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Fractional governing equations for coupled random walks

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ABSTRACT

In a continuous time random walk (CTRW), a random waiting time precedes each random jump. The CTRW is coupled if the waiting time and the subsequent jump are dependent random variables. The CTRW is used in physics to model diffusing particles. Its scaling limit is governed by an anomalous diffusion equation. Some applications require an overshoot continuous time random walk (OCTRW), where the waiting time is coupled to the previous jump. This paper develops stochastic limit theory and governing equations for CTRW and OCTRW. The governing equations involve coupled space–time fractional derivatives. In the case of infinite mean waiting times, the solutions to the CTRW and OCTRW governing equations can be quite different.

1. Introduction

The continuous time random walk (CTRW) model was developed in physics to represent diffusing particles. A random waiting time $J_n > 0$ precedes the nth random jump Y_n of the particle. Typically we assume that (Y_n, J_n) are i.i.d. random vectors in space–time with possible dependence between the waiting time J_n and the jump Y_n. This coupling can be used to enforce certain physical constraints, e.g., particle velocity Y_n/J_n should not exceed the speed of light [1]. The jumps can represent movements of tracer particles in underground aquifers [2–4], downstream movements of gravel particles along river beds [5], biological cell movements [6], motion of DNA-binding proteins along a chromosome [7], or movements of animals in search of a food source [8]. In finance, the jumps represent changes in price (or log-returns) [9].

In certain applications, it is useful to consider the overshoot continuous time random walk (OCTRW), where the waiting time J_n follows the jump Y_n. The OCTRW can be used to model dielectric relaxation phenomena in complex systems. The OCTRW scenario, with the jump coupled to the subsequent waiting time through random clustering, provides a physical explanation for the empirical Havriliak–Negami dielectric response, widely observed in relaxing dielectric materials [10,11]. In applications of the OCTRW to finance, the jump Y_n is the nth price change (log-return), J_n is the waiting time between the nth and the $(n+1)$st trades, and the OCTRW represents the logarithm of the current price [12]. Coupling between log returns and waiting times is rather common in finance [12,13]. In this paper, we develop limit theory and governing equations for CTRW and OCTRW with infinite mean waiting times. We emphasize the general setting, where (Y_n, J_n) are i.i.d., but we allow dependence between the waiting time J_n and the subsequent jump Y_n.

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2. Preliminaries

Let (Y_n, J_n) be i.i.d. with (Y, J) on $\mathbb{R} \times \mathbb{R}_+$ and set

$$T(n) = J_1 + \cdots + J_n \quad \text{and} \quad S(n) = Y_1 + \cdots + Y_n \tag{2.1}$$

so that $(S(n), T(n))$ is a random walk on $\mathbb{R} \times \mathbb{R}_+$. For $t \geq 0$ we define the continuous time random walk (CTRW)

$$S(N(t)) = Y_1 + \cdots + Y_{N(t)} \tag{2.2}$$

where

$$N(t) = \max\{n \geq 0 : T(n) \leq t\} \tag{2.3}$$

is the number of jumps by time t. The OCTRW

$$S(N(t) + 1) = Y_1 + \cdots + Y_{N(t)} + Y_{N(t)+1} \tag{2.4}$$

involves one additional jump.

Assume (Y, J) belongs to the strict generalized domain of attraction of some operator stable law [14] with exponent $E = \mathrm{diag}(1/\alpha, 1/\beta)$, so that for some $b_n > 0$ and $B_n > 0$ we have

$$(B_n S(n), b_n T(n)) \Rightarrow (A, D) \tag{2.5}$$

where $D > 0$ almost surely. Here $\Rightarrow$ denotes convergence in distribution. The distribution μ of (A, D) is strictly operator stable with index E, meaning that $\mu^t = t^E \mu$ for all $t > 0$, where μ^t is the convolution power of the infinitely divisible law μ, $t^E = \exp(E \log t)$ using the usual matrix exponential, and $(t^E \mu)(dx) = \mu(t^{-E} dx)$ is the probability distribution of $t^E(A, D) = (t^{1/\alpha} A, t^{1/\beta} D)$ for $t > 0$. Then a standard result [15, Theorem 4.1] shows that

$$\{(B(c)S(ct), b(c)T(ct))\}_{t \geq 0} \Rightarrow \{(A(t), D(t))\}_{t \geq 0} \quad \text{as } c \to \infty \tag{2.6}$$

in the Skorohod space $D([0, \infty), \mathbb{R} \times \mathbb{R}_+)$ with the J_1 topology, where $b(t) = b_{[t]}$, $B(t) = B_{[t]}$, and $(A(t), D(t))$ is a Lévy process with $(A(1), D(1)) = (A, D)$. In view of [14, Theorem 8.3.24] we may assume without loss of generality that $B(t)$, $b(t)$ vary regularly with index $-1/\alpha$, $-1/\beta$ respectively. Then $1/b(t)$ is regularly varying with index $1/\beta > 0$ so by [16, Property 1.5.5] there exists a regularly varying function $\tilde{b}$ with index β such that $1/b(\tilde{b}(c)) \sim c$ as $c \to \infty$. Here $f \sim g$ means that $f(c)/g(c) \to 1$ as $c \to \infty$. Define $\tilde{B}(c) = B(\tilde{b}(c))$, a regularly varying function with index $-\beta/\alpha$.

For suitable functions g on $\mathbb{R} \times \mathbb{R}_+$ we define the Fourier–Laplace transform (FLT)

$$\bar{g}(k, s) = \int_{\mathbb{R}} \int_0^\infty e^{ikx} e^{-st} g(x, t) dt\, dx \tag{2.7}$$

where $(k, s) \in \mathbb{R} \times \mathbb{R}_+$. Similarly, if μ is a bounded Borel measure on $\mathbb{R} \times \mathbb{R}_+$,

$$\bar{\mu}(k, s) = \int_{\mathbb{R}} \int_0^\infty e^{ikx} e^{-st} \mu(dx, dt)$$

is the FLT of μ. If ρ is a probability measure on $\mathbb{R}$, the Fourier transform (FT)

$$\hat{\rho}(k) = \int_{\mathbb{R}} e^{ikx} \rho(dx).$$

If ρ_t is a probability measure on $\mathbb{R}$ for each $t > 0$ such that $t \mapsto \hat{\rho}_t(k)$ is Borel measurable, then

$$\bar{\rho}(k, s) = \int_0^\infty \int_{\mathbb{R}} e^{-st} e^{ikx} \rho_t(dx)\, dt$$

is the FLT of $(\rho_t)_{t>0}$.

Any infinitely divisible distribution is characterized by the Lévy–Khintchine formula. This concept carries over to the FLT setting [17, Lemma 2.1] so that

$$\mathbb{E}[e^{-sD(u)+ikA(u)}] = \exp(-u\psi(k, s)) \tag{2.8}$$

for all $(k, s) \in \mathbb{R} \times \mathbb{R}_+$. We call ψ the Fourier–Laplace symbol of (A, D). Moreover, there exist uniquely determined $(a, b) \in \mathbb{R} \times \mathbb{R}_+$, a positive constant σ^2 and a measure ϕ on $\mathbb{R} \times \mathbb{R}_+ \setminus \{(0, 0)\}$ such that

$$\psi(k, s) = iak + bs + \frac{1}{2}\sigma^2 k^2 + \int_{\mathbb{R} \times \mathbb{R}_+ \setminus \{(0,0)\}} \left(1 - e^{ikx} e^{-st} + \frac{ikx}{1 + x^2}\right) \phi(dx, dt). \tag{2.9}$$

The Lévy measure ϕ is finite outside every neighborhood of the origin and

$$\int_{0 < x^2 + t \leq 1} (x^2 + t)\phi(dx, dt) < \infty.$$

We denote by $\phi_A(dx) = \phi(dx, \mathbb{R}_+)$ the Lévy measure of the Lévy process $\{A(u)\}_{u\geq 0}$. By setting $s = 0$ in the representation (2.8) we see that

$$\int_{\mathbb{R}} e^{ikx} P_{A(u)}(dx) = e^{-u\psi_A(k)} \tag{2.10}$$

so that

$$\psi_A(k) = iak + \frac{1}{2}\sigma^2 k^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{-ikx} + \frac{ikx}{1+x^2}\right) \phi_A(dx) \tag{2.11}$$

is the Fourier symbol of the Lévy process $\{A(u)\}$. Similarly, we let $\phi_D(dt) = \phi(\mathbb{R}, dt)$ denote the Lévy measure of $\{D(u)\}$. By setting $k = 0$ in the representation (2.8) we see that

$$\int_0^\infty e^{-st} P_{D(u)}(dt) = e^{-u\psi_D(s)} \tag{2.12}$$

where

$$\psi_D(s) = \int_0^\infty \left(1 - e^{-sv}\right) \phi_D(dv) \tag{2.13}$$

is the Laplace symbol of the Lévy process $\{D(u)\}$. Note that $\{D(u)\}$ is a stable subordinator with drift term $b = 0$ in (2.9). Since the sample paths of $D(t)$ are càdlàg and strictly increasing with $D(0) = 0$ and $D(t) \to \infty$ as $t \to \infty$, the first passage time process

$$E(t) = \inf\{x : D(x) > t\} \tag{2.14}$$

is well-defined.

