On ARMA(1,q) models with bounded and periodically correlated solutions

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Abstract: In this paper, motivated by [2], we derive necessary and sufficient conditions for bounded and periodically correlated solutions to the system of equations described by ARMA(1,q) model.
On ARMA(1,q) models with bounded and periodically correlated solutions

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Abstract

In this paper, motivated by [2], we derive necessary and sufficient conditions for bounded and periodically correlated solutions to the system of equations described by ARMA(1,q) model.

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1 Introduction

We consider a system ARMA(1,q) given by the formula:

\[ X_n - b_n X_{n-1} = a_n \xi_n + a_{n-1} \xi_{n-1} + \ldots + a_{n-(q-1)} \xi_{n-(q-1)}, \]

where

- \((X_n)\) is a sequence of complex random variables with mean 0 and finite variance in the space with the inner product \(\langle \cdot, \cdot \rangle\) and \(M_X = sp \{X_k : k \in \mathbb{Z} \}\),
- \((b_n)\) and \((a_n)\) are sequences of non zero complex numbers,
- \((\xi_n)\) is a sequence of uncorrelated complex random variables with mean 0 and variance 1 and \(M_{\xi} = sp \{\xi_k : k \in \mathbb{Z} \}\).

In a recent paper H. Hurd, A. Makagon and A. G. Miamee [2] gave necessary and sufficient conditions for boundedness in the general case of AR(1) model and then specifically for periodic and almost periodic coefficients \((a_n)\). The present effort is an attempt to understand the situation in an important for applications case of ARMA(1,q) models [1,3]. Such systems arise in climatology, economics, hydrology, electrical engineering and other disciplines. In Section 2 we discuss the relationship between existence of bounded solutions to the system equations described by ARMA(1,q) model and conditions on their coefficients (Theorem 1). Next, periodically correlated solutions are examined (Theorem 2). In Section 3 we simplify the consideration for \(q = 2\). The final Example provides a negative answer to the question: Whether or not for the fact that system (1) has a PC solution it follows that the sequences of coefficients \((b_n)\) and \((a_n)\) are periodic? This gives a partial solution to the question (in the case of AR(1) system) studied in [2].

Let us denote:

\[ B^s_r = \prod_{j=r}^s b_j \]

with the convention that \(B^s_r = 1\) if \(r > s\). It is easy to show, that iterating \(k\)-times the equation (1) we obtain:

\[ X_{n+k} = B^{n+k}_{n+1} X_n + \sum_{j=1}^k B^{n+k}_{n+j+1} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s}, \]

(2)
\[ X_{n-k} = \frac{X_n}{B_{n-k+1}^n} - \sum_{j=1}^{k} \frac{1}{B_{n-k+1}^{n-j}} \sum_{s=0}^{q-1} \xi_{n-k+j-s} a_{n-k+j-s}. \]  

(3)

2 The ARMA(1,q) model

**Definition 1** A stochastic sequence is called bounded if 

\[ \sup_{n} ||X_n|| = \infty. \]

**Lemma 1** If \( \sup_r |B_r^1| = \infty \) and system (1) has a bounded solution in \( M\xi \), then:

\[ \sup_{n} \left| \sum_{j=1}^{q} \sum_{k=0}^{q-1} \frac{|a_{n+j-k}|^2}{B_{n+j}^{n-j-(k-s)}} \right| < \infty. \]  

(4)

**PROOF:** If \( \sup_r |B_r^1| = \infty \) then there exist subsequence \( k_r \) of positive integers such that 

\[ \lim_{r} |B_r^{k_r}| = \infty. \]

So we have for all \( n \in Z \):

\[ \lim_{r} |B_{n+1}^{n+k_r}| = \infty. \]

If system (1) has a bounded solution then from (2) we obtain:

\[ X_n + \sum_{j=1}^{k_r} B_{n+j}^{n+k_r} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s} = \frac{X_{n+k_r}}{B_{n+1}^{n+k_r}} \to 0. \]

Hence:

\[ X_n = - \lim_{r} \sum_{j=1}^{k_r} \frac{1}{B_{n+j}^{n+1}} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s}. \]

We obtain:

\[ || - \sum_{j=1}^{k_r} \frac{1}{B_{n+j}^{n+1}} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s} ||^2 = \sum_{j=1}^{k_r} \sum_{k=0}^{q-1} \frac{|a_{n+j-k}|^2}{B_{n+j}^{n+j-(k-s)}} \]

