Asymptotic behavior of measures of dependence for ARMA(1,2) models with stable innovations. Stationary and non-stationary coefficients

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Abstract

We derive the asymptotic behavior of two measures of dependence (Codifference and Covariation) for ARMA(1,2) models with symmetric α-stable innovations and non-stationary coefficients.

1 Introduction

Definition 1 The system ARMA(1,q) is given by the following formula:

\[ X_n - b_n X_{n-1} = \sum_{j=0}^{q-1} a_{n-j} \xi_{n-j}, \]  

(1)

where the innovations (\(\xi_n\)) are independent, symmetric α-stable with the scale parameter \(1\), i.e. with the characteristic function given by:

\[ E \exp(i\theta \xi_n) = \exp(-|\theta|^\alpha), 0 < \alpha \leq 2. \]

Moreover the coefficients \((a_n)\) and \((b_n)\) are nonzero and complex for all \(n \in \mathbb{Z}\).

In [7] Weron and Wylomanska show the conditions, which give a bounded solutions of system ARMA(1,q), where the innovations are uncorrelated complex random variables with mean 0 and variance 1. In this case we have three conditions, which give bounded solution of the system, but only two give its unique solution. Therefore we consider two cases:

I. \( \sup_q |B_q^0| = \infty \)

II. \( \sup_q |B_q^0|^{-1} = \infty. \)

In this paper \(X_n = Y\) (\(X_n\)-the sequence of random variables, \(Y\)- the random variable) means \(\lim_n ||X_n - Y|| = 0\) and \(B_r^s = \prod_{j=r}^{s} b_j\) (with the convention that \(B_s^r = 1\) if \(r > s\)). In the analysis we consider the system ARMA(1,2) given by the following equation:

\[ X_n - b_n X_{n-1} = a_n \xi_n + a_{n-1} \xi_{n-1}, \]

(3)

where the innovations and coefficients have the same properties like in the general model.
Definition 2 Measures of dependence of jointly symmetric $\alpha$–stable random variables $X_1$ and $X_2$ \cite{[6]}. 

- **Covariation** $\text{CV}(X_1, X_2)$ of $X_1$ on $X_2$ defined for $1 < \alpha \leq 2$ is the real number

$$\text{CV}(X_1, X_2) = \int_{S^2} s_1 s_2^{<\alpha-1>} \Gamma(ds),$$

where $\Gamma$ is the spectral measure of the random vector $(X_1, X_2)$, $z^{<p>} = |z|^{p-1} \bar{z}$.

- **Codifference** $\text{CD}(X_1, X_2)$ of $X_1$ on $X_2$ defined for $0 < \alpha \leq 2$ equals

$$\text{CD}(X_1, X_2) = \ln E\exp\{i(X_1 - X_2)\} - \ln E\exp\{iX_1\} - \ln E\exp\{-iX_2\}. \quad (5)$$

Unlike the codifference, the covariation is not symmetric, but is linear in the first argument. The covariation is closely related to the quantity $E[X_1X_2^{<p>}]$. In this paper the norm $||X||$, where $X$ is a symmetric $\alpha$–stable random variable, is defined $||X|| = (\text{CV}(X, X))^{1/\alpha}$ (the covariation norm \cite{[6]}). If $\alpha = 2$ the following identities hold

$$\text{CV}(X_1, X_2) = \frac{1}{2} \text{Cov}(X_1, X_2),$$

$$\text{CD}(X_1, X_2) = \text{Cov}(X_1, X_2).$$

In contrast to \cite{[3, 4]} we study here time series ARMA(1,2) with non-stationary coefficients. The presented new general proofs use the form of the bounded solution of ARMA(1,2) model from \cite{[7]}. This leads us to consider two separately cases described by Condition 1 and Condition 2. The main results are included in Theorem 1 and Theorem 2. They give the asymptotic behavior of the quotients $\text{CD}(X_1, X_{n+kq})/\text{CV}(X_1, X_{n+kq})$ and $\text{CD}(X_n, X_{n-kq})/\text{CV}(X_n, X_{n-kq})$ for both cases, respectively. Formulas 10 and 20 generalize the earlier result of Nowicka in \cite{[4]}. However formulas 11 and 21 produce a new type of result even in the case of stationary coefficients.