Given any $\lambda > 0$ let $L^1_\lambda(\mathbb{R} \times \mathbb{R}_+)$ denote the collection of real-valued measurable functions on $\mathbb{R} \times \mathbb{R}_+$ for which the integral and hence the norm

$$\|f\|_\lambda = \int_0^\infty \int_{\mathbb{R}} e^{-\lambda t} |f(x, t)|\, dx\, dt$$

exists. With this norm, $L^1_\lambda(\mathbb{R} \times \mathbb{R}_+)$ is a Banach space that contains $L^1(\mathbb{R} \times \mathbb{R}_+)$. The symbol $\psi(k, s)$ defines a pseudo-differential operator $\psi(i\partial_x, \partial_t)$ on this space, and the negative generator of the corresponding Feller semigroup, see [18] for more details. Theorem 3.2 in [19] shows that the domain of this operator contains any $f \in L^1_\lambda(\mathbb{R} \times \mathbb{R}_+)$ whose weak first and second order spatial derivatives as well as weak first order time derivatives are in $L^1_\lambda(\mathbb{R} \times \mathbb{R}_+)$, and that in this case we have

$$\begin{aligned}\psi(i\partial_x, \partial_t) f(x, t) &= -a\partial_x f(x, t) - \frac{1}{2}\sigma^2 \partial_x^2 f(x, t) \\ &\quad - \int_{\mathbb{R}\times\mathbb{R}_+\setminus\{(0,0)\}} \left(H(t-u) f(x-y, t-u) - f(x, t) + \frac{y\partial_x f(x, t)}{1+y^2}\right) \phi(dy, du)\end{aligned} \tag{2.15}$$

where $H(t) = I(t \geq 0)$ is the Heaviside step function.

3. Limit theorems

This section derives the long-time scaling limit of the coupled CTRW and OCTRW processes. Recall from Section 2 that $\tilde{B}(c) = B(\tilde{b}(c))$ where $B(c)$ is the norming sequence for the random walk of jumps, and $\tilde{b}(c)$ is the asymptotic inverse of $1/b(c)$, the norming sequence for the random walk of waiting times, in the joint random walk convergence (2.5).

Theorem 3.1. *Suppose (Y_n, J_n) are i.i.d. random vectors on $\mathbb{R} \times \mathbb{R}_+$ such that (2.5) holds. Then*

$$\{\tilde{B}(c) S(N(ct) + 1)\}_{t\geq 0} \Rightarrow \{A(E(t))\}_{t\geq 0} \tag{3.1}$$

as $c \to \infty$ in the J_1 topology on $D([0, \infty), \mathbb{R})$; Also

$$\{\tilde{B}(c) S(N(ct))\}_{t\geq 0} \Rightarrow \{A(E(t)-)\}_{t\geq 0} \tag{3.2}$$

as $c \to \infty$ in the J_1 topology on $D([0, \infty), \mathbb{R})$.

Proof. Note that the CTRW scaling limit in (3.2) has to be interpreted as the right-continuous version of $\{A(E(t)-)\}_{t\geq 0}$ so that its sample paths are proper elements of $D([0,\infty),\mathbb{R})$. The triangular array $(\kappa_{\varepsilon,k},\xi_{\varepsilon,k})$ with $\varepsilon=c^{-1}$, $n_\varepsilon=\tilde{b}(c)$, $\kappa_{\varepsilon,k}=c^{-1}J_k$ and $\xi_{\varepsilon,k}=\tilde{B}(c)Y_k$ has i.i.d. rows for any $\varepsilon>0$. Hence condition $\mathcal{T}_4$ on [20, p. 287] holds. Next define

$$\xi_\varepsilon(t)=\sum_{k=1}^{[n_\varepsilon t]}\xi_{\varepsilon,k}=\tilde{B}(c)S(\tilde{b}(c)t)$$

and

$$\kappa_\varepsilon(t)=\sum_{k=1}^{[n_\varepsilon t]}\kappa_{\varepsilon,k}=c^{-1}T(\tilde{b}(c)t).$$

Since (2.6) holds, condition $\mathcal{A}_{66}$ on [20, p. 288] also holds with $(\kappa_0(t),\xi_0(t))=(D(t),A(t))$. Finally, note that condition $\mathcal{J}_{20}$ on [20, p. 285] holds with $\pi_1(0+)=\phi_D(0,\infty)=\infty$, by the standard convergence criteria for triangular arrays (e.g., see [14, Theorem 3.2.2]). Define the renewal process

$$\begin{aligned}\nu_\varepsilon(t)&=\sup(s:\kappa_\varepsilon(s)\leq t)=\tilde{b}(c)^{-1}\min\left\{n\geq 0:\sum_{k=1}^{n}\kappa_{\varepsilon,k}>t\right\}\\&=\tilde{b}(c)^{-1}\min\left\{n\geq 0:\sum_{k=1}^{n}J_k>ct\right\}=\tilde{b}(c)^{-1}\big(N(ct)+1\big)\end{aligned}$$

and the corresponding limit process

$$\nu_0(t)=\sup\{s\geq 0:\kappa_0(s)\leq t\}=\sup\{s\geq 0:D(s)\leq t\}=\inf\{s\geq 0:D(s)>t\}=E(t).$$

The random walk process subordinated to the renewal process is

$$\begin{aligned}\zeta_\varepsilon(t)&=\xi_\varepsilon(\nu_\varepsilon(t))=\tilde{B}(c)S(\tilde{b}(c)\nu_\varepsilon(t))\\&=\tilde{B}(c)S\big(\tilde{b}(c)\big(\tilde{b}(c)^{-1}(N(ct)+1)\big)\big)=\tilde{B}(c)S(N(ct)+1)\end{aligned}$$

which is the left-hand side of (3.1). Then [20, Theorem 4.5.6] yields

$$\tilde{B}(c)S(N(ct)+1)=\zeta_\varepsilon(t)\to\zeta_0(t)=\xi_0(\nu_0(t))=A(E(t))\tag{3.3}$$

as $c\to\infty$ in the J_1 topology on $D([0,\infty),\mathbb{R})$.

Next we consider the CTRW limit (3.2). Following [20, page 282], we consider the so-called modified renewal process

$$\nu'_\varepsilon(t)=\tilde{b}(c)^{-1}\max\left\{n\geq 0:\sum_{k=1}^{n}\kappa_{\varepsilon,k}\leq t\right\}=\tilde{b}(c)^{-1}N(ct)$$

and

$$\zeta'_\varepsilon(t)=\xi_\varepsilon(\nu'_\varepsilon(t))=\tilde{B}(c)S(N(ct))$$

which is the left-hand side of (3.2). Since [20, Theorem 4.5.6] is an application of [20, Theorem 4.5.1], the remarks on [20, page 282] show that, under the same conditions we have already checked, we also get process convergence

$$\tilde{B}(c)S(N(ct))=\zeta'_\varepsilon(t)\to\zeta'_0(t)=\xi_0(\nu_0(t)-)=A(E(t)-)$$

as $c\to\infty$ in the J_1 topology on $D([0,\infty),\mathbb{R})$. □

Remark 3.2. The CTRW and OCTRW convergence results in Theorem 3.1 can also be obtained from Straka and Henry [21, Theorem 3.6], which yields

$$\{\tilde{B}(c)S(N(ct)+1),b(\tilde{b}(c))T(N(ct)+1)\}_{t\geq 0}\Rightarrow\{A(E(t)),D(E(t))\}_{t\geq 0}\tag{3.4}$$

and

$$\{\tilde{B}(c)S(N(ct)),b(\tilde{b}(c))T(N(ct))\}_{t\geq 0}\Rightarrow\{A(E(t)-),D(E(t)-)\}_{t\geq 0}\tag{3.5}$$

as $c\to\infty$ in the J_1 topology on $D([0,\infty),\mathbb{R})$. The proof of [21, Theorem 3.6] uses a continuous mapping approach. The convergence (3.4) was also proven by Silvestrov and Teugels [22, Theorem 3.2] by arguments similar to Theorem 3.1.

A stochastic process $\{X(t)\}_{t\geq 0}$ is *self-similar* with index H if, for any $r>0$, $\{X(rt)\}=\{r^HX(t)\}$ in the sense of finite-dimensional distributions, e.g., see [23].

