Pricing on electricity market based on coupled-continuous-time-random-walk concept

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A B S T R A C T

In this paper we propose a model of electricity market based on the forward rate dynamics described by a diffusion with jumps as a generalization of the classical diffusion approach. We consider jump components resulting from a coupled continuous-time random walk (CTRW) with jump lengths proportional to the corresponding inter-jump time intervals. In the framework of the model we derive a formula for the EURO-price of a standard European call option, showing applicability of CTRW processes for pricing of financial instruments. The result, obtained by an advance theory of semimartingales, is an essential extension of the pricing formula derived in the classical diffusion model of the forward rate dynamics. It indicates an influence of both, the continuous and the jump parts of the forward rate process on the option price.

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1. Introduction

Liberalisation of energy market has yielded electricity being quoted on many exchanges on the world where the power forward contract has recently become one of the most popular instruments connected with electricity, usually used as an underlying instrument for other derivatives. Evolution of electricity exchanges implicates a demand for new pricing methods which can help us to valuate the derivatives.

Because of the specific structure and behavior of electricity spot prices, i.e. high volatility, seasonality and frequent spikes, modelling on electricity market is a very interesting and challenging topic, see e.g. Refs. [1–6]. In a huge group of models the spot prices are described by some stochastic process (like diffusion or Lévy process), and then the derivatives are priced [4, 5]. This methodology, however, has some disadvantages since electricity is not storable, and a connection between its spot and forward prices is relatively weak. Also, another attempt at modelling electricity market that ignores a behavior of the spot prices but begins with a modelling of the power forward prices [5,7] does not give a satisfactory description of reality. Recently, both approaches mentioned above have been combined in the model presented in Ref. [8] that refers to the interest rate (Heath–Jarrow–Morton) model [9], well known from financial markets, and is based on the forward rate dynamics described by a diffusion process. In the present paper we propose generalization of this model where a diffusion with jumps given by a coupled continuous-time random walk (CTRW) is considered to describe the forward rate dynamics.

CTRW processes (i.e. random walks with random time intervals between subsequent jumps or, equivalently, the walks subordinated to renewal counting processes) have already been successfully used in physics to model a wide variety of physical phenomena, see e.g. Refs. [10–16].
of phenomena connected with anomalous diffusion (like relaxation processes, fully developed turbulence, transport in disordered or fractal media) [10–19]. Recently, CTRW’s have been applied also in finance to model the movement of log-prices or the risk process [20–22]. Usually, in the applications of the CTRW ideology analysis of the large-time asymptotic behavior of the diffusion front is of great importance [11–13,23–26]. In the present paper a coupled CTRW process with the length of the jump proportional to the corresponding inter-jump time interval (i.e. a process which does not belong to the class of Lévy processes) is proposed to describe the jump component of the forward rate dynamics, and other properties of the walk, like being a semimartingale which can be obtained by random time change, are taken into account. For such a form of CTRW-like jump component a formula for the EURO-price of a standard European call option is derived and discussed. The obtained pricing formula is an essential generalization of the one obtained in Ref. [8] for the classical diffusion model of the forward rate dynamics.

The article is structured as follows: In Section 2 we present main assumptions of the proposed model for electricity market with the forward rate dynamics given by a diffusion with jumps. In Section 3 we derive a formula for the EURO-price of a standard European call option (with details of the proof contained in Appendix). Concluding remarks are given in Section 4.

