed random variables with mean 0 and variance 1 and define the sequence of partial sums

\[
S_0 = 0; \quad S_k = \sum_{j=1}^{k} \xi_j, \quad \text{for } k = 1, 2, \ldots
\]

By applying a linear interpolation technique we can define the continuous time process

\[
Y(t) = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)\xi_{\lfloor t \rfloor + 1}, \quad \text{for } t \in [0, \infty),
\]
where \([t]\) denotes the greatest integer not greater than \(t\). Next, let us define the sequence of random walk processes

\[
X^{(n)}(t) = \frac{1}{\sqrt{n}} Y(nt), \quad \text{for} \quad t \in [0, \infty).
\] (2.2.4)

Note that \(\{X^{(n)}(t)\}\) can be interpreted as a good approximation of the Brownian motion because, with \(s = k/n\) and \(t = (k + 1)/n\), increments \(X^{(n)}(t) - X^{(n)}(s) = (1/\sqrt{n})\xi_{k+1}\) are independent of \(\mathcal{F}^{X^{(n)}} = \sigma(\xi_1, \xi_2, ..., \xi_k)\) with zero mean and variance \(t - s\).

An example of realization of this method is visualized on Fig. 2.2.3 below. Random variables \(\xi_i\) are chosen to be independent discrete random variables with masses concentrated on points \(0, \sqrt{3}, -\sqrt{3}\) with probabilities \(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\), respectively. (Notice that five first moments of \(\xi_i\) are the same as those of the standard normal random variable.)

![Figure 2.2.3. Trajectories of a random walk process approximating Brownian motion.](image)

The result on convergence of this approximate construction is proved in the following theorem.

**Theorem 2.2.2** With \(\{X^{(n)}(t) : t \geq 0\}\) defined by (2.2.4) and for any \(0 \leq t_1 < t_2 < ... < t_d < \infty\), we have

\[
(X^{(n)}(t_1), X^{(n)}(t_2), ..., X^{(n)}(t_d)) \overset{D}{\to} (B(t_1), B(t_2), ..., B(t_d)), \quad \text{as} \quad n \to \infty,
\]

where \(\{B(t) : t \geq 0\}\) is the standard, one-dimensional Brownian motion.
PROOF. For the sake of simplicity we take the case $d = 2$. Set $s = t_1$, $t = t_2$. We have to show that

$$(X^{(n)}(s), X^{(n)}(t)) \xrightarrow{D} (B(s), B(t)).$$

Since

$$\left| X^{(n)}(t) - \frac{1}{\sqrt{n}}S_{[tn]} \right| \leq \frac{1}{\sqrt{n}} |\xi_{[tn]+1}|,$$

so, by the Čebyshev Inequality, we have

$$P\left[ \left| X^{(n)}(t) - \frac{1}{\sqrt{n}}S_{[tn]} \right| > \epsilon \right] \leq \frac{1}{\epsilon^2 n} \to 0,$$

as $n \to \infty$. It is clear then that

$$\left\| (X^{(n)}(s), X^{(n)}(t)) - \frac{1}{\sqrt{n}} (S_{[sn]}, S_{[tn]}) \right\| \to 0 \quad \text{in probability},$$

so it suffices to show that

$$\frac{1}{\sqrt{n}} (S_{[sn]}, S_{[tn]}) \xrightarrow{D} (B(s), B(t)),$$

which is equivalent to

$$\frac{1}{\sqrt{n}} \left( \sum_{j=1}^{[sn]} \xi_j, \sum_{j=[sn]+1}^{[tn]} \xi_j \right) \xrightarrow{D} (B(s), B(t) - B(s)).$$

The independence of the random variables $\{\xi_j\}_{j=1}^\infty$ implies

$$\lim_{n \to \infty} \mathbb{E} \left[ \exp \left\{ \frac{iu}{\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j + \frac{iv}{\sqrt{n}} \sum_{j=[sn]+1}^{[tn]} \xi_j \right\} \right]$$

$$= \lim_{n \to \infty} \mathbb{E} \left[ \exp \left\{ \frac{iu}{\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j \right\} \right] \cdot \lim_{n \to \infty} \mathbb{E} \left[ \exp \left\{ \frac{iv}{\sqrt{n}} \sum_{j=[sn]+1}^{[tn]} \xi_j \right\} \right], \quad (2.2.5)$$

