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# Ergodic properties of anomalous diffusion processes

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In this paper we study ergodic properties of some classes of anomalous diffusion processes. Using the recently developed measure of dependence called the Correlation Cascade, we derive a generalization of the classical Khinchin theorem. This result allows us to determine ergodic properties of Lévy-driven stochastic processes. Moreover, we analyze the asymptotic behavior of two different fractional Ornstein–Uhlenbeck processes, both originating from subdiffusive dynamics. We show that only one of them is ergodic.


## 1. Introduction

The foundation of statistical mechanics and the explanation of the success of its methods rest on the fact that the theoretical values of physical quantities (phase averages) may be compared with the results of experimental measurements (infinite time averages) [1]. L. Boltzmann in his papers on the kinetic theory of gases introduced a special hypothesis according to which leaving a system in free evolution and waiting for a sufficient long time, the system will pass through all the states consistent with its general conditions, namely with given value of the total energy. This hypothesis was later called the Boltzmann ergodic hypothesis [2].

With J.W. Gibbs's work and the subsequent arrangement by P. and T. Ehrenfests, this hypothesis acquired a central position in statistical mechanics. At the beginning of the 1930s, a complete new and original approach was attempted by G.D. Birkhoff, B. Koopmann and J. von Neumann. They proposed the idea of proving the equality of the phase average with the infinite time average without using the Boltzmann hypothesis [3]. Khinchin in 1949 [4] proposed a new approach to the ergodic problem and maintained that furnishing an approximate method for evaluating phase averages is part of the solution to the ergodic problem. The celebrated Khinchin theorem shows that the measure of phase

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points for which the infinite time average of whatever function differs from the phase average by more than a number as small as we please, tends to zero. In other words Khinchin linked the ergodicity of a physical system with the irreversibility of the corresponding autocorrelation function. However, the Khinchin theorem cannot be successfully applied to processes with infinite second moments, in particular to the relevant class of Lévy flights [5].

In this paper, we solve the challenging problem of verifying ergodicity for processes with infinite second moments, in full generality. Also, we clarify the role of ergodicity and ergodicity breaking in the context of anomalous diffusion processes. Recently, ergodic properties of systems exhibiting anomalous behavior have attracted growing attention of researchers in various fields of physics and related sciences. Ergodicity breaking was reported in blinking quantum dots [6,7]. Analysis of time averages for non-ergodic systems was introduced in [8]. Ergodicity breaking for systems described by the fractional Fokker–Planck equation was studied in [9–11]. For the ergodic properties of generalized Langevin equations, see [12–17]. The relationship between ergodicity and irreversibility of anomalous systems was studied in [18]. The analysis of the time average mean-square displacement of fractional Brownian motion was presented in [19,20].

Consider a dynamical system $\{S_t\}_{t\in\mathbb{R}}$ describing the temporal evolution of a physical system on a measure space $(X, \mathcal{A}, \mu)$. Here, $X$ is the phase space, $\mathcal{A}$ is the $\sigma$-algebra on $X$, $\mu$ is the probability measure on $X$, and $\{S_t\}$ is the group of measurable transformations $S_t : X \to X$, $t \in \mathbb{R}$, satisfying the following two group conditions: $S_0(x) = x$ and $S_t(S_{t'}(x)) = S_{t+t'}(x)$ for $x \in X$, $t, t' \in \mathbb{R}$.

One of the most fundamental concepts in the theory of dynamical systems is *ergodicity*. Intuitively, a system is ergodic if the phase space $X$ cannot be divided into two regions such that a phase point starting in one region will always stay in that region. More formally, we say that a system is ergodic if every invariant set $A \in \mathcal{A}$ is such that either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ (in other words, for an ergodic system all invariant sets are trivial). Recall that a set $A \in \mathcal{A}$ is called invariant if $S_t(A) = A$ for all $t \in \mathbb{R}$. The celebrated Birkhoff ergodic theorem states that if $\{S_t\}$ is ergodic and measure-preserving (i.e. $\mu(S_t(A)) = \mu(A)$ for all $A \in \mathcal{A}$), then the temporal and ensemble averages coincide:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(S_t(x))dt = \int_X f(x)\mu(dx). \tag{1}$$

Here, $f$ is an arbitrary integrable function.

Another concept arising naturally in the studies of dynamical systems is *mixing*. We say that a measure-preserving dynamical system $\{S_t\}$ on a measure space $(X, \mathcal{A}, \mu)$ is mixing if

$$\lim_{t\to\infty} \mu(A \cap S_t(B)) = \mu(A)\mu(B) \tag{2}$$

for all $A, B \in \mathcal{A}$. The above condition for mixing has a simple interpretation. It can be viewed as an asymptotic independence of the sets $A$ and $B$ under the transformation $S_t$. Alternatively, condition (2) states that the fraction of points starting in $A$ that end up in $B$ after long time $t$, is equal to the product of the measures of $A$ and $B$. It is straightforward to prove that mixing is a stronger property than ergodicity [21]. Thus, it is enough to verify condition (2) in order to show that the system is ergodic.