2. Diffusion-with-CTRW-jump model for electricity market

On real electricity market, prices \( P(t, T) \) at time \( t \leq T \leq T^* \) of the power forward contracts are given in EURO (or other money currency), while the contracts concern electricity in MWh constantly delivered within the interval \([T, T + \Delta]\) for some fixed \( \Delta > 0 \). (Here \( T^* \) is the considered time horizon.) Hence, MWh should be considered as a currency for the internal electricity market where MWh-price \( p(t, T) \) of the power forward contract with delivery of 1 MWh in the interval \([T, T + \Delta]\) is taken into account. Note that \( p(t, T) \) is equivalent to the zero-coupon bond on the financial market. A relationship between the internal and financial electricity markets can be provided by an exchange process \( N_t \) that is the MWh-value of the EURO-bank amount \( e^{-\int_0^t r(s) \, ds} \) with deterministic interest rate \( r(t) \). Namely, we have

\[
P(t, T) = \frac{p(t, T)}{e^{-\int_0^t r(s) \, ds} N_t}.
\]

In the stochastic approach to model electricity market proposed in Ref. [8] and generalized in this paper one begins with modelling of the term structure of the internal electricity market. Then, in order to find EURO-prices for derivatives, one returns to the financial market using the exchange process \( e^{-\int_0^t r(s) \, ds} N_t \). The term structure of the internal market is expressed in a classical way [9,27] by means of the forward rate \( f(t, T) \) that determines the form of basic market processes, MWh-prices \( p(t, T) \) and process \( B_t \) describing the short-term investing in the power forward contracts. Namely, we have

\[
p(t, T) = \exp \left( -\int_t^T f(t, s) \, ds \right),
\]

\[
B_t = \exp \left( \int_0^t f(s, s) \, ds \right).
\]

Let us note that it seems quite natural to apply diffusion processes with jumps to describe the forward rate dynamics. Processes of this kind are very often used for modelling of electricity spot prices that tend to have high volatility and many spikes caused, for example, by some technical drawbacks or weather movements (see e.g. Refs. [4,5,28]), and although the forward prices behave in a different way, they still describe the same commodity (as in Fig. 1). The discounting process \( N_t \) is assumed to be a numeraire asset in relation to which we choose the most accurate martingale measure \( Q \) on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) connected with the considered market, i.e. the stochastic basis. (Note that in order to choose \( Q \) methodology based on utility maximisation for incomplete markets [29] is usually applied.) According to the First Fundamental Asset Pricing Theorem, [30], the market with numeraire \( N_t \) is arbitrage-free if and only if there exists (at least one) measure \( Q \) (martingale measure) equivalent to \( \mathbb{P} \) such that the discounted price of any instrument on this market is a \( Q \)-martingale. In particular,

\[
p(t, T) = N_tE_Q (N_t^{-1} \mid \mathcal{F}_t).
\]
On the other hand, asset $B_t$ also can be taken as a numeraire, and for the equivalent probability measure $\hat{Q}$ on space $(\Omega, \mathcal{F}_T)$ with the density function of the form
\[
\frac{d\hat{Q}}{dQ} = \frac{N_0 R_T}{N_T B_0} \quad Q - a.s.
\]
the discounted instruments $\hat{p}(t, T) = p(t, T)/B_t$ and $\hat{N}_t = N_t/B_t$ have the martingale property. The probability space $(\Omega, \mathcal{F}, \hat{Q})$ with measure $\hat{Q}$ associated with $B_t$ is the most convenient for us to model the forward rate dynamics. Namely, let us consider the forward rate $f(t, T)$ that on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \hat{Q})$ with the filtration generated by the considered process has the following form
\[
df(t, T) = \alpha(t, T)dt + \sigma(t, T) dW_t + \gamma(t, T) df^K_t,
\]
where for $C = \{(s, t) : 0 \leq s \leq t \leq T^*\}$, factors $\alpha : C \rightarrow \mathbb{R}, \sigma : C \rightarrow \mathbb{R}^d$ and $\gamma : C \rightarrow \mathbb{R}$ are deterministic, $W_t$ is a classical $d$-dimensional Wiener process, and $f^K_t$ is the compensated form of some jump process $J_t$ independent of $W_t$. Note that process $f^K_t$ is given by
\[
f^K_t = \int_0^t \int_\mathbb{R} xd(\mu_j - v_j),
\]
where $d\mu_j = \mu_j(\omega; ds, dx)$ denotes a measure describing the jump structure of process $J_t$ [30–32] such that
\[J_t = \int_0^t \int_\mathbb{R} xd\mu_j,
\]
and $dv_j = v_j(\omega; ds, dx)$, the compensating measure for $\mu_j$, i.e. a predictable measure with the property that $\mu_j - v_j$ is a martingale measure [30–32]. This means that $f^K_t$ given by (2.2) is a $\hat{Q}$-martingale.