Corollary 3.3. *The limit processes $A(E(t))$ and $A(E(t)-)$ in Theorem 3.1 are both self-similar with index β/α.*

Proof. Recall that $\tilde{B}(c)$ varies regularly with index $-\beta/\alpha$, i.e., $\tilde{B}(rc)\tilde{B}(c)^{-1} \to r^{-\beta/\alpha}$ as $c \to \infty$ for every $r > 0$. From (3.1) we get

$$\{\tilde{B}(c)S(N(c \cdot rt) + 1)\}_{t\geq 0} \Rightarrow \{A(E(rt))\}_{t\geq 0}$$

while a continuous mapping argument along with (3.1) yields

$$\{\tilde{B}(c)S(N(crt) + 1)\} = \{\tilde{B}(c)\tilde{B}(cr)^{-1} \cdot \tilde{B}(cr)S(N(crt) + 1)\} \Rightarrow \{r^{\beta/\alpha}A(E(t))\}$$

so that $\{A(E(rt))\}$ and $\{r^{\beta/\alpha}A(E(t))\}$ are identically distributed as elements of $D([0, \infty), \mathbb{R})$. A similar argument using (3.2) shows that $\{A(E(rt)-)\}$ and $\{r^{\beta/\alpha}A(E(t)-)\}$ are identically distributed as elements of $D([0, \infty), \mathbb{R})$. Then we also have equality in the sense of finite dimensional distributions. □

Remark 3.4. Eq. (3.1) corrects the result in [17, Theorem 3.4]. Examples 5.2–5.6 in [17] provide valid governing equations for the CTRW limit process $A(E(t)-)$.

Remark 3.5. It is not hard to extend Theorem 3.1 to the more general case of triangular array convergence. Let $(J_n^{(c)}, Y_n^{(c)})$ be i.i.d. on $\mathbb{R} \times \mathbb{R}_+$ for each $c > 0$ and set

$$T^{(c)}(n) = \sum_{j=1}^{n} J_j^{(c)} \quad \text{and} \quad S^{(c)}(n) = \sum_{i=1}^{n} Y_i^{(c)} \tag{3.6}$$

and let $N^{(c)}(t) = \max\{n \geq 0 : T^{(c)}(n) \leq t\}$. Assume that

$$\{(S^{(c)}(cu), T^{(c)}(cu))\}_{u\geq 0} \Rightarrow \{(A(u), D(u))\}_{u\geq 0} \quad \text{as } c \to \infty \tag{3.7}$$

in the J_1 topology on $D([0, \infty), \mathbb{R} \times \mathbb{R}_+)$, where $\{(A(u), D(u))\}_{u\geq 0}$ is a Lévy process on $\mathbb{R} \times \mathbb{R}_+$ such that $\phi_D(0, \infty) = \infty$ and $b = 0$ in (2.9). Triangular array convergence is useful in applications to finance, because the limit is more flexible. For example, $A(t)$ can be a Brownian motion with drift, or a CGMY (tempered stable) process with finite moments but probability tails that follow a power law at some scale. An explicit triangular array scheme for the CGMY process was developed in [24]. Let $\varepsilon = c^{-1}$, $n_\varepsilon = c$, $\kappa_{\varepsilon,k} = J_k^{(c)}$ and $\xi_{\varepsilon,k} = Y_k^{(c)}$. Then it follows exactly as in the proof of Theorem 3.1 that

$$\begin{aligned} &\{S^{(c)}(N^{(c)}(t) + 1)\}_{t\geq 0} \Rightarrow \{A(E(t))\}_{t\geq 0} \\ &\{S^{(c)}(N^{(c)}(t))\}_{t\geq 0} \Rightarrow \{A(E(t)-)\}_{t\geq 0} \end{aligned} \tag{3.8}$$

as $c \to \infty$ in the J_1 topology on $D([0, \infty), \mathbb{R})$. This corrects certain results in [18]: Theorem 3.6, Corollary 3.8, and the governing equation (4.5) in [18] pertain to the CTRW limit process $A(E(t)-)$.

4. Governing equations

This section develops the governing pseudo-differential equations of the OCTRW limit process $A(E(t))$, and the CTRW limit process $A(E(t)-)$, from Theorem 3.1. Theorem 4.1 shows that the governing equations of the CTRW and OCTRW limits differ only in their initial/boundary conditions. While this may seem like a minor difference, the examples in Section 5 will demonstrate that the effect can be quite dramatic. Recall that the pseudo-differential operator $\psi(i\partial_x, \partial_t)$ was defined in (2.15). Also note that, since the set $\mathbb{R} \times (t, \infty)$ is bounded away from $(0, 0)$, $\phi(dx, (t, \infty))$ is a finite measure on $\mathbb{R}$ for any $t > 0$. Given a weakly measurable family $h(dx, t)$ of bounded measures on $\mathbb{R}$, we will say that a function $f(x, t)$ is a *mild solution* to the pseudo-differential equation $\psi(i\partial_x, \partial_t)f(x, t) = h(dx, t)$ if its FLT solves the corresponding algebraic equation.

Theorem 4.1. *If the OCTRW limit $A(E(t))$ in (3.1) has a Lebesgue density $a(x, t)$, then this density is a mild solution to the governing equation*

$$\psi(i\partial_x, \partial_t)a(x, t) = \phi(dx, (t, \infty)). \tag{4.1}$$

If the CTRW limit $A(E(t)-)$ in (3.2) has a Lebesgue density $c(x, t)$, then this density is a mild solution to the governing equation

$$\psi(i\partial_x, \partial_t)c(x, t) = \delta(x)\phi_D(t, \infty). \tag{4.2}$$

The proof of Theorem 4.1 is based on the following result, which computes the Fourier–Laplace transforms of the CTRW and OCTRW limit processes. Recall that the Fourier symbol $\psi_A(k)$, the Laplace symbol $\psi_D(s)$, and the Fourier–Laplace symbol

$\psi(k,s)$ were defined in Section 2. For any fixed $x \in \mathbb{R}$ define the translation $T_x(y) = y + x$. Define the image measure

$$T_x(\phi)(B,(t,\infty)) = \phi(T_x^{-1}(B),(t,\infty)) = \phi(B - x,(t,\infty))$$

for any Borel set $B \subset \mathbb{R}$. We will also use the notation

$$\hat{P}_Y(k) = \mathbb{E}\left[e^{ikY}\right] \quad k \in \mathbb{R}$$

for the Fourier transform of the distribution of a random variable Y on $\mathbb{R}$,

$$\tilde{P}_J(s) = \mathbb{E}\left[e^{-sJ}\right] \quad s \geq 0$$

for the Laplace transform of a nonnegative random variable J, and

$$\bar{P}_{(Y,J)}(k,s) = \mathbb{E}\left[e^{-sJ+ikY}\right] \quad (k,s) \in \mathbb{R} \times \mathbb{R}_+$$

for the FLT of a random vector (Y, J) on $\mathbb{R} \times \mathbb{R}_+$.

Proposition 4.2. *Assume that (Y_n, J_n) are i.i.d. random vectors on $\mathbb{R} \times \mathbb{R}_+$ such that (2.5) holds. Then*

$$\rho_t(dy) = \int_0^\infty \int_{\mathbb{R}} \int_0^t T_x(\phi)(dy,(t-\tau,\infty)) P_{(A(u),D(u))}(dx, d\tau)\, du \tag{4.3}$$

is the distribution of the OCTRW limit $A(E(t))$ in (3.1), and its FLT is given by

$$\int_0^\infty e^{-st} \hat{P}_{A(E(t))}(k)\, dt = \frac{1}{s} \frac{\psi(k,s) - \psi_A(k)}{\psi(k,s)}. \tag{4.4}$$

Furthermore,

$$\eta_t(dy) = \int_0^\infty \int_0^t \phi_D((t-\tau,\infty)) P_{(A(s),D(s))}(dy, d\tau)\, ds \tag{4.5}$$

is the distribution of the CTRW limit $A(E(t)-)$ in (3.2), and its FLT is given by

$$\int_0^\infty e^{-st} \hat{P}_{A(E(t)-)}(k)\, dt = \frac{1}{s} \frac{\psi_D(s)}{\psi(k,s)}. \tag{4.6}$$

The proof of Proposition 4.2 requires some preliminary lemmas. Recall that (Y_n, J_n) are i.i.d. with (Y, J).

Lemma 4.3.