In this paper we concentrate on the ergodic properties of some classes of anomalous diffusion processes. Consider a stationary stochastic process $\{Y(t), t \in \mathbb{R}\}$. In its canonical representation [22], $Y(t)$ can be treated as a probability measure $\mathbb{P}$ on the space $\mathbb{R}^{\mathbb{R}}$. Here, by $\mathbb{R}^{\mathbb{R}}$ we denote the space of all the functions $f : \mathbb{R} \to \mathbb{R}$. Additionally, on $\mathbb{R}^{\mathbb{R}}$ we consider a $\sigma$-algebra $\mathcal{B}$ generated by cylinder sets [23], and a group of left-shift transformations $\{S_t\}_{t\in\mathbb{R}}$, i.e. $S_t(f)(s) = f(s+t)$ for any $f \in \mathbb{R}^{\mathbb{R}}$. Observe that the group $\{S_t\}_{t\in\mathbb{R}}$ on the measure space $(\mathbb{R}^{\mathbb{R}}, \mathcal{B}, \mathbb{P})$ is a typical object of study in the theory of dynamical systems. Therefore, ergodic properties of stationary stochastic processes can be successfully studied as a part of the general theory of dynamical systems. The assumption about stationarity of the process $Y(t)$ is absolutely crucial. Its physical meaning is obvious — the system is in thermal equilibrium. From the mathematical point of view, stationarity implies that the shift transformations $\{S_t\}$ are measure-preserving. Consequently, if $Y(t)$ is ergodic, then the Birkhoff theorem applies and we obtain the equality of time and ensemble averages

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Y(t)dt = \mathbb{E}(Y(0)), \tag{3}$$

provided that $\mathbb{E}(|Y(0)|) < \infty$. Property (3) is fundamental in statistical mechanics. Let us underline that for non-stationary processes the Birkhoff theorem does not apply and the above equality in general does not hold.

This paper is structured as follows. In the next section we discuss the ergodic properties of stationary Gaussian processes. We show that ergodicity of the fractional Brownian motion and the fractional Langevin equation follows immediately from the very old results of Maruyama [24] and Grenander [25]. Next, using the so-called Lévy autocorrelation function, which is an analogue of autocorrelation for Lévy-driven processes, we derive a generalized Khinchin theorem. This result allows us to verify ergodic properties of Lévy-driven stochastic processes, in particular Lévy flights. In the final section we investigate the ergodic properties of two fractional Ornstein–Uhlenbeck (O–U) processes, both exhibiting subdiffusive behavior. The first one is defined as the stationary solution of the fractional Fokker–Planck equation, the second one is derived via Lamperti transformation of the force-free subdiffusion process. We prove that the second process is mixing (and therefore ergodic), whereas the first process is known to display ergodicity breaking.

## 2. Ergodic properties of Gaussian processes

Consider a stationary Gaussian process $\{Y(t), t \in \mathbb{R}\}$. The problem of ergodicity of $Y(t)$ was completely solved over sixty years ago by Maruyama [24] and Grenander [25]. They proved independently that the stationary Gaussian process $Y(t)$ is ergodic if and only if its spectral measure is continuous. Recall that the autocorrelation function of $Y(t)$ can be written in the form

$$r(t) = \frac{\mathbb{E}[(Y(0)-m)(Y(t)-m)]}{\mathbb{E}[Y^2(0)]} = \int_{-\infty}^{\infty} e^{it\lambda}\nu(d\lambda),$$

where $\nu$ is the spectral measure of $Y(t)$ and $m = E(Y(0))$. It can be assumed without loss of generality that $\nu$ is a probability measure. Thus, it is sufficient and necessary for $Y(t)$ to be ergodic that $\nu$ admits a density function.

An even simpler condition has to be checked in order to verify that the Gaussian process is mixing. Following the classical result of Itô [26] (see also [27]), the stationary Gaussian process $Y(t)$ is mixing if and only if its autocorrelation function satisfies

$$\lim_{t\to\infty} r(t) = 0. \tag{4}$$

Since mixing is a stronger property than ergodicity, condition (4) implies that the process $Y(t)$ is ergodic. This implication is known as the *Khinchin theorem* [4].

In general, condition (4) is rather easy to verify. Taking advantage of this fact, one can prove ergodicity of many Gaussian anomalous diffusion processes.