Since our aim is to find the EURO-prices for the derivatives, we need to assume also some convenient form of the exchange process $N_t$. Hence, let the $\hat{Q}$-martingale $\hat{N}_t = \frac{N_t}{N_0}$ take the form
\[
\hat{N}_t = \hat{N}_0 \mathcal{E}(H_t),
\]
where $\mathcal{E}(\cdot)$ means the stochastic exponent [30–32], and
\[
H_t = \int_0^t v(s) \circ dW_s + \int_0^t \int_\mathbb{R} [e^{\beta(s)x} - 1]d(\mu_j - v_j)
\]
for some deterministic volatility functions $v : [0, T^*] \rightarrow \mathbb{R}^d$ and $\beta : [0, T^*] \rightarrow \mathbb{R}$. Equivalently, formula (2.3) means that $\hat{N}_t$ is a solution of the following stochastic differential equation
\[
d\hat{N}_t = \hat{N}_t dH_t.
\]
Observe that such a process $\hat{N}_t$ depends on the same Wiener and jump processes as the forward rate $f(t, T)$.

In the proposed model we assume additionally that the jump process $J_t$ takes the form of the following coupled CTRW process [11,23,26]. Namely, let $J_t$ be given by
\[
J_t = \sum_{i=1}^{J(t)} R_i, \quad t \geq 0
\]
with the counting process defined as
\[
L_R(t) = \min \left\{ n : \sum_{i=0}^n R_i > t \right\}
\]
where $R_0 = a$ is a positive constant, and $(R_n)_{n \geq 1}$ on the probability space $(\Omega, \mathcal{F}, \hat{Q})$ is a sequence of positive independent and identically distributed random variables with finite expected value.

In Figs. 2 and 3 exemplary trajectories of the processes responsible for the jump component in (2.1) with $J_t$ defined by (2.4) are shown. As we see, although $J_t$ is positive at each time $t$, the corresponding compensated process $f^K_t$ can take both positive and negative values (Fig. 2). Moreover, for the full jump component $f^K_t \gamma(u, T) df^K_t$ the picture (Fig. 3) becomes more realistic and interesting.

In the next section, in the framework of the introduced model we shall derive and discuss a pricing formula for the call option written on the power forward contract.
3. Call option pricing with the internal forward rate given by a coupled CTRW

In order to evaluate EURO-price $C_0$ at time $t = 0$ of a standard European call option with the maturity time $T \in [0, T^*]$ and the strike price $K > 0$ written on the power forward contract with the maturity time $T_1 \in [T, T^*]$ we begin with the following general pricing formula \[8,33\]

$$C_0 = E_Q \left( \left[ \tilde{p}(T, T_1) - e^{-\int_0^T r(s) ds} K \right]^+ \right),$$

where measure $Q$ is that associated to numeraire $N_t$, and

$$\tilde{p}(T, T_1) = \frac{p(T, T_1)}{N_T} = \tilde{p}(T, T_1)$$

for $\tilde{p}(T, T_1) = p(T, T_1)/B_T$. We need to get representation of the process $\tilde{p}(t, T)$ with respect to the measure $Q$ for the forward rate dynamics (2.1) with the CTRW-like process $J_t$ given by formula (2.4). We begin with deriving the stochastic-exponent form of the discounted MWh-prices $\tilde{p}(t, T)$ with respect to the measure $Q$. Observe that the counting process $L_R(t)$,
Eq. (2.5), is the first-passage process. Hence, it is a stopping time with respect to the natural filtration $\mathcal{F}_t = \sigma(R_1, \ldots, R_n)$, and can be used to construct processes with continuous time by the so-called random change of time. The CTRW-like process $J_t$ given by (2.4), as well as its transformation $\int_0^T h(s, x) \, d\mu_j$ for some Borel function $h$, are examples of continuous-time processes obtained in such a way. Therefore we can easily obtain [30] that the compensator of $\int_0^T h(s, x) \, d\mu_j$ with respect to measure $\mathcal{Q}$ has the form