2 Condition 1

If $\sup_q |B^q_0| = \infty$, then there exist sequence $(k_q)$ of positive integers such that $\lim_q |B^q_n| = \infty$, and for all $n \in Z$ $\lim_q |B^q_{n+1}| = \infty$. In this case the solution of (3) is given by (see \cite{[7]}):

$$X_n = -\lim_{q} \frac{a_{n} \xi_{n} + \xi_{n+1}}{B^q_{n+1}} \left(1 + \frac{1}{b_{n+1}} + \frac{a_{n+k-q} \xi_{n+k-q}}{B^q_{n+1}}\right).$$

If we denote:

$$c_n(\xi) = \begin{cases} 
0 & j > k_q \\
\frac{-a_{n+k-q}}{B^q_{n+1}} & j = k_q \\
\frac{a_{n} \xi_{n+1}}{b_{n+1}} + \frac{a_{n+k-q} \xi_{n+k-q}}{B^q_{n+1}} & 0 < j < k_q \\
\frac{-a_{n}}{b_{n+1}} & j = 0 \\
0 & j < 0,
\end{cases}$$

\[2\]
then we can write:

$$X_n = \lim_{q} \left[ \sum_{j=n}^{k_q + n} c_n(j - n) \xi_j \right].$$

In this case the covariance of $X_n$ on $X_{n+k_q}$ is given by:

$$CV(X_n, X_{n+k_q}) = c_n(k_q)c_{n+k_q}^{<\alpha-1>}(0) = \frac{|a_{n+k_q}|^\alpha}{b_{n+k_q+1}|a-2B_{n+k_q+1}^{n+1}B_{n+1}^n}. \quad (6)$$

However the covariance of $X_{n+k_q}$ on $X_n$ has the following form:

$$CV(X_{n+k_q}, X_n) = c_{n+k_q}(0)c_n^{<\alpha-1>}(k_q) = \frac{|a_{n+k_q}|^\alpha}{b_{n+k_q+1}|a-2B_{n+k_q}^{n+1}B_{n+1}^n}. \quad (7)$$

If the coefficients satisfy condition 1, then the codifference of $X_n$ on $X_{n+k_q}$ is given by the formula:

$$CD(X_n, X_{n+k_q}) = |c_n(k_q)|^\alpha + |c_{n+k_q}(0)|^\alpha - |c_n(k_q) - c_{n+k_q}(0)|^\alpha.$$

Therefore:

$$CD(X_n, X_{n+k_q}) = \frac{a_{n+k_q}}{b_{n+k_q+1}}^\alpha + \frac{a_{n+k_q}}{b_{n+k_q+1}}^\alpha - \frac{a_{n+k_q}}{b_{n+k_q+1}}^\alpha - \frac{a_{n+k_q}}{b_{n+k_q+1}}^\alpha. \quad (8)$$

The codifference takes the form:

$$CD(X_n, X_{n+k_q}) = \frac{a_{n+k_q}}{b_{n+k_q+1}}^\alpha (1 + \frac{b_{n+k_q+1}}{B_{n+1}^{n+1}}^\alpha - |1 - \frac{b_{n+k_q+1}}{B_{n+1}^{n+1}}^\alpha|). \quad (9)$$

The codifference is symmetric, therefore:

$$CD(X_n, X_{n+k_q}) = CD(X_{n+k_q}, X_n).$$

If $1 < \alpha \leq 2$ and $\sup_q |B_{n+1}^{n+1+k_q}| = \infty$, then the following fact is true for all $n \in Z$:

$$\lim_{k_q \to \infty} \frac{b_{n+k_q+1}}{b_{n+k_q+1}} [1 + \frac{b_{n+k_q+1}}{B_{n+1}^{n+1}}^\alpha - |1 - \frac{b_{n+k_q+1}}{B_{n+1}^{n+1}}^\alpha|] = \alpha. \quad (9)$$

**Theorem 1** Suppose $(X_n)$ is the solution of system (3) and $\sup_q |B_{n+1}^q| = \infty$, then for $1 < \alpha \leq 2$ and for all $n \in Z$ the following are fulfilled:

$$\lim_{k_q \to \infty} \frac{CD(X_n, X_{n+k_q})}{CV(X_n, X_{n+k_q})} = \alpha. \quad (10)$$

$$\lim_{k_q \to \infty} \frac{|b_{n+k_q+1}|^{\alpha-2}CD(X_n, X_{n+k_q})}{|B_{n+1}^{n+1+k_q}|^{\alpha-2}CV(X_{n+k_q}, X_n)} = \alpha \quad (11)$$

if $CV(X_n, X_{n+k_q}) \neq 0$ and $CV(X_{n+k_q}, X_n) \neq 0$. 

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PROOF: We use formulas (6), (8) and (9) to compute (10). We obtain (11) from (7), (8) and (9).