Because \( X_n \) is a bounded solution of system (1), so we obtain:

\[ \sup_n \left| \lim_{r} \sum_{j=1}^{k_r} \frac{1}{B_{n+j}^{n+1}} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s} \right|^2 = \sup_{n} ||X_n||^2 < \infty, \]

hence

\[ \sup_n \sum_{j=1}^{k_r} \sum_{k=0}^{q-1} \frac{|a_{n+j-k}|^2}{B_{n+j}^{n+j-(k-s)}} < \infty. \]

\[ \square \]

**Lemma 2** If \( \sup_r |B_r^0|^{-1} = \infty \) and system (1) has a bounded solution in \( M\xi \), then:

\[ \sup_n \sum_{j=1}^{q} \sum_{k=0}^{q-1} \frac{|a_{n-j-k}|^2}{B_{n-j}^{n+j-1} B_{n-j+1-(k-s)}} < \infty. \]  

(5)
PROOF: If \( \sup_r |B_r^0|^{-1} = \infty \), then there is a subsequence \((k_r)\) of positive integers such that:

\[
\lim_{r \to \infty} |B_{k_r}^0|^{-1} = \infty.
\]

For all \( n \in \mathbb{Z} \) we have:

\[
\lim_{r \to \infty} |B_{n+k_r}^0|^{-1} = \infty.
\]

Because system (1) has the bounded solution we have from (3):

\[
X_n - \sum_{j=1}^{k_r} q^{-1} \sum_{s=0}^{q-1} B_{n-k_r+1+j}^n \xi_{n-k_r+j-s} a_{n-k_r+j-s} = X_n - k_r B_{n-k_r+1}^n \to 0.
\]

So we obtain:

\[
X_n = \lim_{r \to \infty} \left[ \sum_{j=1}^{k_r} q^{-1} \sum_{s=0}^{q-1} B_{n-k_r+1+j}^n \xi_{n-k_r+j-s} a_{n-k_r+j-s} \right] =
\]

\[
= \lim_{r \to \infty} \left[ \sum_{j=-k_r+1}^{0} q^{-1} \sum_{s=0}^{q-1} B_{n+1-j}^n \xi_{n-j-s} a_{n-j-s} \right].
\]

Since \( X_n \) is the bounded solution of system (1) and \( \xi_n \) is the orthonormal basis in \( M_\xi \) hence we have:

\[
||X_n||^2 = \lim_{r \to \infty} \left| \sum_{j=-k_r+1}^{0} q^{-1} \sum_{s=0}^{q-1} B_{n+1-j}^n \xi_{n-j-s} a_{n-j-s} \right|^2 =
\]

\[
= \sum_{j=1}^{\infty} \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} |a_{n-j-k}|^2 B_{n-j+1}^k B_{n-j+1-(k-s)}^n.
\]

We obtain then:

\[
\sup_{n} \sum_{j=1}^{\infty} \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} |a_{n-j-k}|^2 B_{n-j+1}^k B_{n-j+1-(k-s)}^n = \sup_{n} ||X_n||^2 < \infty.
\]

\[\square\]

If \( \sup_r |B_r^r| = \infty \) and \( \sup_r |B_r^0|^{-1} = \infty \), then system (1) has a bounded solution. But there is a third possible condition, which gives a bounded solution of (1):

\[
\sup_r |B_r^n| < \infty \quad \text{and} \quad \sup_r |B_r^0|^{-1} < \infty.
\]

**Lemma 3** If condition (6) holds and system (1) has a bounded solution, then:

\[
\sup_k \left[ \sum_{j=1}^{k} q^{-1} \sum_{u=0}^{q} |a_{j-w}|^2 B_{j+1}^u B_{j-w+u}^k \right] < \infty
\]

and

\[
\sup_k \left[ \sum_{j=1}^{k} q^{-1} q^{-1} |a_{-k+j-w}|^2 B_{-k+1}^{-k+j+w+u} B_{-k+1}^{-k+j+w+u} \right] < \infty.
\]
PROOF: We use (2) and (3) (and provide n=0). We assume for all \( k \in \mathbb{Z} \) and some \( C \) that we have \( |B_k^1| < C \) and \( |B_{-k}^0|^{-1} < C \). For all \( k > 0 \) we then have:

\[
\sup_{k \geq 1} \sum_{j=1}^{k} \sum_{u=0}^{q-1} \sum_{s=1}^{q} |a_{j-u}|^2 B_{j+1}^k \frac{B_k}{B_{j-w+s}} = \sup_{k \geq 1} \|X_n - B_k^1 X_0\|^2 \leq \sup_{k \geq 1} \|X_k\|^2 (1 + C)^2 < \infty,
\]

\[
\sup_{k \geq 1} \sum_{j=1}^{k} \sum_{u=0}^{q-1} \sum_{s=0}^{q-1} \frac{|a_{-k+j+w}|^2}{B_{k+1}^0 B_{-k+1}^0 B_{-k+j-w+s}} = \sup_{k \geq 1} \|X_{-k} - \frac{X_0}{B_{-k+1}^0}\|^2 \leq \sup_{k \geq 1} \|X_{-k}\|^2 (1 + C)^2 < \infty.
\]

The solution of system (1) is given by:

\[
X_k = \begin{cases} 
B_k^1 X + \sum_{j=1}^\infty B_{j+1}^k \sum_{s=0}^{q-1} \xi_{j-s} a_{j-s} & \text{if } k > 0, \\
X_{-k} \frac{X}{B_{k+1}^0} - \sum_{j=k+1}^\infty \frac{1}{B_{k+1}^0} \sum_{s=0}^{q-1} \xi_{j-s} a_{j-s} & \text{if } k < 0,
\end{cases} \quad \text{(9)}
\]

where \( X \) is a random variable in \( M \xi \).

\[\square\]

**Theorem 1** System (1) has a bounded solution if and only if one of the following holds:

(I) \( \sup_r |B_r^1| = \infty \) and

\[
\sup_n \left( \sum_{j=1}^{\infty} \sum_{k=0}^{q-1} \sum_{s=0}^{q} \frac{|a_{n+j-k}|^2}{B_{n+j+1}^n B_{n-j+1-k-s}} \right) < \infty.
\]

(II) \( \sup_r |B_r^0|^{-1} = \infty \) and

\[
\sup_n \left( \sum_{j=1}^{\infty} \sum_{k=0}^{q-1} \sum_{s=0}^{q} \frac{|a_{n-j-k}|^2}{B_{n+j+1}^n B_{n-j+1-k-s}} \right) < \infty.
\]

(III) \( \sup_r |B_r^1| < \infty, \sup_r |B_r^0|^{-1} < \infty \) and

\[
\sup_{k \geq 1} \left( \sum_{j=1}^{k} \sum_{u=0}^{q-1} \sum_{s=1}^{q} |a_{j-u}|^2 B_{j+1}^k \frac{B_k}{B_{j-w+s}} \right) < \infty,
\]

\[
\sup_{k \geq 1} \left( \sum_{j=1}^{k} \sum_{u=0}^{q-1} \sum_{s=0}^{q-1} \frac{|a_{-k+j+w}|^2}{B_{k+1}^0 B_{-k+1}^0 B_{-k+j-w+s}} \right) < \infty.
\]

**PROOF:** If condition (I) holds, then the solution of system (1) given by the following formula:

\[
X_n = -\sum_{j=1}^{\infty} \frac{1}{B_{n+j+1}^n} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s} \quad \text{(10)}
\]

is bounded.

If condition (II) holds, then \( X_n \) defined by:

\[
X_n = \sum_{j=1}^{\infty} B_{n+j+1}^n \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s}
\]

is the bounded solution of system (1).

If condition (III) holds, then \( X_n \) given by formula (9) is bounded and is a solution of system (1). In lemmas 1, 2 and 3 it is shown that if \( X_n \) is a bounded solution of system (1), then one of conditions (I), (II) or (III) holds.
**Definition 2** A stochastic sequence \((X_n)\) is called **periodically correlated (PC)** with period \(T\) if for all \(k\) sequence \((X_{n+k}, X_n)\) is periodic in \(n\) with period \(T\), i.e., \((X_{n+k}, X_n) = (X_{n+k+T}, X_{n+T})\).