provided both limits on the right-hand side exist. We have

$$\lim_{n \to \infty} \mathbb{E} \left[ \exp \left\{ \frac{iu}{\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j \right\} \right] = e^{-u^2 s/2},$$

because

$$\left| \frac{1}{\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j - \frac{\sqrt{s}}{\sqrt{[sn]}} \sum_{j=1}^{[sn]} \xi_j \right| \to 0, \quad \text{in probability}$$
and, thanks to the Central Limit Theorem, the sequence $\sqrt{s/[3n]} \sum_{j=1}^{[3n]} \xi_j$ converges in distribution to the normal random variable with mean 0 and variance $s$. Similarly,

$$\lim_{n \to \infty} E \left[ \exp \left\{ \frac{iv}{\sqrt{n}} \sum_{j=[3n]+1}^{[3n]} \xi_j \right\} \right] = e^{-v^2(t-s)^2/2}.$$ 

Substitution in (2.2.5) completes the proof.

**Canonical probability space for Brownian motion.** Let us denote by $C[0, \infty)$ the space of all real-valued continuous functions $\omega = \omega(t)$ defined on $[0, \infty)$ with the metric

$$\rho(\omega_1, \omega_2) \overset{df}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq n} (|\omega_1(t) - \omega_2(t)| \wedge 1).$$

Let $\mathcal{C}$ be the collection of finite-dimensional cylinder sets of the form

$$C = \{ \omega \in C[0, \infty); (\omega(t_1), ..., \omega(t_n)) \in D \},$$

for any natural $n \geq 1$, $0 < t_1 < ... < t_n < \infty$ and any $D \in \mathcal{B}(\mathbb{R}^n)$.

The smallest $\sigma$-field containing $\mathcal{C}$ is equal to the Borel $\sigma$-field generated by all open sets in $C[0, \infty)$. We denote it by $\mathcal{B}(C[0, \infty))$.

If $\mu$, acting from $\mathcal{C}$ into $[0,1]$, is defined by

$$\mu(C) = \int_D \left( \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi} (t_j - t_{j-1})} e^{-\frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})^2}} \right) dx_1...dx_n,$$

for $C \in \mathcal{C}$ (with $t_0 = 0$ and $x_0 = 0$), then $\mu$ extends to a probability measure on the space $(C[0, \infty), \mathcal{B}(C[0, \infty)))$. The measure $\mu$ is called the Wiener measure and the probability space $(C[0, \infty), \mathcal{B}(C([0, \infty)), \mu)$ is known as the Wiener space. Then $B(t, \omega) = \omega(t)$ is a Brownian motion and this probability space is called the canonical probability space for Brownian motion.

We want to end this section by recalling the theorem, which asserts convergence of random walks defined by (2.2.4).

**Theorem 2.2.3 (The Invariance Principle).** Let $(\Omega, \mathcal{F}, P)$ be a probability space on which is given a sequence $\{\xi_j\}_{j=1}^{\infty}$ of independent, identically distributed random variables with mean 0 and variance 1. Define the sequence of processes $X^{(n)} = \{X^{(n)}(t) : t \in [0, \infty)\}$ by (2.2.4). Let $P_n$ be the measure induced by $X^{(n)}$ on $(C[0, \infty), \mathcal{B}(C[0, \infty)))$. Then the sequence of measures $\{P^{(n)}\}_{n=1}^{\infty}$ converges weakly to a measure $P$, under which the coordinate mapping process $W(t, \omega) \overset{df}{=} \omega(t)$ on $C[0, \infty)$ is the standard one-dimensional Brownian motion.

For the proof we refer the interested reader to Karatzas and Shreve (1988).
Remark 2.2.1 The standard one-dimensional Brownian motion defined on any probability space can be thought of as a random variable with values in $C[0, \infty)$. Regarded this way, the Brownian motion induces the Wiener measure $P$ on the measure space $(C[0, \infty), \mathcal{B}(C[0, \infty]))$. This explains why the probability space of the form $(C[0, \infty), \mathcal{B}(C[0, \infty)), P)$ is called the canonical probability space for the Brownian motion.

This characterization of the Brownian motion and more sophisticated constructions describing processes with jumps (and further - semimartingales) play an important role in the modern approach to the theory of stochastic processes, but are not so crucial in approximate constructions of stochastic processes, applicable in computer simulations.