*Fractional Brownian motion*. The first example considered here is the celebrated fractional Brownian motion (FBM). FBM is the mean-zero Gaussian process (denoted here by $B_H(t)$), whose autocovariance function is given by

$$\mathbb{E}[B_H(s)B_H(t)] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad t, s \geq 0.$$

Here $0 < H < 1$ is the Hurst index. For $H = 1/2$ FBM reduces to the standard Brownian motion. The mean-square dispacement of FBM equals $\mathbb{E}[B_H^2(t)] = t^{2H}$, which for $H < 1/2$ gives the subdiffusive dynamics, whereas for $H > 1/2$ we obtain superdiffusion. Moreover, FBM has stationary increments. The stationary sequence of FBM increments $b_H(j) = B_H(j+1) - B_H(j)$ is very strongly correlated. One can show that the autocorrelation function of $b_H(j)$ satisfies

$$r(j) = \mathbb{E}[b_H(j)b_H(0)] \sim H(2H-1)j^{2H-2}$$

as $j \to \infty$. The last result immediately implies that $r(j) \to 0$ as $j \to \infty$. Thus, by condition (4), the stationary increments of FBM are mixing, and therefore also ergodic.

*Langevin equation with fractional Gaussian noise*. In a similar way we prove ergodicity of the stochastic process defined by the following Langevin equation with fractional Gaussian noise:

$$dW_H(t) = -\lambda W_H(t)dt + \sigma dB_H(t), \quad \lambda, \sigma > 0. \tag{5}$$

This equation was studied in detail in [28]. The corresponding Kramers problem was considered in [29]. The stationary solution of (5) has the form

$$W_H(t) = \sigma \int_{-\infty}^{t} e^{-\lambda(t-s)} dB_H(s).$$

Moreover, its autocorrelation function satisfies [28]

$$r(t) \propto t^{2H-2}$$

as $t \to \infty$. Therefore $r(t) \to 0$ as $t \to \infty$, which, by (4), implies that $W_H(t)$ is mixing and ergodic.

*Fractional Langevin equation.* Another important Gaussian model of anomalous diffusion (subdiffusion) is the fractional Langevin equation (FLE). FLE is the generalized Langevin equation (developed in [30,31]) with power-law memory kernel. FLE for a single particle of mass $m$ in the absence of external force has the form

$$m\frac{dV}{dt} = -\gamma \int_0^t \frac{1}{(t-u)^\beta} V(u)du + \sigma \frac{dB_H(t)}{dt}, \tag{6}$$

where $\gamma > 0$ is the friction constant and $0 < \beta < 1$ is the fractional exponent. Moreover, $\frac{dB_H(t)}{dt}$ is the fractional Gaussian noise with the Hurst parameter $H > 1/2$ satisfying the relation $\beta = 2 - 2H$.

Solution of the above FLE is a stationary Gaussian process [32] with the autcovariance function $c(t) = \mathbb{E}[V(t)V(0)]$ whose Laplace transform yields [31]

$$\widetilde{c}(\omega) = \frac{1}{\omega + c\omega^{\beta-1}}.$$

Here, $c > 0$ is the appropriate constant. Consequently, from the Tauberian theorem, the corresponding autocorrelation function satisfies $r(t) \to 0$ as $t \to \infty$. Thus, the process $V(t)$ is mixing and ergodic. Similarly, one can prove ergodicity of FLE in the presence of a harmonic potential [33].

We underline that an analogous method of verifying ergodic properties, based on the analysis of the asymptotic behavior of the autocorrelation function, can be applied to other stationary Gaussian processes.

## 3. Ergodic properties of Lévy-driven processes

The classical Khinchin theorem can be successfully applied under the assumption that the second moment (and thus the autocorrelation function) of the considered process is finite. This assumption is obviously fulfilled by the previously discussed family of Gaussian processes. However, different methods need to be used in order to analyze processes with infinite second moment, in particular $\alpha$-stable processes with $0 < \alpha < 2$ [22,34,35]. These processes, being the natural models for Lévy flight dynamics, have found widespread applications in various areas of physics (see [36] and references therein).

In what follows, we consider stochastic processes of the general form

$$Y(t) = \int_{-\infty}^{\infty} K(t,x)dL(x), \quad t \in \mathbb{R}. \tag{7}$$

Here, $K(t,x)$ is the nonnegative deterministic integration kernel and $L(x)$ is the driving Lévy process with stationary and independent increments [37]. The Fourier transform of $L(x)$ is given by (Lévy–Khinchin formula)

$$\mathbb{E}[\exp(izL(x))] = \exp(x\Psi(z)),$$

where

$$\Psi(z) = i\mu z - \sigma^2 z^2/2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{\{|x|<1\}})Q(dx).$$

Here, $\mu \in \mathbb{R}$ is the drift parameter, $\sigma^2$ is the variance of the Gaussian part of $Y(t)$ and $Q$ is the so-called Lévy measure of $Y(t)$, see [37] for the details. Since the case of Gaussian processes was discussed in detail in the previous section, we further assume that the Gaussian part of $Y(t)$ disappears

($\mu = \sigma^2 = 0$). Additionally, we use the assumption that $Y(t)$ is stationary (the system is in thermal equilibrium).