(a) *For the OCTRW process $S(N(t)+1)$ we have for $s > 0$, $k \in \mathbb{R}$ that*

$$\int_0^\infty e^{-st} \hat{P}_{S(N(t)+1)}(k)\, dt = \frac{1}{s} \frac{\hat{P}_Y(k) - \bar{P}_{(Y,J)}(k,s)}{1 - \bar{P}_{(Y,J)}(k,s)}. \tag{4.7}$$

(b) *For the CTRW process $S(N(t))$ we have for $s > 0$, $k \in \mathbb{R}$ that*

$$\int_0^\infty e^{-st} \hat{P}_{S(N(t))}(k)\, dt = \frac{1}{s} \frac{1 - \tilde{P}_J(s)}{1 - \bar{P}_{(Y,J)}(k,s)}. \tag{4.8}$$

Proof. Observe first that

$$\begin{aligned}\int_0^\infty e^{-st} \int 1_{\{T(n) \leq t\}} e^{ikS(n)}\, dP\, dt &= \int \left(\int_{T(n)}^\infty e^{-st}\, dt \right) e^{ikS(n)}\, dP \\ &= \frac{1}{s} \int e^{-sT(n)+ikS(n)}\, dP = \frac{1}{s} \left(\bar{P}_{(Y,J)}(k,s)\right)^n.\end{aligned} \tag{4.9}$$

Note that $1_{\{N(t)=n\}} = 1_{\{T(n) \leq t\}} - 1_{\{T(n+1) \leq t\}}$ and hence

$$\begin{aligned}\hat{P}_{S(N(t)+1)}(k) &= \int e^{ikS(N(t)+1)}\, dP \\ &= \sum_{n=0}^\infty \int 1_{\{N(t)=n\}} e^{ikS(n+1)}\, dP \\ &= \sum_{n=0}^\infty \left[\int 1_{\{T(n) \leq t\}} e^{ikS(n+1)}\, dP - \int 1_{\{T(n+1) \leq t\}} e^{ikS(n+1)}\, dP \right].\end{aligned}$$

Therefore we have in view of (4.9) and independence that

$$\begin{aligned}
\int_0^\infty e^{-st}\hat{P}_{S(N(t)+1)}(k)\,dt &= \sum_{n=0}^\infty\left[\int_0^\infty e^{-st}\int 1_{\{T(n)\le t\}}e^{ikS(n+1)}\,dP\,dt - \int_0^\infty e^{-st}\int 1_{\{T(n+1)\le t\}}e^{ikS(n+1)}\,dP\,dt\right]\\
&= \sum_{n=0}^\infty\left[\int_0^\infty e^{-st}\int 1_{\{T(n)\le t\}}e^{ikS(n)}e^{ikY_{n+1}}\,dP\,dt - \frac{1}{s}\big(\bar{P}_{(Y,J)}(k,s)\big)^{n+1}\right]\\
&= \sum_{n=0}^\infty\left[\frac{1}{s}\big(\bar{P}_{(Y,J)}(k,s)\big)^{n}\hat{P}_Y(k) - \frac{1}{s}\big(\bar{P}_{(Y,J)}(k,s)\big)^{n+1}\right]\\
&= \frac{1}{s}\big(\hat{P}_Y(k)-\bar{P}_{(Y,J)}(k,s)\big)\sum_{n=0}^\infty\big(\bar{P}_{(Y,J)}(k,s)\big)^{n} = \frac{1}{s}\frac{\hat{P}_Y(k)-\bar{P}_{(Y,J)}(k,s)}{1-\bar{P}_{(Y,J)}(k,s)}
\end{aligned}$$

which proves (4.7).

For the proof of (4.8) note first that

$$\begin{aligned}
\int 1_{\{T(n+1)\le t\}}e^{ikS(n)}\,dP &= \int 1_{\{T(n)+J_{n+1}\le t\}}e^{ikS(n)}\,dP\\
&= \int\int_0^t 1_{\{T(n)\le t-\tau\}}e^{ikS(n)}\,dP_J(\tau)\,dP.
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_0^\infty e^{-st}\int 1_{\{T(n+1)\le t\}}e^{ikS(n)}\,dP\,dt &= \int_0^\infty e^{-st}\int\int_0^t 1_{\{T(n)\le t-\tau\}}e^{ikS(n)}\,dP_J(\tau)\,dP\,dt\\
&= \int e^{ikS(n)}\int_0^\infty e^{-st}\int_0^t 1_{\{T(n)\le t-\tau\}}\,dP_J(\tau)\,dt\,dP\\
&= \int e^{ikS(n)}\int_0^\infty\int_\tau^\infty e^{-st}1_{\{T(n)\le t-\tau\}}\,dt\,dP_J(\tau)\,dP\\
&= \int e^{ikS(n)}\int_0^\infty\int_{T(n)+\tau}^\infty e^{-st}\,dt\,dP_J(\tau)\,dP\\
&= \frac{1}{s}\int e^{-sT(n)+ikS(n)}\,dP\int_0^\infty e^{-s\tau}\,dP_J(\tau)\\
&= \frac{1}{s}\tilde{P}_J(s)\big(\bar{P}_{(Y,J)}(k,s)\big)^{n}.
\end{aligned}$$

In view of (4.9) we obtain

$$\begin{aligned}
\int_0^\infty e^{-st}\hat{P}_{S(N(t))}(k)\,dt &= \sum_{n=0}^\infty\left[\int_0^\infty e^{-st}\int 1_{\{T(n)\le t\}}e^{ikS(n)}\,dP\,dt - \int_0^\infty e^{-st}\int 1_{\{T(n+1)\le t\}}e^{ikS(n)}\,dP\,dt\right]\\
&= \frac{1}{s}\sum_{n=0}^\infty\Big[\big(\bar{P}_{(Y,J)}(k,s)\big)^{n} - \tilde{P}_J(s)\big(\bar{P}_{(Y,J)}(k,s)\big)^{n}\Big] = \frac{1}{s}\frac{1-\tilde{P}_J(s)}{1-\bar{P}_{(Y,J)}(k,s)}
\end{aligned}$$

and the proof is complete. □

Lemma 4.4.

(a) *For the OCTRW process $S(N(t)+1)$ we have for all $k\in\mathbb{R}$ and $s>0$*

$$\int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct)+1)}(k)\,dt \to \frac{1}{s}\frac{\psi(k,s)-\psi_A(k)}{\psi(k,s)}\quad \text{as } c\to\infty. \tag{4.10}$$

(b) *For the CTRW process $S(N(t))$ we have for all $k\in\mathbb{R}$ and $s>0$*

$$\int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct))}(k)\,dt \to \frac{1}{s}\frac{\psi_D(s)}{\psi(k,s)}\quad \text{as } c\to\infty. \tag{4.11}$$

Proof. Recall from Section 2 that $\tilde{B}(c) = B(\tilde{b}(c))$ is a regularly varying function with index $-\beta/\alpha$. From (2.6) we get

$$\left(\tilde{B}(c)S(\tilde{b}(c)), c^{-1}T(\tilde{b}(c))\right) \Rightarrow (A, D) \quad \text{as } c \to \infty.$$

By the continuity theorem for the FLT for probability distributions, this is equivalent to

$$\left(\bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)\right)^{\tilde{b}(c)} \to \bar{P}_{(A,D)}(k, s) = e^{-\psi(k,s)} \quad \text{as } c \to \infty \tag{4.12}$$

for all $k \in \mathbb{R}$ and $s \ge 0$. Take logs and apply a Taylor expansion to see that (4.12) is equivalent to

$$\tilde{b}(c)\left(1 - \bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)\right) \to \psi(k, s) \quad \text{as } c \to \infty. \tag{4.13}$$

Using $\bar{P}_{(Y,J)}(0, s) = \tilde{P}_J(s)$ and $\bar{P}_{(Y,J)}(k, 0) = \hat{P}_Y(k)$ as well as $\psi(k, 0) = \psi_A(k)$ in (2.11) and $\psi(0, s) = \psi_D(s)$ in (2.13), we get from (4.13)

$$\begin{aligned} &\tilde{b}(c)\left(1 - \hat{P}_Y(\tilde{B}(c)k)\right) \to \psi_A(k) \\ &\tilde{b}(c)\left(1 - \tilde{P}_J(c^{-1}s)\right) \to \psi_D(s) \end{aligned} \tag{4.14}$$

as $c \to \infty$.