Multidimensional Brownian motion. In order to construct $d$-dimensional Brownian motion it is enough to take $d$ independent copies of standard, one-dimensional Brownian motion $B^{(i)} = \{B^{(i)}(t) : t \in [0, \infty)\}$, $i = 1, \ldots, d$, defined on probability spaces $(\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)})$ (with filtrations $\mathcal{F}^{(i)}_t$) and on the product space

$$(\Omega^{(1)} \times \ldots \times \Omega^{(d)}, \mathcal{F}^{(1)} \otimes \ldots \otimes \mathcal{F}^{(d)}, P^{(1)} \times \ldots \times P^{(d)})$$

to define

$$B(t, \omega) = (B^{(1)}(t, \omega_1), \ldots, B^{(d)}(t, \omega_d)),$$

with the filtration $\mathcal{F}_t = \mathcal{F}^B_t$.

2.3 The Poisson Process

Along with Brownian motion the Poisson process plays a fundamental role in the theory of continuous-time stochastic processes, so taking as given a complete probability space $(\Omega, \mathcal{F}, P)$ (together with a filtration $\{\mathcal{F}_t\}$), let us recall the definition of this process.

Suppose we are given a strictly increasing sequence $\{T_n\}_{n \geq 0}$ of positive random variables with $T_0 = 0$ a.s. and $\sup_n T_n = \infty$ a.s.

Definition 2.3.1 The process $N = \{N(t) : t \geq 0\}$ defined by

$$N(t) = \sum_{n=1}^{\infty} I_{(T_n, \infty)}(t),$$

with values in $\mathbb{N} \cup \{0\}$ is called the counting process (without explosion and associated to the sequence $\{T_n\}_{n \geq 1}$).

($I_A = I_A(x)$ denotes the characteristic function of the set $A$.)

It is not difficult to notice that

$$[T_n, \infty) = \{N \geq n\} = \{(t, \omega) : N(t, \omega) \geq n\},$$
\[ [T_n, T_{n+1}) = \{ N = n \}. \]

Note that for \( 0 \leq s < t < \infty \) we have

\[ N(t) - N(s) = \sum_{n=1}^{\infty} I_{(T_n, \infty)}(t) \cdot I_{(0, T_n)}(s). \]

The increment \( N(t) - N(s) \) counts the number of random times \( T_n \) that occur between the fixed times \( s \) and \( t \).

**Remark 2.3.1** In order to have this process adapted to the filtration \( \{ \mathcal{F}_t \} \) it is enough to assume that \( T_n \) are stopping times.

**Definition 2.3.2** An adapted counting process \( N \) without explosion is a Poisson process if

1. \( P\{\omega : N(0, \omega) = 0\} = 1; \)

2. \( \{ N(t, \omega) \} \) has independent increments, i.e. for any sequence \( 0 = t_0 < t_1 < \ldots < t_n \), the random variables \( N(t_j) - N(t_{j-1}), \ j = 1, 2, \ldots, n, \) are independent (in other words, for any \( 0 \leq s < t \), the increment \( N(t) - N(s) \) is independent of \( \mathcal{F}_s \));

3. \( \{ N(t, \omega) \} \) has stationary increments, i.e. for any \( 0 \leq s < t < \infty \) and \( 0 \leq u < v < \infty \) random variables \( N(t) - N(s) \) and \( N(v) - N(u) \) have the same distribution.

**Theorem 2.3.1** Let \( N \) be a Poisson Process. Then

\[ P(N(t) = k) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}; \quad k = 0, 1, \ldots, \]

for some \( \lambda > 0 \), which means that \( N(t) \) has the Poisson distribution with parameter \( \lambda t \).

For the proof we refer the reader to Protter (1990).

In two figures below (Fig.2.3.1 and Fig.2.3.2) we present (exact, not approximated !) trajectories of the Poisson process with the compensator \( \{ N(t) - \lambda t \}_{t \geq 0} \), for two values of \( \lambda : 1.0 \) and \( 0.1 \), respectively. The random variables \( T_n \) in the sequence \( \{ T_n \} \) have been chosen as arrival times, i.e. \( T_n \) is a sum of \( n \) independent, exponentially distributed random variables with the distribution function \( F(x) = 1 - e^{-\lambda x} \).
Figure 2.3.1. Trajectories of the Poisson process with compensator for $\lambda = 1.0$.

Figure 2.3.2. Trajectories of the Poisson process with compensator for $\lambda = 0.1$. 
2.4 \(\alpha\)-Stable Random Variables

As the title of this monograph indicates, we focus our attention on the class of \(\alpha\)-stable (or stable) random variables. From the literature on this topic let us mention among others: Feller (1966) and (1971), Lévy (1937) and (1948), Zolotarev (1986) or Samorodnitsky and Taqqu (1993).