$$\int_0^T h(s, x) \, dv_j = \sum_{0 < \tau \leq T} 1_{\{|\Delta h(s) - 1\}|} \mathcal{E}_0 h(s, R_1)$$

with a simple jump structure. Moreover $v_j(\omega; \{s \times \mathbb{R}\} \in [0, 1]$ that yields similar results of the change of measure (which we shall do in the next step) as in the case of continuous compensators (for example, for Lévy processes), see th. 5.10, th. 5.19, ch. III of Ref. [32] and the conclusion in Ref. [31]. After the changing of measure process $L_0(t)$ is still a stopping time, and the compensator has a form similar to (3.2).

**Lemma 1.** Assuming the following conditions for the factors in (2.1) and (2.3):

$$\int_0^T |\alpha(s, T)| \, ds < \infty, \quad \int_0^T \|\sigma(s, T)\|^2 \, ds < \infty, \quad \int_0^T \left(1 - e^{-\gamma(s, T)x} \right) \, dv_j < \infty,$$

$$\forall (t, a) \in \mathbb{C}^2 \int_t^T \alpha(t, s) \, ds = \left\| \int_t^T \sigma(t, s) \, ds \right\|^2,$$

$$\int_0^T \|v(s)\|^2 \, ds + \int_0^T \int_0^\infty \left(1 - e^{-\rho(s, x)} \right) \, dv_j < \infty,$$

process $\hat{p}(t, T)$ has the following stochastic-exponent form

$$\hat{p}(t, T) = \hat{p}(0, T) \mathcal{E}(D(t, T))$$

with

$$D(t, T) = - \int_0^t a(s, T) \circ dW_t^r + \int_0^t \int_0^\infty \left( e^{-b(t, T)x} - 1 \right) d(\mu_j - v_j),$$

for

$$a(t, T) = \int_t^T \sigma(t, s) \, ds + v(t) \quad \text{and} \quad b(t, T) = \int_t^T \gamma(t, s) \, ds + \beta(t),$$

where $W_t^r$ is a standard $d$-dimensional Wiener process independent of the jump process $J_t$ on $(\Omega, \mathcal{F}, \mathcal{Q})$, and $dv_j = e^{b(t, x)} \, dv_j$ is a new compensating measure for $J_t$ on this space.

**Proof.** By means of the Fubini theorem for semimartingales [34], assuming (3.3) we get that

$$\hat{p}(t, T) = \hat{p}(0, T) \mathcal{E}(Z(t, T)),$$

where

$$Z(t, T) = - \int_0^t \sigma^*(s, T) \circ dW_t + \int_0^t \int_0^\infty \left[ e^{-\gamma^*(s, T)x} - 1 \right] d(\mu_j - v_j).$$

with

$$\sigma^*(t, T) = \int_t^T \sigma(t, s) \, ds, \quad \gamma^*(t, T) = \int_t^T \gamma(t, s) \, ds.$$

Moreover, to be in accordance with the $\mathcal{Q}$-martingale property of $\hat{p}(t, T)$, factors $\alpha$ and $\sigma$ have to be in relation (3.4).