Proof of (a). In view of Lemma 4.3(a) we get by a simple change of variables for all $k \in \mathbb{R}$ and $s > 0$

$$\begin{aligned} \int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct)+1)}(k)\, dt &= c^{-1}\int_0^\infty e^{-(sc^{-1})t}\hat{P}_{S(N(t)+1)}(\tilde{B}(c)k)\, dt \\ &= \frac{1}{s}\frac{\hat{P}_Y(\tilde{B}(c)k) - \bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)}{1 - \bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)} \\ &= \frac{1}{s}\frac{\tilde{b}(c)\left(\hat{P}_Y(\tilde{B}(c)k) - 1\right) + \tilde{b}(c)\left(1 - \bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)\right)}{\tilde{b}(c)\left(1 - \bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)\right)} \\ &\to \frac{1}{s}\frac{-\psi_A(k) + \psi(k, s)}{\psi(k, s)} \end{aligned}$$

as $c \to \infty$, using (4.13) and (4.14). □

Proof of (b). Similarly, we get from Lemma 4.3(b) that for all $k \in \mathbb{R}$ and $s > 0$

$$\begin{aligned} \int_0^\infty e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct))}(k)\, dt &= c^{-1}\int_0^\infty e^{-(sc^{-1})t}\hat{P}_{S(N(t))}(\tilde{B}(c)k)\, dt \\ &= \frac{1}{s}\frac{1 - \tilde{P}_J(c^{-1}s)}{1 - \bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)} \\ &= \frac{1}{s}\frac{\tilde{b}(c)\left(1 - \tilde{P}_J(c^{-1}s)\right)}{\tilde{b}(c)\left(1 - \bar{P}_{(Y,J)}(\tilde{B}(c)k, c^{-1}s)\right)} \to \frac{1}{s}\frac{\psi_D(s)}{\psi(k, s)} \end{aligned}$$

as $c \to \infty$, using (4.13) and (4.14) again. The proof is complete. □

Remark 4.5. In the uncoupled case where A, D are independent, we have $\psi(k, s) = \psi_A(k) + \psi_D(s)$ and hence the limits in (4.10) and (4.11) are equal. Hence it follows from Lemma 4.4 that the FLT limits of $\tilde{B}(c)S(N(ct) + 1)$ and $\tilde{B}(c)S(N(ct))$ are equal if and only if A and D are independent.

Lemma 4.6. *Let $(\rho_t)_{t>0}$ and $(\eta_t)_{t>0}$ be two families of probability measures on $\mathbb{R}$ such that $t \mapsto \rho_t$ and $t \mapsto \eta_t$ are weakly right-continuous. If*

$$\int_0^\infty e^{-st}\hat{\rho}_t(k)\, dt = \int_0^\infty e^{-st}\hat{\eta}_t(k)\, dt$$

for all $s > 0$ and $k \in \mathbb{R}$, then $\rho_t = \eta_t$ for all $t > 0$.

Proof. For any fixed $k \in \mathbb{R}$, the uniqueness theorem for Laplace transforms implies that $\hat{\rho}_t(k) = \hat{\eta}_t(k)$ for Lebesgue-almost all $t > 0$. By the continuity theorem for the Fourier transform, both $t \mapsto \hat{\rho}_t(k)$ and $t \mapsto \hat{\eta}_t(k)$ are right-continuous. It follows that $\hat{\rho}_t(k) = \hat{\eta}_t(k)$ for all $t > 0$. Since $k \in \mathbb{R}$ is arbitrary, the uniqueness theorem of the Fourier transform implies $\rho_t = \eta_t$ for all $t > 0$, and the proof is complete. □

Lemma 4.7. *For any $t>0$, $k\in\mathbb{R}$ and $s>0$ we have*

$$\int_0^\infty e^{-st}\int_{\mathbb{R}} e^{ikx}\phi(dx,(t,\infty))\,dt=\frac{1}{s}\big(\psi(k,s)-\psi_A(k)\big) \tag{4.15}$$

where $\psi(k,s)$ is the log-FLT of (A,D) as in (2.8).

Proof. Since $\phi(dx,(t,\infty))$ is a finite measure on $\mathbb{R}$, the Fourier-transform of $\phi(dx,(t,\infty))$ is well defined for any $t>0$. Moreover

$$\left|\int_{\mathbb{R}} e^{ikx}\phi(dx,(t,\infty))\right|\le\phi(\mathbb{R},(t,\infty))=\phi_D(t,\infty)$$

and by [18, Eq. (3.12)] we know that

$$\int_0^\infty e^{-st}\phi_D(t,\infty)\,dt=\frac{1}{s}\psi_D(s)$$

for $s>0$. Therefore, we can apply Fubini's theorem to get

$$\begin{aligned}\int_0^\infty e^{-st}\int_{\mathbb{R}} e^{ikx}\phi(dx,(t,\infty))\,dt&=\int_{\mathbb{R}}\int_0^\infty e^{ikx}\left(\int_0^\infty 1_{(t,\infty)}(u)e^{-st}\,dt\right)\phi(dx,du)\\&=\frac{1}{s}\int_{\mathbb{R}}\int_0^\infty(1-e^{-su})e^{ikx}\,\phi(dx,du)\\&=\frac{1}{s}\int_{\mathbb{R}}\int_0^\infty\left[\left(e^{ikx}-1-\frac{ikx}{1+x^2}\right)+\left(1-e^{ikx}e^{-su}+\frac{ikx}{1+x^2}\right)\right]\phi(dx,du)\\&=\frac{1}{s}\big(-\psi_A(k)+\psi(k,s)\big)\end{aligned}$$

and the proof is complete. □

Lemma 4.8. *Eq. (4.3) defines a probability measure $\rho_t(dy)$ on $\mathbb{R}$ such that*

$$\int_0^\infty e^{-st}\hat\rho_t(k)\,dt=\frac{1}{s}\frac{\psi(k,s)-\psi_A(k)}{\psi(k,s)}$$

for any $s>0$ and $x\in\mathbb{R}$. Moreover, the mapping $t\mapsto\rho_t$ is right continuous with respect to weak convergence.

Proof. Observe first that $T_x(\phi)(\mathbb{R},(t-\tau,\infty))=\phi_D(t-\tau,\infty)$ and hence

$$\begin{aligned}\rho_t(\mathbb{R})&=\int_0^\infty\int_0^t\phi_D(t-\tau,\infty)\,P_{(A(u),D(u))}(\mathbb{R},d\tau)\,du\\&=\int_0^\infty\int_0^t\phi_D(t-\tau,\infty)\,P_{D(u)}(d\tau)\,du=1\end{aligned}$$

by [18, Theorem 3.1], so that ρ_t is a probability measure on $\mathbb{R}$ for any $t>0$. Observe that for $k\in\mathbb{R}$ we have using Fubini that

$$\hat\rho_t(k)=\int_{u=0}^\infty\int_{x\in\mathbb{R}}\int_{\tau=0}^t\int_{y\in\mathbb{R}}e^{ik(x+y)}\phi(dy,(t-\tau,\infty))\,P_{(A(u),D(u))}(dx,d\tau)\,du. \tag{4.16}$$

Then, by Fubini's theorem we get for any $s>0$ and $k\in\mathbb{R}$, using (4.15) that

$$\begin{aligned}\int_0^\infty e^{-st}\hat\rho_t(k)\,dt&=\int_{u=0}^\infty\int_{x\in\mathbb{R}}\int_{\tau=0}^\infty\int_{t=\tau}^\infty\int_{y\in\mathbb{R}}e^{-st}e^{ik(x+y)}\phi(dy,(t-\tau,\infty))\,dt\,P_{(A(u),D(u))}(dx,d\tau)\,du\\&=\int_{u=0}^\infty\int_{x\in\mathbb{R}}\int_{\tau=0}^\infty\left(\int_{v=0}^\infty\int_{y\in\mathbb{R}}e^{-s(v+\tau)}e^{ik(x+y)}\phi(dy,(v,\infty))\,dv\right)P_{(A(u),D(u))}(dx,d\tau)\,du\\&=\frac{1}{s}\big(\psi(k,s)-\psi_A(k)\big)\int_{u=0}^\infty\left(\int_{x\in\mathbb{R}}\int_{\tau=0}^\infty e^{-s\tau}e^{ikx}P_{(A(u),D(u))}(dx,d\tau)\right)du\\&=\frac{1}{s}\big(\psi(k,s)-\psi_A(k)\big)\int_0^\infty e^{-u\psi(k,s)}\,du=\frac{1}{s}\frac{\psi(k,s)-\psi_A(k)}{\psi(k,s)}.\end{aligned}$$

Note that the last equality is justified since $\operatorname{Re}\psi(k,s)\ge\psi_D(s)>0$, as in [18, p. 1619].