**Characteristic function.** The most common and convenient way to introduce \(\alpha\)-stable random variable is to define its characteristic function.

**Definition 2.4.1** The characteristic function of an \(\alpha\)-stable random variable involves four parameters: \(\alpha\) – index of stability, \(\beta\) – skewness parameter, \(\sigma\) – scale parameter and \(\mu\) – shift. This function is given by

\[
\log \phi(\theta) = -\alpha |\theta|^\alpha \{ 1 - i \beta \text{sgn}\(\theta\) \tan(\alpha \pi / 2) \} + i \mu \theta,
\]

when \(\alpha \in (0, 1) \cup (1, 2)\), \(\beta \in [-1, 1]\), \(\sigma \in \mathbb{R}_+\), \(\mu \in \mathbb{R}\); and by

\[
\log \phi(\theta) = -\sigma |\theta| + i \mu \theta,
\]

when \(\alpha = 1\), which gives a very well-known symmetric Cauchy distribution.

In spite of the belief of some authors recalling "reparametrization that removes discontinuity", it seems to us that the only acceptable value of parameter \(\beta\) for \(\alpha = 1\) is \(\beta = 0\), (see Figures 2.4.5, 2.4.6 below).

For the random variable \(X\) distributed according to the rule described above we use the notation \(X \sim S_\alpha(\sigma, \beta, \mu)\). When \(\mu = \beta = 0\), i.e., \(X\) is a symmetric \(\alpha\)-stable random variable, we will write \(X \sim S_\alpha S\). In order to shorten the notation we denote by \(S_\alpha\) any random variable \(X\) such that \(X \sim S_\alpha(1, 0, 0)\).

**Domain of attraction of \(X\).** A random variable \(X\) has a stable distribution if and only if it has a domain of attraction, i.e., if there exist a sequence \(Y_1, Y_2, \ldots\) of i.i.d. random variables and sequences \(\{d_n\}\) and \(\{a_n\}\) of positive real numbers such that

\[
\frac{Y_1 + Y_2 + \ldots + Y_n}{d_n} + a_n \xrightarrow{d} X.
\]

According to Feller (1971), we have in general \(d_n = n^{1/\alpha}h(n)\), where the function \(h = h(x), x \geq 0\) varies slowly at infinity; the sequence \(\{Y_i\}\) is said to belong to the normal domain of attraction of \(X\) when \(d_n = n^{1/\alpha}\). Observe that if \(Y_i\)'s are i.i.d. random variables with finite variance, then \(X\) is Gaussian and we obtain an ordinary version of the Central Limit Theorem.

**Lévy measure.** To justify what was said above one can recall the Lévy–Khinchin Representation Theorem (see Feller (1971)).

Let us introduce the Lévy measure

\[
d\nu_\alpha(x) = \frac{P}{x^{1+\alpha}}I_{(0,\infty)}(x) \, dx + \frac{Q}{|x|^{1+\alpha}}I_{(-\infty,0)}(x) \, dx
\]
with non-negative numbers $P$, $Q$, and a function

$$
\psi(\theta, x) = e^{i\theta x} - 1 - \frac{i\theta x}{1 + x^2}.
$$

Then for $X$ we have the following representation

$$
E\ exp(i\theta X) = \begin{cases}
\exp\{ib\theta - c^2\theta^2\} & \text{if } \alpha = 2, \\
\exp\{ib\theta + \int_{\mathbb{R}\setminus\{0\}} \psi(\theta, x) \, dv_\alpha(x)\} & \text{if } 0 < \alpha < 2,
\end{cases}
$$

where $b$ and $c$ are real numbers (looking for a more general form of the Lévy–Khintchine Formula, see Definition 4.4.2).

**Some arithmetic properties.** Now we are going to recall a few simple but important properties of random variables $S_\alpha(\sigma, \beta, \mu)$.