**Theorem 2** If \((b_n)\) and \((a_n)\) are periodic with the same period \(T\) and \(P = b_1b_2\ldots b_T\), then system (1) has a bounded solution if and only if \(|P| \neq 1\). Moreover, the solution is PC with the same period \(T\) and:

(i) If \(|P| > 1\), then the solution is given by (10).

(ii) If \(|P| < 1\), then the solution is given by (11).

**PROOF:**

(i) If \(|P| > 1\), then for all \(n \in Z\) we have:

\[
\sum_{j=1}^{\infty} \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} \left| a_{n-j-k} \right|^2 B_{n+j+1} B_{n+j+1-(k-s)} = \sum_{N=0}^{\infty} \sum_{w=1}^{T} \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} \left| a_{n+NT+w-k} \right|^2 B_{n+NT+w+1} B_{n+NT+w+1-(k-s)} = \sum_{N=0}^{\infty} \frac{1}{|P|^{2N}} \sum_{w=1}^{T} \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} \left| a_{n+w-k} \right|^2 B_{n+w+1} B_{n+w+1-(k-s)} < \infty.
\]

Therefore (4) holds and \(X_n\) defined by (10) is the bounded solution of system (1).

(ii) If \(|P| < 1\), then for all \(n \in Z\) we obtain:

\[
\sum_{j=1}^{\infty} \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} \left| a_{n-j-k} \right|^2 B_{n-j+1} B_{n-j+1-(k-s)} = \sum_{N=0}^{\infty} \sum_{w=1}^{T} \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} \left| a_{n-NT-w-k} \right|^2 B_{n-NT-w+1} B_{n-NT-w+1-(k-s)} = \sum_{N=0}^{\infty} \frac{1}{|P|^{2N}} \sum_{w=1}^{T} \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} \left| a_{n-w-k} \right|^2 B_{n-w+1} B_{n-w+1-(k-s)} < \infty.
\]

Therefore (5) holds. \(X_n\) defined by formula (11) is bounded and satisfies formula (1).

In the next section it is shown for \(q = 2\) that \((X_n)\) defined by formulas (10) or (11) is periodically correlated and the condition \(|P| = 1\) violates the conditions (I), (II) and (III) of Theorem 1. Therefore, system (1) has no bounded solution if \(|P| = 1\).
3 The ARMA(1,2) model

For simplicity of notation we consider here only the ARMA (1,2) case:

\[ X_n - b_n X_{n-1} = a_n \xi_n + a_{n-1} \xi_{n-1}. \] (12)

**Theorem 3** If \((b_n)\) and \((a_n)\) are periodic with the same period \(T\) and \(P = b_1 b_2 \ldots b_T\), then system (12) has a bounded solution if and only if \(|P| \neq 1\). Moreover, the solution is PC with the same period \(T\) and is given by (10) if \(|P| > 1\) and is given by (11) if \(|P| < 1\).

**PROOF:** We will split the proof in 3 cases.

(i) In view of Theorem 2 we have that if \(|P| > 1\), then for all \(n \in Z\) condition (4) holds. Hence there is a bounded solution of (12). The solution is given by formula (10) for \(q = 2\). Now we want to show, that the stochastic sequences \((X_n)\) in formula (10) are PC with period \(T\). We take any \(k, n\) and we have:

\[ (X_{n+k}, X_n) = \]

Therefore from (13) we obtain:

\[ (X_{n+k}, X_n) = (X_{n+T+k}, X_{n+T}). \]

(ii) Similarly, from Theorem 2 we have that if \(|P| < 1\), then for all \(n \in Z\) condition (5) holds. Hence there is a bounded solution of (12). The solution is given by formula (11) for \(q = 2\). The correlation function is given by:

\[ (X_{n+k}, X_n) = \]

Because the correlation function is bounded and \((b_n)\) and \((a_n)\) are periodic with period \(T\), therefore from (14) we obtain:

\[ (X_{n+k}, X_n) = (X_{n+T+k}, X_{n+T}). \]

Thus by the above conditions \((X_n)\) is PC with period \(T\).