For \( \alpha = 2 \) naturally we have:

\[
\lim_{k_q \to \infty} \frac{CD(X_n, X_{n+k_q})}{CV(X_n, X_{n+k_q})} = \frac{CD(X_{n+k_q}, X_n)}{CV(X_{n+k_q}, X_n)} = \alpha.
\]

**Corollary 1** If we take \( n - k_q \) instead \( n \) in formulas (10) and (11), then we obtain the following formulas:

\[
\lim_{k_q \to \infty} \frac{CD(X_{n-k_q}, X_n)}{CV(X_{n-k_q}, X_n)} = \alpha
\]

and

\[
\lim_{k_q \to \infty} \frac{|b_{n+1}|^{\alpha-2}CD(X_n, X_{n-k_q})}{|B_{n-k_q+1}|^{\alpha-2}CV(X_n, X_{n-k_q})} = \alpha.
\]

**Remark 1** System ARMA(1,2) with the time varying coefficients given by the formula:

\[
X_n - b_nX_{n-1} = a_0(n)\xi_n + a_1(n)\xi_{n-1}
\]

do not have property (10), because for the system we obtain the following formulas:

\[
CV(X_n, X_{n+k_q}) = \frac{a_0(n+k_q)}{B_{n+1}^{n+k_q}} \frac{|a_1(n+k_q+1)|^{\alpha-2}a_1(n+k_q+1)}{b_{n+k_q+1}^{n+k_q}} - \frac{|a_0(n+k_q)+1|^{\alpha-2}a_0(n+k_q+1)}{b_{n+k_q+1}^{n+k_q}}.
\]

Therefore the asymptotic behaviour of the measures \( CV \) and \( CD \) has the following forms:

\[
\lim_{k_q \to \infty} \frac{|a_1(n+k_q+1)|^{\alpha-2}a_1(n+k_q+1)CD(X_n, X_{n+k_q})}{|a_0(n+k_q)|^{\alpha-2}a_0(n+k_q)CV(X_n, X_{n+k_q})} = \alpha.
\]

**Example 1** We consider now ARMA(1,2) model given by the equation:

\[
X_n + 2X_{n-1} = \sqrt{2^{n-1}} \xi_n + \sqrt{2^{n-1}} \xi_{n-1}.
\]

The coefficients \((b_n)\) satisfy Condition 1, i.e. \( \sup q |B_q| = \sup q 2^q = \infty \). On Figure 1 we show the plot of \( \frac{CD(X_n, X_{n+k_q})}{\alpha CV(X_n, X_{n+k_q})} \) for \( k_q = 0, 1, \ldots, 50 \), \( n = 10 \) and \( \alpha = 1.2 \) and \( \alpha = 1.5 \).
Remark 2 For the comparison we consider system ARMA(1,2) with the stationary coefficients, given by the equation:

\[ X_n - b_1 X_{n-1} = a_1 \xi_n + a_2 \xi_{n-1}, \quad (15) \]

where the innovations and coefficients are like in the definition 1. In this case the formulas (10) and (11) have the following forms respectively:

\[
\lim_{k_q \to \infty} \frac{CD(X_{-k_q}, X_0)}{CV(X_{-k_q}, X_0)} = \alpha
\]

and

\[
\lim_{k_q \to \infty} \frac{CD(X_0, X_{-k_q})}{|b_1|(|\alpha-2)(k_q-1)|CV(X_0, X_{-k_q})} = \alpha.
\]

3 Condition 2

If \( \sup_q |B_q^0|^{-1} = \infty \), then there exist a sequence \( (k_q) \) of positive integers such that \( \lim_{k_q} |B_{n-k_q}^0|^{-1} = \infty \), and for all \( n \in \mathbb{Z} \) \( \lim_{k_q} |B_{n-k_q}^n|^{-1} = \infty \). In this case the solution of system (3) is given by the formula (see [7]):

\[
X_n = \lim_{q} \left[ \sum_{j=1}^{k_q-1} a_{n-j} B_{n+2-j}^n \right] (1 + b_{n-j+1}) \xi_{n-j} + a_{n-k_q} B_{n-k_q+2}^n \xi_{n-k_q} + a_n \xi_n].
\]

If we assume:

\[
c_n(j) = \begin{cases} 0 & j > k_q \\ a_{n-k_q} B_{n-k_q+2}^n & j = k_q \\ a_{n-j} B_{n+2-j}^n (1 + b_{n+1-j}) & 0 < j < k_q \\ a_n & j = 0 \\ 0 & j < 0, \end{cases}
\]

then the solution of (3) takes the form:

\[
X_n = \lim_{q} \left[ \sum_{j=n-k_q}^{n} c_n(n-j) \xi_j \right].
\]
If the coefficients $b_n$ fulfill condition 2, then the covariation of $X_n$ on $X_{n-k_q}$ has the form:

$$CV(X_n, X_{n-k_q}) = c_n(k_q)c_n^{\frac{\alpha-1}{\alpha-2}}(0) = |a_{n-k_q}|^\alpha B_{n-k_q}^n + 2. \quad (16)$$