1. If we have $X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i)$ for $i = 1, 2$ and $X_1$, $X_2$ are independent random variables, then

$$
X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu),
$$

with

$$
\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, \quad \beta = \frac{\beta_1 \sigma_1^\alpha + \beta_2 \sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \quad \mu = \mu_1 + \mu_2.
$$

2. If we have $X_1, X_2 \sim S_\alpha(\sigma, \beta, \mu)$ and $A$, $B$ are real positive constants and $C$ is a real constant, then

$$
AX_1 + BX_2 + C \sim S_\alpha(\sigma(A^\alpha + B^\alpha)^{1/\alpha}, \beta, \mu(A^\alpha + B^\alpha)^{1/\alpha} + C).
$$

3. $X \sim S_\alpha(\sigma, \beta, \mu)$ is a symmetric random variable if and only if $\beta = 0$ and $\mu = 0$. It is symmetric about $\mu$ if and only if $\beta = 0$.

1. If we have $X \sim S_\alpha(\sigma, \beta, \mu)$ and $\alpha \in (0, 2)$ and $p \in (0, \alpha)$, then

$$
E|X|^p < \infty,
$$

and if $p \in [\alpha, 2)$, then

$$
E|X|^p = \infty.
$$

5. If we have $X \sim S_\alpha(\sigma, 0, \mu)$ and $\alpha \in (1, 2]$, then

$$
EX = \mu.
$$

**Covariation.** Let $(X_1, X_2)$ denote a jointly $S\alpha S$ random vector, where $\alpha \in (1, 2]$. Considering the $S\alpha S$ random variable $Y = \theta_1 X_1 + \theta_2 X_2$ for any real
\[ Y \sim S_\alpha(\sigma, 0, 0) \quad \text{with} \quad \sigma = \sigma(\theta_1, \theta_2). \]

**Definition 2.4.2** The **covariation** \([X_1, X_2]_\alpha\) of the jointly \(S\alpha S\) random vector \((X_1, X_2)\) is

\[
[X_1, X_2]_\alpha = \frac{1}{\alpha} \frac{\partial}{\partial \theta_1} \sigma^\alpha(\theta_1, \theta_2) \big|_{\theta_1 = 0, \theta_2 = 1}.
\]

The covariation is designed to replace the covariance when \(\alpha \in (1, 2)\). In the case of \(\alpha = 2\) we have the following relation between these two expressions

\[ [X_1, X_2]_2 = \frac{1}{2} \operatorname{Cov}(X_1, X_2). \]

**Asymptotic behavior of tail probabilities.** Using the Central-Limit-Theorem-type argument, one can prove that if \(X \sim S_\alpha(\sigma, \beta, \mu)\) and \(\alpha \in (0, 2)\), then

\[
\lim_{y \to -\infty} y^\alpha P\{X > y\} = C_\alpha \frac{1 + \beta}{2} \sigma^\alpha,
\]

\[
\lim_{y \to -\infty} y^\alpha P\{X < -y\} = C_\alpha \frac{1 - \beta}{2} \sigma^\alpha,
\]

where

\[
C_\alpha = \left( \int_0^\infty x^{-\alpha} \sin(x) \, dx \right)^{-1}.
\]

**Density functions.** When we start to work with \(\alpha\)-stable distributions, the main problem is that except for a few values of four parameters describing the characteristic function, their density functions are not known explicitly. The most interesting exceptions are the following:

- the Gaussian distribution \(S_2(\sigma, 0, \mu)\), whose density is

\[
f(x) = \frac{1}{2\sigma\sqrt{\pi}} \exp \left\{ -\frac{(x - \mu)^2}{4\sigma^2} \right\};
\]

- the Cauchy distribution \(S_1(\sigma, 0, \mu)\), whose density is

\[
f(x) = \frac{2\sigma}{\pi((x - \mu)^2 + 4\sigma^2)};
\]

- the Lévy distribution \(S_{1/2}(\sigma, 1, \mu)\), whose density

\[
f(x) = \left( \frac{\sigma}{2\pi} \right)^{1/2} (x - \mu)^{-3/2} \exp \left\{ -\frac{\sigma}{2(x - \mu)} \right\}
\]

is concentrated on \((\mu, \infty)\), i.e., \(f(x) = 0\) for \(x \in (-\infty, \mu]\).
So, in order to obtain $\alpha$-stable density functions, we have to take into account the definition describing characteristic functions of $\alpha$-stable random variables and apply the Fourier transform, namely

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) dt.$$  

Using a numerical approximation of this formula we were able to construct such densities in the general case. The result of computer calculations is presented here in the form of a series of graphs of such densities for different values of parameters $\alpha, \beta, \sigma$ (the role of parameter $\mu$ is obvious), obtained with the use of IBM PC/386 graphics.