(iii) If \(|P| = 1\), then

\[ \sum_{j=1}^{\infty} \frac{|a_n+k-j|}{B_{n+1}}^2 (1 + |\frac{1}{b_{n+j+1}}|^2 + |\frac{1}{b_{n+j+1}}|) + \frac{|a_n|^2}{b_{n+1}} (1 + \frac{1}{b_{n+1}}) = \infty \]

and

\[ \sum_{j=2}^{\infty} |a_{n-j} B_{n-j+2}^2 (1 + |b_{n-j}|^2 + b_{n-j} + \overline{b_{n-j}}) + |a_{n-1}|^2 (|b_n|^2 + b_n) = \infty \]
which violates conditions (I) and (II) of Theorem 1. Since
\[
\sum_{j=1}^{NT} |B_{j+2}^{NT} a_j|^2 (1 + |b_{j+1}|^2 + b_{j+1} + \overline{b_{j+1}}) \geq \\
\geq \sum_{k=1}^{N} |B_{kT+2}^{NT} a_k|^2 (1 + |b_{kT+1}|^2 + b_{kT+1} + \overline{b_{kT+1}}) \geq |P|^2 \sum_{k=1}^{N} |a_k|^2 (1 + |b_1|^2 + b_1 + \overline{b_1}) = \\
= |a_0|^2 (1 + |b_1|^2 + b_1 + \overline{b_1}) \ast N \ast |P|^2 \to \infty
\]
then
\[
\sup_{k \geq 1} \sum_{j=1}^{k-1} |B_{j+2}^{k} a_j|^2 (1 + |b_{j+1}|^2 + b_{j+1} + \overline{b_{j+1}}) + |a_0 B_2^k|^2 (b_1 + 1) + |a_k|^2 (b_{k+1} + |b_{k+1}|^2) \geq \\
\geq \sup_{k=NT+1} \sum_{j=1}^{k-1} |B_{j+2}^{k} a_j|^2 (1 + |b_{j+1}|^2 + b_{j+1} + \overline{b_{j+1}}) + |a_0 B_2^k|^2 (b_1 + 1) + |a_k|^2 (b_{k+1} + |b_{k+1}|^2) \to \infty
\]
which violates condition (III) of Theorem 1. Therefore, in view of Theorem 1, system (12) has no bounded solution if |P| = 1.

Finally, we show that there exists an ARMA(1,2) system with bounded and PC solution for which coefficients \((a_n)\) are not periodic.

**Example** Let us consider the system ARMA(1,2) given by:
\[
X_n + 2X_{n-1} = (\sqrt{2})^n \xi_n + (\sqrt{2})^{n-1} \xi_{n-1}
\]
We have:
\[
b_n = -2, \\
a_n = (\sqrt{2})^n.
\]
The coefficients fulfill condition (4):
\[
\sum_{j=1}^{\infty} \frac{|a_{n+j}|^2}{|B_{n+1}^{n+j}|^2} (1 + \frac{1}{b_{n+j+1}}^2 + \frac{1}{b_{n+j+1}} + \frac{1}{b_{n+j+1}}^2) + \frac{|a_n|^2}{b_{n+1}^2} (1 + \frac{1}{b_{n+1}}) = \\
= \sum_{j=1}^{\infty} \frac{2^{n+j}}{4^j} (1 - \frac{1}{2})^2 - 2^{n-1} (1 - \frac{1}{2}) = 0.
\]
Therefore by Theorem 1 the solution of the system is given by formula (10). The correlation \((X_{n+k}, X_n)\) is given by:
\[
(X_{n+k}, X_n) = \frac{1}{B_{n+1}^{n+k}} \sum_{j=1}^{\infty} \frac{|a_{n+k+j}|^2}{|B_{n+k+1}^{n+k+j}|^2} (1 + \frac{1}{b_{n+k+j+1}}^2 + \frac{1}{b_{n+k+j+1}} + \frac{1}{b_{n+k+j+1}}^2) + \frac{|a_{n+k}|^2}{B_{n+1}^{n+k}} (\frac{1}{b_{n+k+1}} + \frac{1}{b_{n+k+1}}^2).
\]
So we have:
\[
(X_{n+k}, X_n) = \frac{(-1)^k}{2^k} \sum_{j=1}^{\infty} \frac{2^{n+k+j}}{4^j+1} - (-1)^k \frac{2^{n+k}}{4^k} = \frac{(-1)^k 2^n}{4} - \frac{(-1)^k 2^n}{4} = 0.
\]
Therefore \((X_n)\) is PC (as the correlation function is constant), but the coefficients \((a_n)\) are not periodic.

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References


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