On the support of the spectral measure of a harmonizable sequence

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Abstract

In this note we discuss a relationship between the correlation function of a harmonizable sequence and support of its spectral measure.

1 Introduction

Let $Z$ denote the group of integers. The dual $\hat{Z}$ of $Z$ is in this paper identified with the interval $[0, 2\pi)$ with addition modulo $2\pi$.

A second order zero-mean stochastic sequence is often viewed as a sequence \{\(x(n), n \in Z\)\} of elements of a Hilbert space $H$ over the field of complex numbers. The covariance function $R(n, m)$ of $x(n)$ is then defined as $R(n, m) = (x(n), x(m))$, where $(\cdot, \cdot)$ is an inner product in $H$. A sequence $x(n), n \in Z$, is called strongly harmonizable if there is a measure $F$ on $[0, 2\pi)^2$, called the spectral measure of the sequence \{\(x(n), n \in Z\)\}, such that

$$R(n, m) = \int_0^{2\pi} \int_0^{2\pi} e^{i(ns-mt)} F(ds, dt).$$

(1)

A sequence \{\(x(n), n \in Z\)\} is periodically correlated (PC) with period $T$, if for every $p \in Z$, the function

$$k \rightarrow B_p(k) = R(p+k, k)$$

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is periodic with the same period $T$. Every PC sequence is strongly harmonizable and its spectral measure $F$ is supported on $T$ lines (see [1]):

$$L_\lambda = \{(s,t) \in [0,2\pi]^2 : t = s + \lambda\}, \quad \text{where } \lambda \in \left\{ 0, \frac{2\pi}{T}, \frac{2 \times 2\pi}{T}, \ldots, \frac{(T-1) \times 2\pi}{T} \right\}.$$ 

Note that, since addition in $[0,2\pi)$ is modulo $2\pi$, each line $L_\lambda$, $\lambda \neq 0$, if drawn on the square $[0,2\pi) \times [0,2\pi)$ comprises two segments; for example if $\lambda = \frac{2\pi}{T}$ then $L_\lambda = \{(s,t) : s \in [0,2\pi) \} \cup \{(s,s + 2\pi/T - 2\pi) : s \in [2\pi - 2\pi/T, 2\pi)\}$. If the functions $k \rightarrow B_p(k)$ are merely almost-periodic and $\{x(n), n \in \mathbb{Z}\}$ is strongly harmonizable, then its spectrum is supported on lines $L_\lambda = \{(s,t) \in [0,2\pi]^2 : t = s + \lambda\}$, $\lambda \in \Lambda$, where $\Lambda$ is the set of non-zero frequencies of sequences $B_p(\cdot)$, $p \in \mathbb{Z}$ (see [3]).

In both cases the location of spectral lines (that is the set $\Lambda$) is a Borel support of measures whose Fourier transforms are sequences $k \rightarrow B_p(k)$, $p \in \mathbb{Z}$. The purpose of this note is to show that this phenomenon holds true for any harmonizable sequence $\{x(n), n \in \mathbb{Z}\}$, namely, that a Borel support of the spectrum $F$ of $\{x(n), n \in \mathbb{Z}\}$ is on parallel to the diagonal stripes determined by the common support of measures which are the inverse Fourier transforms of sequences $k \rightarrow B_p(\cdot)$, $p \in \mathbb{Z}$.

## 2 Borel Support

In this paper by a measure on a topological space $G$ we will understand a finite complex $\sigma$-additive function defined on Borel $\sigma$-algebra of $G$. If $\mu$ is a measure then any Borel set $D$ with the property that $\mu(\Delta) = 0$ for every Borel $\Delta$ disjoint with $D$ will be called a Borel support of $\mu$. Note that this definition differs from the standard notion of support (c.f. [2], p.124), where $D$ is assumed to be closed and the smallest in the sense that $|\mu|(D \cap U) > 0$ for every open $U$ such that $D \cap U \neq \emptyset$.

It turns out that it is easier to work with some transformation of the spectral measure $F$, namely with the measure

$$H(\Delta) = F(\Psi(\Delta)), \quad \text{where } \Psi(u,w) = (u,u+w).$$

Since $\Psi$ is a homeomorphism of $[0,2\pi)^2$ onto itself, $H$ is a measure.
Theorem 1. Let \( \{x(n), n \in \mathbb{Z}\} \) be a strongly harmonizable sequence, \( F \) be its spectral measure, and \( \Psi \) be the function defined in (2). Let \( \mu_p \) denote the measure on \([0, 2\pi)\) such that

\[
R(p + k, k) = \int_0^{2\pi} e^{-iku} \mu_p(du), \quad k \in \mathbb{Z}
\]

(the existence of \( \mu_p \) is proved below in (6)). Then for every Borel subset \( D \) of \([0, 2\pi)\) the following conditions are equivalent:

1. for every \( p \) the set \( D \) is a Borel support of \( \mu_p \),
2. the set \( \Psi([0, 2\pi) \times D) \) is a Borel support of \( F \).