Figures 2.4.1 - 2.4.6 below demonstrate in a very precise way how $\alpha$-stable density functions depend on parameters:

- $\alpha$ - the index of stability from the interval $(0, 2]$,
- $\beta$ - the skewness parameter from the interval $[-1, 1]$,
  with the exception: $\alpha = 1 \Rightarrow \beta = 0$,
- $\sigma$ - the scale parameter from $(0, \infty)$.

Figure 2.4.1 shows the dependence of densities on $\alpha$ and Figure 2.4.2 - the dependence on $\sigma$. Figures 2.4.3 and 2.4.4 demonstrate how densities fluctuate when parameter $\beta$ goes throughout the set of different values from -1 to 1, for two different values of $\alpha$, respectively. Each of these two figures contains one example of the so called totally skewed density (in Figure 2.4.3 we have the density of $X \sim S_{0.8}(1, -1, 0)$ and in Figure 2.4.4 - $X \sim S_{0.5}(1, 1, 0)$. Figures 2.4.5 and 2.4.6 show what is going on, when parameter $\alpha$ approaches 1 with fixed, different from 0 value of parameter $\beta$. This corresponds to the remark on the definition of the characteristic function of $\alpha$-stable random variable (exceptional case of $\alpha = 1$) at the beginning of this section. We believe that it is an easy and instructive task to find out which curves correspond to which values of parameters in all figures presented here. Figures 2.4.1 - 2.4.5 preserve the same scaling of both axes; the next (Figure 2.4.6) has the vertical one changed.
Figure 2.4.1. The case of $\alpha \in \{2.0, 1.2, 0.8, 0.5\}$, $\beta = 0.0$, $\sigma = 1.0$, $\mu = 0.0$.

Figure 2.4.2. The case of $\alpha = 1.2$, $\beta = 0.0$, $\sigma \in \{0.5, 0.7, 1.0, 1.5\}$, $\mu = 0.0$. 
Figure 2.4.3. The case of $\alpha = 0.8$, $\beta \in \{-1.0, -0.8, -0.5, 0.0\}$, $\sigma = 1.0$ and $\mu = 0.0$.

Figure 2.4.4. The case of $\alpha = 0.5$, $\beta \in \{-0.5, 0.0, 1.0\}$, $\sigma = 1.0$, $\mu = 0.0$. 
Figure 2.4.5. The case of $\alpha \in \{0.5, 0.8, 0.9, 0.95\}$, $\beta = 0.8$, $\sigma = 1.0$, $\mu = 0.0$.

Figure 2.4.6. The case of $\alpha \in \{1.8, 1.2, 1.1, 1.05\}$, $\beta = 0.8$, $\sigma = 1.0$, $\mu = 0.0$. 
2.5 $\alpha$-Stable Lévy Motion

Roughly speaking, an $\alpha$-stable process is a random element whose finite dimensional distribution is $\alpha$-stable, where $0 < \alpha \leq 2$.

**Definition 2.5.1** A stochastic process $\{X(t) : t \in \mathbb{T}\}$, where $\mathbb{T}$ is an arbitrary set, is stable if all its finite dimensional distributions

$$X(t_1), X(t_2), \ldots, X(t_n), \; t_1, t_2, \ldots, t_n \in \mathbb{T}, \; n \geq 1$$

are stable. It is symmetric stable if all its finite-dimensional distributions are symmetric stable.

If the finite dimensional distributions are stable or symmetric stable, then by consistency, they must all have the same index of stability $\alpha$. It is known that $\{X(t) : t \in \mathbb{T}\}$ is symmetric stable if and only if all linear combinations

$$\sum_{k=1}^{n} a_k X(t_k), \; n \geq 1, \; t_1, t_2, \ldots, t_n \in \mathbb{T}, \; a_1, a_2, \ldots, a_n \in \mathbb{R},$$

are symmetric stable. For non-symmetric stable processes this property holds only when $\alpha > 1$.

The best known example of an $\alpha$-stable process is the $\alpha$-stable Lévy motion. Let us recall the definition.

**Definition 2.5.2** A stochastic process $\{X(t) : t \geq 0\}$ is called the (standard) $\alpha$-stable Lévy motion if

1. $X(0) = 0$ a.s.;
2. $\{X(t) : t \geq 0\}$ has independent increments;
3. $X(t) - X(s) \sim S_{\alpha}((t - s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t < \infty$.