In order to show that $t \mapsto \rho_t$ is weakly right-continuous, in view of the continuity theorem for the Fourier transform, it is enough to show that for any fixed $k \in \mathbb{R}$ the function $t \mapsto \hat{\rho}_t(k)$ is right-continuous. Using (4.16) we get for any $t > 0$ and $h > 0$ that

$$\begin{aligned}
\hat{\rho}_t(k) - \hat{\rho}_{t+h}(k) &= \int_{u=0}^{\infty}\int_{x\in\mathbb{R}}\int_{\tau=0}^{t}\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t-\tau,\infty))\,P_{(A(u),D(u))}(dx,d\tau)\,du \\
&\quad - \int_{u=0}^{\infty}\int_{x\in\mathbb{R}}\int_{\tau=0}^{t+h}\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t+h-\tau,\infty))\,P_{(A(u),D(u))}(dx,d\tau)\,du \\
&= \int_{u=0}^{\infty}\int_{x\in\mathbb{R}}\int_{\tau=0}^{t}\left(\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t-\tau,\infty))\right. \\
&\quad \left. - \int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t+h-\tau,\infty))\right)P_{(A(u),D(u))}(dx,d\tau)\,du \\
&\quad - \int_{u=0}^{\infty}\int_{x\in\mathbb{R}}\int_{\tau=t}^{t+h}\int_{y\in\mathbb{R}} e^{ik(x+y)}\phi(dy,(t+h-\tau,\infty))\,P_{(A(u),D(u))}(dx,d\tau)\,du \\
&= I_h - J_h.
\end{aligned}$$

Then we get

$$\begin{aligned}
|I_h| &\le \int_{u=0}^{\infty}\int_{\tau=0}^{t}\big[\phi(\mathbb{R},(t-\tau,\infty)) - \phi(\mathbb{R},(t+h-\tau,\infty))\big]\,P_{(A(u),D(u))}(\mathbb{R},d\tau)\,du \\
&= \int_0^{\infty}\int_0^{t}\big[\phi_D(t-\tau,\infty) - \phi_D(t+h-\tau,\infty)\big]\,P_{D(u)}(d\tau)\,du \\
&\to 0
\end{aligned}$$

as $h \downarrow 0$ by a dominated convergence argument along with [18, Eq. (3.1)], as in [18, p. 1625]. Moreover

$$\begin{aligned}
|J_h| &\le \int_0^{\infty}\int_t^{t+h}\phi(\mathbb{R},(t+h-\tau,\infty))\,P_{(A(u),D(u))}(\mathbb{R},d\tau)\,du \\
&= \int_0^{\infty}\int_t^{t+h}\phi_D(t+h-\tau,\infty)\,P_{D(u)}(d\tau)\,du \\
&\to 0
\end{aligned}$$

as $h \downarrow 0$ using some results in [25], as in [18, pp. 1615–1616]. This concludes the proof. □

Remark 4.9. Although it is not required for the proof of Proposition 4.2, it is also true that the distribution ρ_t of the OCTRW limit process $A(E(t))$ is weakly left-continuous, thus it is weakly continuous. The proof is similar to Lemma 4.8.

Proof of Proposition 4.2. Lemma 4.8 shows that $\rho_t(dy)$ is right-continuous with FLT

$$\frac{1}{s}\,\frac{\psi(k,s) - \psi_A(k)}{\psi(k,s)}. \tag{4.17}$$

Theorem 3.1 shows that $\tilde{B}(c)S(N(ct)+1)$ converges in J_1 to $A(E(t))$, and Lemma 4.4 shows that the FLT of $\tilde{B}(c)S(N(ct)+1)$ converges to the same limit (4.17). Note that J_1 convergence implies convergence in distribution on the set of all points of stochastic continuity of the limit process, e.g. see [20, p. 44]. Moreover, all but countably many points of a càdlàg process are points of stochastic continuity, e.g. see [20, Lemma 1.6.2]. Then

$$P_{\tilde{B}(c)S(N(ct)+1)}(dx) \Rightarrow P_{A(E(t))}(dx)$$

as $c \to \infty$ for all but countably many $t > 0$. Then the continuity theorem for the Fourier transform yields

$$\hat{P}_{\tilde{B}(c)S(N(ct)+1)}(k) \to \hat{P}_{A(E(t))}(k)$$

as $c \to \infty$ for all $k \in \mathbb{R}$, for dt-almost every $t > 0$. Then we have for each $k \in \mathbb{R}$ that

$$\int_0^{\infty} e^{-st}\hat{P}_{\tilde{B}(c)S(N(ct)+1)}(k)\,dt \to \int_0^{\infty} e^{-st}\hat{P}_{A(E(t))}(k)\,dt$$

as $c \to \infty$, and this together with (4.10) shows that the FLT of $A(E(t))$ equals (4.17). Since $A(t)$ is càdlàg and $E(t)$ is continuous and nondecreasing, $A(E(t))$ is a càdlàg process. Then it is right-continuous almost surely, and hence it is also right-continuous in distribution. Then Lemma 4.6 implies that $\rho_t(dy)$ equals the distribution of $A(E(t))$, which finishes the proof of (a). Part (b) follows from [18, Theorem 3.6] and Remark 3.5. The arguments are similar. □

Proof of Theorem 4.1. If the OCTRW limit $A(E(t))$ in (3.1) has a density $a(x, t)$, then it follows from Proposition 4.2 and Lemma 4.7 that

$$\bar{a}(k, s) = \frac{1}{s}\frac{\psi(k, s) - \psi_A(k)}{\psi(k, s)}.$$

Rewrite in the form

$$\psi(k, s)\bar{a}(k, s) = \frac{\psi(k, s) - \psi_A(k)}{s}$$

and invert the FLT using Lemma 4.7 to see that (4.1) holds. If the CTRW limit $A(E(t)-)$ in (3.2) has a density $c(x, t)$, then it follows from [18, Eq. (4.5)] and Remark 3.5 that (4.2) holds, with a different initial/boundary condition on the right-hand side. □

Remark 4.10. In order to avoid distributions in the OCTRW limit governing equation (4.1), one can impose a smooth initial condition as in [26]. Suppose that X_0 is a random variable with C^∞ density $p(x)$, independent of the process $(A(t), D(t))$. Physically, the random variable X_0 represents the particle position at time $t = 0$. Then the OCTRW limit $A(E(t)) + X_0$ has a density $a(x, t) = \int p(x - y)\rho_t(dy)$ with Fourier transform $\hat{a}(k, t) = \hat{\rho}_t(k)\hat{p}(k)$ and FLT

$$\bar{a}(k, s) = \frac{s^{-1}[\psi(k, s) - \psi_A(k)]\hat{p}(k)}{\psi(k, s)}. \tag{4.18}$$

Lemma 4.7 shows that the Fourier transform $\hat{q}(k, t) = \int e^{ikx}\phi(dx, (t, \infty))$ exists for all $t > 0$, and that the Laplace transform of $\hat{q}(k, t)$ is given by (4.15). It follows easily that the FLT of $\int p(x - y)\phi(dy, (t, \infty))$ is given by the numerator in (4.18). Inverting the FLT in (4.18) reveals the governing equation

$$\psi(i\partial_x, \partial_t)a(x, t) = \int p(x - y)\phi(dy, (t, \infty)). \tag{4.19}$$

Using the same smooth initial condition for the CTRW limit is equivalent to replacing $\delta(x)$ by $p(x)$ in the governing equation (4.2).

5. Examples

This section provides several concrete examples of coupled CTRW and OCTRW convergence, and computes and solves the corresponding governing equations.

Example 5.1. If Y_n and J_n are independent, then so are $A(t)$ and $D(t)$. The FL-symbol is $\psi(k, s) = \psi_A(k) + \psi_D(s)$, and $\phi(dx, (t, \infty)) = \varepsilon_0(dx)\phi_D(t, \infty)$, where ε_0 is the point mass at zero. Suppose that the stable Lévy motion $A(t)$ is totally positively skewed with Fourier symbol $\psi_A(k) = -b(-ik)^\alpha$ for some $0 < \alpha \le 2, \alpha \ne 1$. Suppose that J_n belongs to the domain of attraction of a standard β-stable subordinator D with Laplace symbol

$$\psi_D(s) = s^\beta = \int_0^\infty \left(1 - e^{-su}\right)\phi_D(du). \tag{5.1}$$

A calculation similar to [14, Lemma 7.3.7] shows that

$$\phi_D(t, \infty) = \frac{t^{-\beta}}{\Gamma(1 - \beta)}. \tag{5.2}$$

Since $\delta(x) = \varepsilon_0(dx)$, the OCTRW limit governing equation (4.1) reduces to

$$\partial_t^\beta a_1(x, t) = b\partial_x^\alpha a_1(x, t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1 - \beta)} \tag{5.3}$$

where $b < 0$ if $0 < \alpha < 1$ and $b > 0$ for $1 < \alpha \le 2$. In this case, the CTRW limit Eq. (4.2) reduces to the same form. In fact, the limit processes $A(E(t))$ and $A(E(t)-)$ in Theorem 3.1 are the same in this case, since $A(t)$ and $D(t)$ have (almost surely) no simultaneous jumps. The stable subordinator D has a smooth density $g_\beta(u)$ supported on $u > 0$, and the stable Lévy motion $A(t)$ has a smooth density $p(x, t)$ for all $t > 0$. Using the self-similarity of D, a simple conditioning argument shows that $A(E(t))$ has density

$$a_1(x, t) = \int_0^\infty p(x, (t/s)^\beta)g_\beta(s)ds = \frac{t}{\beta}\int_0^\infty p(x, u)g_\beta(tu^{-1/\beta})u^{-1/\beta-1}du, \tag{5.4}$$