Proof: From (2) it follows that for any bounded complex measurable function \( \phi \)

\[
\int_0^{2\pi} \int_0^{2\pi} \phi(s, t) F(ds, dt) = \int_0^{2\pi} \int_0^{2\pi} \phi(s, s + t) H(ds, dt).
\]

(4)

In particular (1) and (4) imply that

\[
R(p + k, k) = \int_0^{2\pi} \int_0^{2\pi} e^{ipu} e^{-ikw} H(du, dw).
\]

(5)

From Fubini’s theorem we therefore conclude that the measures \( \mu_p, p \in \mathbb{Z} \), satisfying (3) exist and are given by

\[
\mu_p(\Delta) = \int_0^{2\pi} e^{ipu} H(du, \Delta).
\]

(6)

Since the Fourier transform determines a measure we conclude that

\[
\mu_p(\Delta) = 0 \text{ iff } H(E \times \Delta) = 0 \text{ for every Borel } E \subset [0, 2\pi).
\]

(7)

Suppose first that \( D \) is a Borel support of each \( \mu_p, p \in \mathbb{Z} \). Then from (7) it follows that for every \( \Delta \) disjoint with \( D, H(E \times \Delta) = 0 \) for every \( E \). Since rectangles determine the measure \( H \), it implies that \([0, 2\pi) \times D\) is a Borel support of \( H \), that is \( \Psi([0, 2\pi) \times D) \) is a Borel support of \( F \). Conversely suppose that \( \Psi([0, 2\pi) \times D) \) is a Borel support of \( F \). Then \([0, 2\pi) \times D\) is a Borel support of \( H \), that is \( H(E \times \Delta) = 0 \) for every Borel \( E \subset [0, 2\pi) \) and \( \Delta \) disjoint with \( D \). From (7) we conclude that for every \( p, \mu_p(\Delta) = 0 \) provided \( \Delta \) is disjoint with \( D \).
Note that two different iterations of (5) give two representations of $R(p + k, k)$:

1. $R(p + k, k) = \int_0^{2\pi} e^{-ikw} \mu_p(dw)$, where $\mu_p(\Delta) = \int_0^{2\pi} e^{ipu} H(du, \Delta)$,

2. $R(p + k, k) = \int_0^{2\pi} e^{ipu} \nu_k(du)$, where $\nu_k(\Delta) = \int_0^{2\pi} e^{-ikw} H(\Delta, dw)$.

The argument above shows, roughly speaking, that the sequences $k \rightarrow R(p + k, k)$ determine the 'vertical' support of $H$, while the sequences $p \rightarrow R(p + k, k)$ its 'horizontal' support. The latter means that if all measures $\nu_k$ vanish on a set $\Delta$, then the spectral measure $F$ of $\{x(n), n \in Z\}$ vanishes on the stripe $\Delta \times [0, 2\pi)$.

3 Comment on Harmonizability

The strong assumption in Section 2 was the the strong harmonizability of $\{x(n), n \in Z\}$, which yields existence of measures $\mu_p$ in (3). However, in some instances the existence of these measures can be deduced from the form of the sequences $B_p(k)$.

For example if $\{x(n), n \in Z\}$ is PC, then just periodicity of $B_p(k)$ implies that

$$B_p(k) = \sum_{j=0}^{T-1} e^{-ik(2\pi j/T)} r_p(j),$$

where

$$r_p(j) = \frac{1}{T} \sum_{k=0}^{T-1} e^{ik(2\pi j/T)} B_p(k),$$

and so (3) holds true with

$$\mu_p = \sum_{j=1}^{T} r_p(j) \delta(2\pi j/T)$$

($\delta_a$ denotes the unit measure concentrated at $\{a\}$).

The immediate question is whether or not (3) itself implies harmonizability of $\{x(n), n \in Z\}$. Following Hurd [3] one can show that the answer is affirmative if the common support of all $\mu_p$ is a finite set.

**Theorem 2** (cf. [3], Proposition 3) Suppose that $\{x(n), n \in Z\}$ is a stochastic sequence such that for every $p \in Z$ there is a measure $\mu_p$ satisfying

$$R(p + k, k) = \int_0^{2\pi} e^{-iku} \mu_p(du), \quad k \in Z.$$  

(8)
Suppose additionally that all measures $\mu_p$ are Borel supported on the same finite set $\Lambda \subset [0, 2\pi)$ containing 0. Then the sequence $\{x(n), n \in \mathbb{Z}\}$ is strongly harmonizable.

We sketch the proof, referring for details to Hurd’s paper [3].