Observe that the $\alpha$-stable Lévy motion has stationary increments. It is the Brownian motion, when $\alpha = 2$. The $\alpha$-stable Lévy motions are $S_{\alpha}S$ when $\beta = 0$ and they are $1/\alpha$ self-similar. That is, for all $c > 0$ the processes $\{X(ct) : t \geq 0\}$ and $\{c^{1/\alpha}X(t) : t \geq 0\}$ have the same finite-dimensional distributions.

The first step toward a description of basic properties of stable processes would be the discussion of some properties of the $\alpha$-stable Lévy motion. In order to illustrate the most important features of this stochastic process distinguishing it from the processes presented in Sections 2.2 and 2.3, we present some computer graphs of trajectories of standard $\alpha$-stable Lévy motion for a few different values of the parameter $\alpha$. At a first glance one can notice a remarkable qualitative and quantitative difference between graphical representations of the Brownian motion (see Section 2.2) and the $\alpha$-stable Lévy motion presented here (with the parameter $\alpha$ taking on smaller values, the "jumps" of the trajectories become bigger, so in Figures 2.2.1 and 2.5.1 - 2.5.3 we had to change constantly the scaling of the vertical axis). We will come back to the discussion of this effect later on.
Figure 2.5.1. Trajectories of $\alpha$-stable Lévy motion in the case of $\alpha = 1.7$.

Figure 2.5.2. Trajectories of $\alpha$-stable Lévy motion in the case of $\alpha = 1.2$. 
Figure 2.5.3. Trajectories of $S\alpha S$ Lévy motion in the case of $\alpha = 0.7$.

Figure 2.5.4. Trajectories of totally skewed Lévy motion in the case of $\alpha = 0.7$. 
Construction of the $S\alpha S$ Lévy Motion from the Brownian Motion. It was Bochner (1955) who noted an interesting relationship between the processes plotted in figures presented in this chapter. It is possible to obtain the $S\alpha S$ Lévy motion process from the Brownian motion by a random time change defined by the totally skewed Lévy motion. Here we present this construction.

Observe that if the measure $\nu_\alpha$ on $(0, \infty)$ is defined by the formula

$$d\nu_\alpha(x) = \frac{P}{x^{1+\alpha}} \, dx,$$

then for $0 < \alpha < 1$ the following integral

$$I(u) = \int_0^\infty (e^{iu x} - 1) \, d\nu_\alpha(x)$$

is finite (see Sections 2.4 and 4.4 for the Lévy–Khintchine Formula in different settings).

Consider the stochastic process $X = \{X(t)\}$ with stationary independent increments, and such that the characteristic function

$$\mathbb{E} e^{iu X(t)} = e^{i \psi(u)}$$

has the exponent function $\psi(u) = I(u)$. Then, the process $X$ is the limit of the sequence of processes $\{X_n\}$ with exponent functions

$$\psi_n(u) = \int_{1/n}^\infty (e^{iu x} - 1) \, d\nu_\alpha(x).$$

These processes have only upward jumps. Hence, all paths of $\{X(t)\}$ are non-decreasing pure jump functions, (see Breiman (1968)).

Take on a given probability space the normalized Brownian motion $\{B(t)\}$ and an $\alpha/2$-stable totally skewed Lévy motion $\{X(t)\}$ with nondecreasing sample paths ($\alpha/2 \in (0, 1)$, $\beta = 1$), and such that $\mathcal{F}\{X(t) : t \geq 0\}$, $\mathcal{F}\{B(t) : t \geq 0\}$ are independent. If we put

$$Y(t) = B(X(t)),$$

then the process $\{Y(t)\}$ is an $S\alpha S$ Lévy motion.

The idea becomes clear when one notice that the process defined by

$$Y(t + s) - Y(t) = B((X(t + s) - X(t)) + X(t)) - B(X(t))$$

looks just as if it were the $Y(s)$, independent of $Y(t)$, $\tau \leq t$.