Representation (3.7) of $\hat{p}(t, T)$ and assumed form (2.3) of the discounting process $N_t$ allow us to derive representation (3.6) of $\hat{p}(t, T)$. Namely, by applying the two-dimensional Itô formula, the theorems concerning the change of measure for semimartingales and assuming additionally (3.5), one obtains that process $\hat{p}(t, T)$ has the stochastic-exponent form (3.6) with

$$W'_t = W_t - \int_0^t v(s) \, ds,$$
being a standard $d$-dimensional Wiener process, independent of the jump process $J_t$ on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ since for any bounded measurable functions $h_1$ and $h_2$ we have

\[
E_{\mathbb{Q}}(h_1(W_t)h_2(J_t)) = E_{\mathbb{Q}}\left\{h_1 \left(W_t - \int_0^t v(s)\,ds\right) \mathcal{E} \left(\int_0^t v(s)\,dW_s\right)\right\} \\
\times E_{\mathbb{Q}}\left\{h_2(J_t)\mathcal{E} \left(\int_0^t (e^{\theta(s)} - 1)\,d(\mu_j - v_j)\right)\right\} = E_{\mathbb{Q}}(h_1(W_t'))E_{\mathbb{Q}}(h_2(J_t)).
\]

The above representation of $\tilde{p}(t, T)$ yields the following theorem.

**Theorem 1.** In the framework of the proposed model under assumptions (3.3), (3.4) and (3.5) and

\[
\int_0^T a^2(s, T_1)\,ds + \int_0^T \int_{\mathbb{R}} \left(1 - e^{-\frac{1}{2}(s, T_1)\mathbb{Q}}\right)\,dv\,dy < \infty
\]

the EURO-price $C_0$, at time $t = 0$, of a standard European call option with the maturity time $T \in [0, T^*]$ and the strike price $K > 0$ written on the power forward contract with the maturity time $T_1 \in [T, T^*]$ is given by

\[
C_0 = P(0, T_1)\bar{F}(\delta_+) - e^{-\int_0^T r(s)\,ds}KF(\delta_-)
\]

where $P(0, T_1)$ is the EURO-price of the underlying instrument,

\[
F = \Phi \ast \theta, \quad \bar{F} = \Phi \ast \bar{\theta}
\]

are the convolutions of the standard normal cumulative distribution function $\Phi(x)$ and $\bar{\theta}(x) = \mathbb{Q}\{Y < x\}, \bar{\theta}(x) = \mathbb{Q}\{Y < x\}$ where

\[
Y = \Sigma^{-1}\left(\int_0^T b(s, T_1)\,ds + \sum_{0 \leq s \leq T} 1_{\{\Delta \theta(s) = 1\}} \ln\left(\frac{\varphi_R(-\int_0^{T_1} \gamma(s, t)\,dt)}{\varphi_R(\beta(s))}\right)\right)
\]

and

\[
\Sigma = \sqrt{\int_0^T \|a(s, T_1)\|^2\,ds}, \quad \delta_+ = \frac{\ln(P(0, T_1)/K) + \int_0^T r(s)\,ds \pm \frac{1}{2} \Sigma^2}{\Sigma},
\]

\[
\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}|_{F_T} = \frac{\tilde{p}(T, T_1)}{\tilde{p}(0, T_1)} \quad \mathbb{Q}\text{-a.s.}
\]

**Proof.** Observe that formula (3.1) can be rewritten as

\[
C_0 = E_{\mathbb{Q}}\left(\tilde{p}(T, T_1)1_A\right) - E_{\mathbb{Q}}\left(e^{-\int_0^T r(s)\,ds}K1_A\right)
\]

for set $A = \{\tilde{p}(T, T_1) > e^{-\int_0^T r(s)\,ds}K\}$. To derive $C_0$ we use the representation (3.6) of process $\tilde{p}(t, T)$ on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{Q})$, given in Lemma 1, and the following formula for its compensator in the exponential form

\[
\sum_{0 \leq s \leq T} 1_{\{\Delta \theta(s) = 1\}} \ln\left(\frac{\hat{\varphi}_R(-\int_0^{T_1} \gamma(s, t)\,dt)}{\hat{\varphi}_R(\beta(s))}\right),
\]