First note that since $R(p+k, k) = \int_0^{2\pi} e^{-ikw} \mu_p(dw)$, from Lebesgue’s Theorem it follows that for every $s \in [0, 2\pi)$, the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} e^{ij\lambda} R(p+j, j) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} R(p+k, q+k)$$

exists and equals $\mu_p(\{\lambda\})$, if $\lambda \in \Lambda$, and 0 otherwise.

We first examine the sequence $\mu_p(\{0\})$, $p \in \mathbb{Z}$. Since

$$\mu_{p-q}(\{0\}) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} R(p-q+j, j) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} R(p+k, q+k)$$

and $R(n, m) = (x(n), x(m))$, 

$$\sum_i \sum_j c_i c_j \mu_{p_i-p_j}(\{0\}) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \left\| \sum_i c_i x(p_i+k) \right\|^2 \geq 0,$$

that is $\mu_p(\{0\})$, $p \in \mathbb{Z}$, is nonnegative definite. From Bochner’s Theorem ([4], Section 1.4.3) it follows that there is a nonnegative measure $\gamma_0$ on $[0, 2\pi)$ such that

$$\mu_p(\{0\}) = \int_0^{2\pi} e^{ipu} \gamma_0(du).$$

Now, repeating arguments presented in [3], p. 32, one can show that for every $\lambda \in \Lambda$ there is a constant $M$ such that for any selection of complex numbers $c_1, \ldots, c_n$ and integers $p_1, \ldots, p_n$

$$\left| \sum_{j=1}^n c_j \mu_p(\{\lambda\}) \right| \leq M \sup_{l} \left| \sum_{j=1}^n c_j e^{ilp_j} \right|.$$

Using [4], Section 1.9.1, we conclude that for every $\lambda \in \Lambda$ there is a measure $\gamma_\lambda$ of $[0, 2\pi)$ such that

$$\mu_p(\{\lambda\}) = \int_0^{2\pi} e^{ipu} \gamma_\lambda(du), \quad p \in \mathbb{Z}.$$
For each \( \lambda \in \Lambda \) let \( H_\lambda \) be the image of \( \gamma_\lambda \) through the mapping \( [0, 2\pi) \ni s \rightarrow (s, \lambda) \in [0, 2\pi)^2 \), and let \( H = \sum_{\lambda \in \Lambda} H_\lambda \). Since \( \Lambda \) is finite, \( H \) is a measure on \( [0, 2\pi)^2 \). Using (9) and (8) it is easy to verify that
\[
\int_0^{2\pi} \int_0^{2\pi} e^{ipu} e^{-i\lambda k} H(du, dw) = \sum_{\lambda} e^{-i\lambda k} \int_0^{2\pi} e^{ipu} \gamma_\lambda(du) = R(p + k, k).
\]
Hence \( F = H \circ \Psi^{-1} \) is the spectral measure of \( \{x(n), n \in \mathbb{Z}\} \), and so \( x(n) \) is strongly harmonizable.

4 Examples

Let us consider the sequence \( \{x(n), n \in \mathbb{Z}\} \), such that for every \( n \)
\[
x(n) = A^2 \int_0^{2\pi} e^{int} g(t) dt,
\]
where \( g(t) \in L^1([0, 2\pi)) \) and \( A \) is a second order random variable of mean zero. The sequence \( \{x(n), n \in \mathbb{Z}\} \) is strongly harmonizable on the Hilbert space with the inner product \( (y, z) = Eyz \). Indeed,
\[
R(n, m) = E \overline{x(n)x(m)} = E \left( \int_0^{2\pi} \int_0^{2\pi} e^{(ns-nt)g(s)\overline{g(t)}} ds dt \right),
\]
where \( \sigma_A = E(A^2) \). Then there exists the measure \( F \) on \( [0, 2\pi)^2 \) such that
\[
R(n, m) = \int_0^{2\pi} \int_0^{2\pi} e^{i(ns-nt)} F(ds, dt),
\]
where \( F(s, t) = \sigma_A g(s)\overline{g(t)} ds dt \). According to (2) the \( H \) measure takes the form
\[
H(s, t) = \sigma_A g(s)\overline{g(s+t)} ds dt.
\]
According to (6) there exist the family of measures \( \{\mu_p, p \in \mathbb{Z}\} \) such that for every \( p \in \mathbb{Z} \) formula (8) holds. Moreover the measures are given by
\[
\mu_p(dt) = \sigma_A \int_0^{2\pi} e^{ipu} g(u)\overline{g(u+t)} du.
\]
As a second example let us consider the strongly harmonizable sequence \( \{x(n), n \in \mathbb{Z}\} \) such that there exists the Borel subset \( D \), which is a Borel support of \( \mu_p \) for every \( p \in \mathbb{Z} \). Moreover let us assume the diagonal line \( L_0 = \{(s, t) \in [0, 2\pi)^2 : t = s\} \) is disjoint with \( \Psi([0, 2\pi) \times D) \). Then, according to Theorem 1, \( F(L_0) = 0 \).
References


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