For a formal proof we will follow Breiman (1968). Take as $X_n(t)$ the process of jumps of $X(t)$ larger than $[0, 1/n)$. According to the above considerations its characteristic function is described by the exponent function

$$\psi_n(u) = \int_{1/n}^\infty (e^{iu x} - 1) \, d\nu_{\alpha/2}(x).$$
The jumps of $X_n(t)$ occur at the jump times of a Poisson process with intensity
\[ \lambda_n = \int_{1/n}^{\infty} d\nu_{\alpha/2}(x). \]
and the jumps have magnitude $Y_1, Y_2, \ldots$ independent of one another and of the jump times, and are identical distributed. Thus $B(X_n(t))$ has jumps only at the jump times of the Poisson process. The size of the $k$-th jump is
\[ U_k = \begin{cases} B(Y_k + \ldots + Y_1) - B(Y_{k-1} + \ldots + Y_1) & k > 1, \\ B(Y_1) & k = 1. \end{cases} \]
By an argument almost exactly the same as that used in the proof of the strong Markov property for the Brownian motion, $U_k$ is independent of $U_{k-1}, \ldots, U_1$ and has the same distribution as $U_1$. Therefore $B(X(t))$ is a process with stationary, independent increments. Take \( \{n'\} \) such that $X_{n'}(t) \to X(t)$ a.s. and uniformly on compact sets. Use continuity of the Brownian motion $B(t)$ to get
\[ \lim_{n' \to \infty} B(X_{n'}(t)) = B(X(t)) \quad \text{a.s. for every } t \in [0, \infty). \]
Thus, $B(X(t))$ is a process with stationary independent increments. To obtain its characteristic function write the conditional expectation
\[ \mathbb{E} \{ \exp[iuB(X(1))] \mid Z(1) = z \} = \mathbb{E} \exp[iuB(z)] = \exp[-zu^2/2]. \]
Therefore,
\[ \mathbb{E} \exp[iuB(X(1))] = \mathbb{E} \exp(-(u^2/2)X(1)) = \exp(-c_X|u|^{\alpha}), \]
so, $Y(t) = B(X(t))$ belongs to the class of $S\alpha S$ Lévy motion processes.
tion, currently the most common, the linear congruential method. It provides a sequence of integers \( I_0, I_1, I_2, \ldots \), each between 0 and \( m - 1 \), by the recurrence relation

\[
I_{n+1} = a \cdot I_n + c \pmod{m},
\]

where \( m, a, c \) are given positive integers. In calculations with IBM PC/386 the following exemplary triples \((m,a,c)\) can serve our purposes:

\[
(134456, 8121, 28411), \quad (243000, 4561, 51349),
\]

\[
(259200, 7141, 54733), \quad (714025, 4096, 150889).
\]

Here, in all four cases, the period of the sequence \( I_0, I_1, I_2, \ldots \) is of maximal length, i.e., of length \( m \). Further application of the so-called shuffling procedure provides new efficient algorithms producing random sequences with still much longer periods, helps to break up possible sequential correlations or diminish the effects of any cycle or bias.

**Add-with-carry and subtract-with-borrow generators.**

A description of a new powerful class of methods for generating random numbers with very long periods can be found in Marsaglia and Zaman (1991).

Let us recall here only one example of add-with-carry generator useful for experiments. Fix base 6 to represent any integer \( I \) in the range from 0 up to \( 6^{21} - 1 \) as a sequence of twenty one digits from the set \{0, 1, 2, 3, 4, 5\}. Now, start an iterating formula by choosing any sequence of twenty one seed digits and any initial carry bit \( c \in \{0, 1\} \) (with two exceptions: twenty one 0's for \( c = 0 \) and twenty one 5's for \( c = 1 \)), i.e., let

\[
I_0 = x_0 x_1 \ldots x_{19} x_{20}, \quad c_0 \in \{0, 1\}.
\]

With

\[
I_{n-1} = x_{n-1} x_n \ldots x_{n+18} x_{n+19}, \quad c_{n-1} \in \{0, 1\},
\]

already constructed, one has to calculate

\[
x_{n+20} = x_{n-1} + x_{n+18} + c_{n-1} \pmod{6},
\]

\[
c_n = \begin{cases} 
0, & \text{if } x_{n-1} + x_{n+18} + c_{n-1} < 6, \\
1, & \text{if } x_{n-1} + x_{n+18} + c_{n-1} \geq 6 
\end{cases}
\]

and to output finally

\[
I_n = x_n x_{n+1} \ldots x_{n+19} x_{n+20},
\]

where positive \( n \) can increase up to \( 6^{21} + 6^2 - 1 \), because this number is a prime for which 6 is a primitive root and thus this generator will always have period \( 6^{21} + 6^2 - 2 = 21\,936\,950\,640\,377\,890 \). (Every possible set of twenty one successive "throws" from the set \{0, 1, 2, 3, 4, 5\} will appear in the sequence, with frequencies for shorter strings consistent with uniformity for the full period!)
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