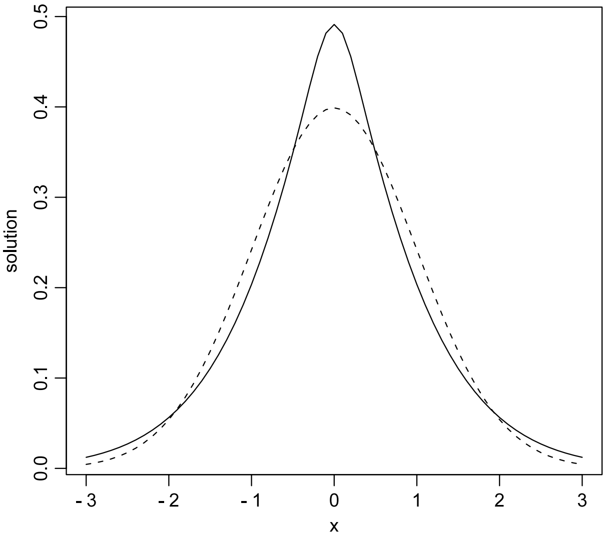

Fig. 1. Solution $a_1(x, t)$ to the uncoupled OCTRW limit Eq. (5.3) with $t = 1.0$, $\alpha = 2$, and $b = 1$ in the case $\beta = 0.6$ (solid line), compared with the solution to (5.3) with $t = 1.0$, $\alpha = 2$, and $b = 1$ in the traditional diffusion case $\beta = 1$ (dashed line). In the uncoupled case, the CTRW and OCTRW are governed by the same equation.

which solves the uncoupled governing equation (5.3). See [26,15] for further details. Eq. (5.3) is called the space–time fractional diffusion equation. It has been used frequently in physics, finance, and hydrology to model anomalous diffusion [2,3,27,28,9,29]. Fig. 1 plots the solution (5.3) at time $t = 1.0$ in the case $\alpha = 2$ and $\beta = 0.6$ with $b = 1.0$, together with the corresponding normal density that solves the same equation with $\beta = 1$. When $\alpha = 2$ and $\beta = 1$, (5.3) is the traditional diffusion equation, and the boundary term $\delta(x)t^{-\beta}/\Gamma(1-\beta)$ reduces to the point source initial condition $\delta(x)\delta(t)$. The solution to the time-fractional diffusion equation has a sharper peak, and broader tails.

The remaining examples are coupled. Suppose that J_n are i.i.d. with D, a standard β-stable subordinator with Lévy measure (5.2). For any probability measure ω on $\mathbb{R}$ and any $p > \beta/2$, suppose that the conditional distribution of Y_n given $J_n = t$ is $t^p\omega$. Then [17, Theorem 2.2] shows that (2.5) holds, the Lévy measure of (A, D) is

$$\phi(dy, dt) = t^p\omega(dy)\phi_D(dt), \tag{5.5}$$

and furthermore, every possible non-normal coupled limit in (2.5) has a Lévy measure of this form. In this case, A is stable with index $\alpha = \beta/p$.

Example 5.2. Suppose $Y_n = J_n$, as in [30]. Take J_n i.i.d. with D, a standard β-stable subordinator. From (5.5) with $p = 1$ and $\omega = \varepsilon_1$, we see that the Lévy measure (jump intensity)

$$\phi(dy, dt) = \varepsilon_t(dy)\phi_D(dt) \tag{5.6}$$

of (A, D) is concentrated on the line $y = t$. Zolotarev [31, Lemma 2.2.1] shows that $\mathbb{E}[e^{ikD}]$ has a unique analytic extension to the complex plane with a branch cut along the ray $\arg(k) = -3\pi/4$, hence $\psi_A(k) = \psi_D(-ik)$. Then an easy computation using (5.1) shows that $\psi(k, s) = (s - ik)^\beta$ where $b = 0$, $\sigma^2 = 0$, and $a = -\int t(1+t^2)^{-1}\phi_D(dt)$ in (2.9). Since $A = D$, the joint distribution of $(A(s), D(s))$ is given by

$$P_{(A(s),D(s))}(dx, du) = \varepsilon_u(dx)P_{D(s)}(du). \tag{5.7}$$

Proposition 4.2(b) shows that the CTRW limit $A(E(t)-) = D(E(t)-)$ in (3.2) has FLT

$$\int_0^\infty e^{-st}\hat{\eta}_t(k)\,dt = \frac{1}{s}\frac{\psi_D(s)}{\psi(k, s)} = \frac{s^{\beta-1}}{(s-ik)^\beta}. \tag{5.8}$$

Following [17, Example 5.4] we can invert the FLT in (5.8) to see that the CTRW limit distribution $\eta_t(dx)$ has a Lebesgue density

$$c_2(x, t) = \frac{x^{\beta-1}(t-x)^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)}, \quad 0 < x < t. \tag{5.9}$$

Formula (5.9) is the density of $t\mathcal{B}$, where $\mathcal{B}$ has a beta distribution with parameters β and $1-\beta$. It solves the coupled governing equation (4.2), which can be written in this case as

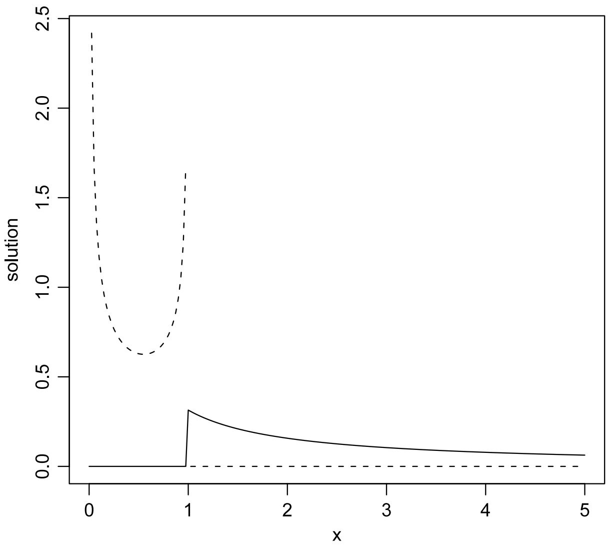

Fig. 2. Solution $a_2(x,t)$ to the coupled OCTRW limit Eq. (5.13) at $t = 1.0$ in the case $\beta = 0.45$ (solid line), compared with the solution $c_2(x,t)$ to the coupled CTRW limit Eq. (5.10) with $t = 1.0$ and $\beta = 0.45$ (dashed line).

$$\left(\partial_t + \partial_x\right)^\beta c_2(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{5.10}$$

with a coupled space–time fractional derivative operator on the left-hand side. It is also possible to derive (5.9) from the general formula (4.5) for the CTRW limit distribution $\eta_t(dx)$.

It follows from (4.4) that the OCTRW limit $A(E(t)) = D(E(t))$ in (3.1) has FLT

$$\int_0^\infty e^{-st}\hat{\rho}_t(k)\,dt = \frac{1}{s}\frac{\psi(k,s)-\psi_A(k)}{\psi(k,s)} = \frac{1}{s}\frac{(s-ik)^\beta-(-ik)^\beta}{(s-ik)^\beta}. \tag{5.11}$$

A straightforward but lengthy computation using (4.3) shows that the OCTRW limit density is

$$a_2(x,t) = \frac{x^{-1}}{\Gamma(\beta)\Gamma(1-\beta)}\left(\frac{t}{x-t}\right)^\beta, \quad x > t. \tag{5.12}$$

Formula (5.12) is the density of $t/\mathcal{B}$, where $\mathcal{B}$ has a beta distribution with parameters β and $1-\beta$. The OCTRW limit density (5.12) solves the governing equation

$$\left(\partial_t + \partial_x\right)^\beta a_2(x,t) = \frac{1}{\Gamma(1-\beta)}\int_t^\infty \delta(x-u)\beta u^{-\beta-1}\,du \tag{5.13}$$

using generalized function notation. It follows from (5.12) that

$$a_2(x,t) \sim \frac{t^\beta}{\Gamma(\beta)\Gamma(1-\beta)}x^{-1-\beta} \quad \text{as } x\to\infty.$$

Fig. 2 compares solutions to the coupled OCTRW limit Eq. (5.13) and coupled CTRW limit Eq. (5.10). Note the striking difference between the CTRW and OCTRW limits in this case: The CTRW limit density (5.9) is supported on $0 < x < t$, so it has moments of all orders. The OCTRW limit density (5.12) is supported on $x > t$, and its moments of order $>\beta$ all diverge.