where $\hat{\varphi}_R(c) = E_{\mathbb{Q}}(e^{cR_t})$ is the jump transform (defined for $c$ from some subset of complex numbers). Assuming additionally (3.3)–(3.5), we get that

\[
E_{\mathbb{Q}}\left(e^{-\int_0^T r(s)\,ds}K1_A\right) = e^{-\int_0^T r(s)\,ds}K\mathbb{Q}\{\xi < \delta_-\},
\]

where the random variable $\xi$ reads

\[
\xi = \Sigma^{-1} \int_0^T a(s, T_1) \,dW_s' + Y
\]

for $Y$ given by (3.10). On the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ the random variable $\Sigma^{-1} \int_0^T a(s, T_1) \,dW_s'$ has the standard normal distribution and is independent of $Y$ resulting from the jump component only. Therefore for any $x$ we have $\mathbb{Q}\{\xi < x\} = (\Phi \ast \theta)(x)$, a convolution of the standard normal cumulative distribution function $\Phi(x)$ and $\theta(x) = \mathbb{Q}\{Y < x\}$. 

\[
\text{(3.8)}
\]

\[
\text{(3.9)}
\]

\[
\text{(3.10)}
\]

\[
\text{(3.11)}
\]
In order to evaluate $E_Q (\tilde{p}(T, T_1) 1_\lambda)$ we introduce measure $\tilde{Q}$ with the probability density function (3.11), well-defined under assumption (3.8). For such a measure we have

$$E_{\tilde{Q}} (\tilde{p}(T, T_1) 1_\lambda) = \tilde{p}(0, T_1) \tilde{Q} \{ \xi < \delta_+ \}$$

where

$$\bar{\xi} = \Sigma^{-1} \int_0^T a(s, T_1) \circ d\tilde{W}_t + Y$$

for a standard Wiener process $\tilde{W}_t = W_t + \int_0^t a(s, T_1)ds$ independent of $J_t$ on the probability space $(\Omega, \mathcal{F}, \tilde{Q})$. As a consequence, we obtain

$$E_{\tilde{Q}} (\tilde{p}(T, T_1) 1_\lambda) = P(0, T_1) (\Phi \ast \bar{\theta})(\delta_+)$$

(3.13)

where $\bar{\theta}(x) = \tilde{Q}\{Y < x\}$ and $P(0, T_1)$ is the EURO-price of the underlying instrument, since $\tilde{p}(0, T_1) = P(0, T_1)$.

Formula (3.9) follows straightforwardly from (3.12) and (3.13). □

Observe that in case $\gamma \equiv 0$, $\beta \equiv 0$, formula (3.9) simplifies to

$$C_0 = P(0, T_1) \Phi(\delta_+) - e^{-\int_0^T r(s)ds} K \Phi(\delta_-)$$

(3.14)

well-known from the classical diffusion model [8]. The pricing formula (3.9) is an essential generalization of (3.14) since distributions $\tilde{F}$ and $F$ differ, in general, from the standard normal law $\Phi$. Examples of density functions corresponding to $\tilde{F}$ and $F$ that result from an exponential jump distribution are presented on Fig. 4 in comparison with the standard normal density function.

4. Conclusions

In this paper, as a generalization of Ref. [8], we have proposed a model of electricity market based on the forward rate dynamics described by a diffusion with jumps. We have considered the jump component resulting from a coupled CTRW processes for modelling spikes in instrument quotations lies in the fact that processes of this kind involve the dependence between the inter-spike time and the spike height, observed on the real market (for example, when after high pick the market is stable longer than after low spike, or when we in fact observe the cumulated spikes resulting from groups of spikes on random intervals of time) [3,6].

In the framework of the model and by means of advanced methods for semimartingales (like measure changing, Itô formula for semimartingales, or compensator searching) we have derived formula (3.9) for the EURO-price of European call options that is an essential generalization of the pricing formula in the classical diffusion model of the forward rate dynamics. Formula (3.9) shows how the continuous and the jump part of the forward rate process influence the option price.

References