Furthermore, the covariation of $X_{n-k_q}$ on $X_n$ is given by the formula:

$$CV(X_{n-k_q}, X_n) = |a_{n-k_q}|^\alpha |B_{n-k_q}^n + 2|^\alpha B_{n-k_q}^n. \quad (17)$$

And the codifference of $X_n$ on $X_{n-k_q}$ is given by the following:

$$CD(X_n, X_{n-k_q}) = |c_n(k_q)|^\alpha + |c_n(k_q)(0)|^\alpha - |c_n(k_q) - c_{n-k_q}(0)|^\alpha =$$

$$= |a_{n-k_q}B_{n-k_q}^n|^{\alpha} + |a_{n-k_q}|^\alpha - a_{n-k_q}B_{n-k_q}^n - a_{n-k_q}|^\alpha. \quad (18)$$

If $1 < \alpha \leq 2$ and $\sup_q |B_{n-k_q}^n|^{-1} = \infty$, then the following equation is fulfilled

$$\lim_{k_q \to \infty} \frac{1}{B_{n-k_q}^n + 2}(1 + |B_{n-k_q}^n + 2|^\alpha - |B_{n-k_q}^n + 2|^\alpha) = \alpha. \quad (19)$$

**Theorem 2** Suppose $(X_n)$ is the solution of system (3) and $\sup_q |B_q^n|^{-1} = \infty$, then for $1 < \alpha \leq 2$ and for all $n \in Z$ the following is fulfilled:

$$\lim_{k_q \to \infty} CD(X_n, X_{n-k_q}) = \alpha \quad (20)$$

$$\lim_{k_q \to \infty} \frac{|B_{n-k_q}^n + 2|^{\alpha} CD(X_{n-k_q}, X_n)}{CD(X_{n-k_q}, X_n)} = \alpha \quad (21)$$

if $CV(X_n, X_{n-k_q}) \neq 0$ and $CV(X_{n-k_q}, X_n) \neq 0$.

**PROOF:** We use formulas (16), (18) and (19) to compute (20). We obtain (21) from (17), (18) and (19).

□

**Corollary 2** We take in formulas (20) and (21) $n + k_q$ instead $n$ and obtain:

$$\lim_{k_q \to \infty} CD(X_{n+k_q}, X_n) = \alpha \quad (22)$$

$$\lim_{k_q \to \infty} \frac{|B_{n+k_q}^{n+2}|^{\alpha} CD(X_n, X_{n+k_q})}{|B_{n+k_q}^{n+2}|} = \alpha. \quad (23)$$

**Example 2** We consider system ARMA(1,2) model given by the equation:

$$X_n + \frac{1}{2}X_{n-1} = \sqrt{2}^n \xi_n + \sqrt{2}^{n-1} \xi_{n-1}. \quad (24)$$
The coefficients $b_n$ satisfy Condition 1, i.e. $\sup_q |B_q^{0\rightarrow\infty}| = \sup_q 2^q = \infty$. On Figure 2 we show the plot of $\frac{CD(X_n, X_{n-k_q})}{\alpha CV(X_n, X_{n-k_q})}$ for $k_q = 0, 1, \ldots, 50$, $n = 10$ and $\alpha = 1.2$ and $\alpha = 1.5$.

![Figure 2. The plot of $\frac{CD(X_n, X_{n-k_q})}{\alpha CV(X_n, X_{n-k_q})}$ for $\alpha = 1.2$ (left) and $\alpha = 1.5$ (right).](image)

**Remark 3** If $X_n$ is the solution of system (15), then formulas (20) and (21) have the following forms:

$$\lim_{k_q \to \infty} \frac{CD(X_{n-k_q}, X_0)}{CV(X_{n-k_q}, X_0)} = \alpha$$

and

$$\lim_{k_q \to \infty} \left| b_1 \right| \frac{CD(X_0, X_{n-k_q})}{\alpha CV(X_0, X_{n-k_q})} = \alpha.$$

In [4] there are given the formulas to the covariation and codifference of stationary ARMA(p,q) models and the author described asymptotic behavior of the measures of dependence. The main result of [4] is given in Corollary 1. It is shown, that the relation \( \lim_{n \to \infty} \frac{CD(X_n, X_0)}{CV(X_n, X_0)} = \alpha \) holds, when some conditions are fulfilled. Therefore, if we assume that the coefficients in ARMA(1,2) model depend on the time (non-stationary ARMA model), then we obtain similar result like in the stationary case.

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