Example 5.3. Suppose D is a stable subordinator with $\mathbb{E}(e^{-sD}) = e^{-s^\beta}$, and the conditional distribution of Y given $D = t$ is normal with mean zero and variance $2t$, as in [1]. Then

$$\mathbb{E}(e^{ikY}) = \mathbb{E}(\mathbb{E}(e^{ikY}|D)) = \mathbb{E}(e^{-k^2D}) = e^{-|k|^{2\beta}}$$

so that Y is symmetric stable with index $\alpha = 2\beta$. If we take (Y_n, J_n) i.i.d. with (Y, D), then (2.5) holds, and it follows from (5.5) that the operator stable limit (A, D) has Lévy measure

$$\phi(dx,dt) = t^{1/2}\omega(dx)\,\phi_D(dt) \tag{5.14}$$

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where ω is a normal distribution with mean zero and variance 2. Take $a = b = \sigma^2 = 0$ in (2.9) to see that

$$\psi(k,s) = \int_0^\infty \int_{-\infty}^\infty \left(1 - e^{ikx}e^{-st} + \frac{ikx}{1+x^2}\right) \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) dx\phi_D(dt)$$
$$= \int_0^\infty \left(1 - e^{-t(s+k^2)}\right) \phi_D(dt) = (s+k^2)^\beta$$

using (5.1). The CTRW limit has FLT

$$\int_0^\infty e^{-st} \hat{P}_{A(E(t)-)}(k)\, dt = \frac{s^{\beta-1}}{(s+k^2)^\beta}. \tag{5.15}$$

Inverting the FLT as in [17, Example 5.2] shows that the CTRW limit $A(E(t)-)$ has Lebesgue density

$$c_3(x,t) = \int_0^t \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{x^2}{4u}\right) c_2(u,t)\, du. \tag{5.16}$$

This density solves the governing equation

$$\left(\partial_t - \partial_x^2\right)^\beta c_3(x,t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{5.17}$$

The OCTRW limit $A(E(t))$ in (3.1) has FLT

$$\int_0^\infty e^{-st} \hat{\rho}_t(k)\, dt = \frac{1}{s} \frac{(s+k^2)^\beta - |k|^{2\beta}}{(s+k^2)^\beta}.$$

The density of (A, D) is given by $p(z,u) = (4\pi u)^{-1/2} \exp(-z^2/(4u)) g_\beta(u)$, where g_β is the density of D. A computation using the self-similarity of $D(s)$ shows that the OCTRW limit density is

$$a_3(x,t) = \int_t^\infty \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{x^2}{4u}\right) a_2(u,t)\, du, \tag{5.18}$$

which solves the governing equation

$$\left(\partial_t - \partial_x^2\right)^\beta a_3(x,t) = \frac{1}{\Gamma(1-\beta)} \int_t^\infty \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{x^2}{4u}\right) \beta u^{-\beta-1}\, du. \tag{5.19}$$

Fig. 3 compares solutions to the coupled OCTRW limit Eq. (5.19) and the coupled CTRW limit Eq. (5.17). It follows easily from (5.16) that the CTRW limit density $c_3(x,t)$ has a finite second moment. A computation using (5.18) shows that $P\{|A(E(t))| > r\}$ varies regularly with index -2β, so that the second moment of $a_3(x,t)$ is infinite. This is reflected in the heavier tails of $a_3(x,t)$ in Fig. 3. Corollary 3.3 shows that both $A(E(t))$ and $A(E(t)-)$ are self-similar with scaling index $\beta/\alpha = 1/2$. Hence, this example provides two very different models for anomalous diffusion that spread at the same rate as a Brownian motion.

Example 5.4. Suppose $X(t)$ is any Lévy process, and that $X(1)$ has distribution ω. Suppose $D(t)$ is a β-stable subordinator with $E[e^{-sD(1)}] = e^{-s^\beta}$, independent of $X(t)$. Define a triangular array with i.i.d. rows such that

$$Y_i^{(c)} \stackrel{d}{=} X(D(c^{-1})) \quad \text{and} \quad J_i^{(c)} \stackrel{d}{=} D(c^{-1}).$$

It is easy to see that

$$(S^{(c)}(ct), T^{(c)}(ct)) \Rightarrow (A(t), D(t))$$

where $A(t) = X(D(t))$. Since $X(t)$ and $D(t)$ are independent, a simple conditioning argument yields

$$P_{A(E(t)-)}(dx) = \int_0^\infty \omega^u(dx)\, P_{D(E(t)-)}(du) = \int_0^\infty \omega^u(dx) c_2(u,t)\, du$$

as well as

$$P_{A(E(t))}(dx) = \int_0^\infty \omega^u(dx)\, P_{D(E(t))}(du) = \int_0^\infty \omega^u(dx) a_2(u,t)\, du.$$

Let $D = D(1)$, $A = A(1)$, and write $\mathbb{E}(e^{ikX(t)}) = e^{-t\psi_0(k)}$. Then

$$\mathbb{E}(e^{-sD}e^{ikA}) = \mathbb{E}(\mathbb{E}(e^{-sD}e^{ikX(D)}|D=t)) = \mathbb{E}(e^{-sD}e^{-D\psi_0(k)}) = e^{-(s+\psi_0(k))^\beta}$$

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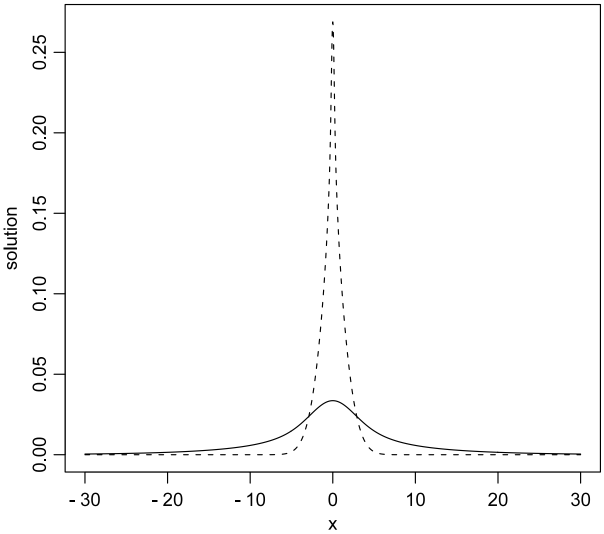

Fig. 3. Solution $a_3(x, t)$ to the coupled OCTRW limit Eq. (5.19) with $t = 1.0$ and $\beta = 0.8$ (solid line), and solution $c_3(x, t)$ to the corresponding CTRW limit Eq. (5.17) with $t = 1.0$ and $\beta = 0.8$ (dashed line).

so that $\psi(k, s) = (s + \psi_0(k))^\beta$ in this case. If $X(t)$ has a density $f_u(t)$, then the CTRW limit density

$$c_4(x, t) = \int_0^t f_u(x) c_2(u, t)\, du \tag{5.20}$$

solves the coupled pseudo-differential equation

$$\left(\partial_t + \psi_0(i\partial_x)\right)^\beta c_4(x, t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}, \tag{5.21}$$

while the OCTRW limit density

$$a_4(x, t) = \int_0^t f_u(x) a_2(u, t)\, du \tag{5.22}$$

solves

$$\left(\partial_t + \psi_0(i\partial_x)\right)^\beta a_4(x, t) = \frac{1}{\Gamma(1-\beta)} \int_t^\infty f_u(x)\, \beta u^{-\beta-1}\, du. \tag{5.23}$$

Remark 5.5. In practical applications, it is useful to consider a CTRW with drift. Extending Example 5.3, suppose that the conditional distribution of Y given $D = t$ is normal with mean $a_0 t$ and variance $2b_0 t$. Example 5.4 with $\psi(k, s) = (s + b_0 k^2 - ika_0)^\beta$ implies that the Lebesgue density

$$c_5(x, t) = \int_0^t \frac{1}{\sqrt{4\pi b_0 u}} \exp\left(-\frac{(x - a_0 u)^2}{4b_0 u}\right) c_2(u, t)\, du$$

of the CTRW limit $A(E(t)-)$ solves the governing equation

$$\left(\partial_t + a_0\partial_x - b_0\partial_x^2\right)^\beta c_5(x, t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}.$$

The OCTRW limit density

$$a_5(x, t) = \int_t^\infty \frac{1}{\sqrt{4\pi b_0 u}} \exp\left(-\frac{(x - a_0 u)^2}{4b_0 u}\right) a_2(u, t)\, du$$

solves the governing equation

$$\left(\partial_t + a_0\partial_x - b_0\partial_x^2\right)^\beta a_5(x, t) = \frac{1}{\Gamma(1-\beta)} \int_t^\infty \frac{1}{\sqrt{4\pi b_0 u}} \exp\left(-\frac{(x - a_0 u)^2}{4b_0 u}\right) \beta u^{-\beta-1}\, du.$$

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