The list of physically relevant processes and distributions, which can be represented in the form (7), includes $\alpha$-stable, tempered $\alpha$-stable, exponential, gamma, Poisson, Pareto, Linnik, Mittag-Leffler, to name only a few.

A challenging problem we are going to discuss now is how to verify ergodicity of the stationary process $Y(t)$ admitting representation (7). Note that, in general, the second moment of $Y(t)$ can be infinite. Thus, the standard autocorrelation function is not an appropriate tool for analyzing ergodicity and mixing of $Y(t)$. This implies that the classical Khinchin theorem cannot be applied for processes of the general form (7). Therefore, it is necessary to use a different mathematical tool, which will substitute the autocorrelation function. In a recent paper by Eliazar and Klafter [38], the authors introduce a new concept of the *Lévy Correlation Cascade*, which is a promising tool for studying the dependence structure of Lévy-driven processes. Lévy Correlation Cascade corresponding to the process $Y(t)$ admitting representation (7) is defined as [38]

$$C_l(t_1, \ldots, t_n) = \int_{-\infty}^{\infty} \Lambda\left(l \cdot \min\{|K(t_1, x)|, \ldots, |K(t_n, x)|\}^{-1}\right) dx, \tag{8}$$

where $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$, $l > 0$ is the resolution level, and $\Lambda(z)$ is the tail of the Lévy measure $Q$, i.e. $\Lambda(z) = Q(\{x : |x| > z\})$. The multi-dimensional function $C_l(t_1, \ldots, t_n)$ takes the role of an $n$-point correlation function. In particular, for stationary $Y(t)$, the function

$$R_l(t) = \frac{C_l(0, t)}{\sqrt{C_l(0)}\sqrt{C_l(t)}} = c_l \int_{-\infty}^{\infty} \Lambda\left(l \cdot \min\{|K(0, x)|, |K(t, x)|\}^{-1}\right) dx \tag{9}$$

is the Lévy analogue of the autocorrelation function. Here $c_l > 0$ is the appropriate constant. $R_l(t)$ is called the Lévy autocorrelation function. It is instructive to discuss the interpretation of $R_l(t)$. Let us denote by $\nu_{0t}$ the Lévy measure of the vector $(Y(0), Y(t))$. It means that the characteristic function of $(Y(0), Y(t))$ can be written as [37]

$$\begin{aligned}&\mathbb{E}[\exp(i(z_1 Y(0) + z_2 Y(t)))]\\ &\quad = \exp\left(\int_{\mathbb{R}^2} (e^{i(z_1x_1+z_2x_2)} - 1 - i(z_1x_1 + z_2x_2)\mathbf{1}_{\{|x_1^2+x_2^2|<1\}})\nu_{0t}(dx_1, dx_2)\right).\end{aligned}$$

As shown in [39], the following relationship between $R_l(t)$ and the Lévy measure $\nu_{0t}$ of the vector $(Y(0), Y(t))$ holds:

$$R_l(t) = c \cdot \nu_{0t}\{(x, y) : \min\{|x|, |y|\} > l\}. \tag{10}$$

Here, $c > 0$ is the appropriate constant. The set $A = \{(x, y) : \min\{|x|, |y|\} > l\}$ is depicted in Fig. 1. The above relationship indicates that $R_l(t)$ tells us how much mass of the measure $\nu_{0t}$ is concentrated beyond the axes $OX$ and $OY$. Noting that independence of the coordinates of the vector $(Y(0), Y(t))$ is equivalent to the fact that the whole mass of $\nu_{0t}$ is concentrated on the axes, $R_l(t)$ actually tells us how dependent $Y(0)$ and $Y(t)$ are. This is just analogous to the interpretation of the autocorrelation function in the Gaussian case! For more details on the general properties of Correlation Cascades, see [38,40,41].

As we will show, $R_l(t)$ plays a fundamental role in determining ergodic properties of Lévy-driven processes (7). We start with the following extension of the Khinchin theorem to Lévy-driven processes.

**Theorem 1** (*Generalized Khinchin Theorem*)**.** *If the Lévy autocorrelation function $R_l(t)$ corresponding to $Y(t)$ given by (7) satisfies*

$$\lim_{t\to\infty} R_l(t) = 0 \quad \textit{for every } l > 0, \tag{11}$$

*then the process $Y(t)$ is ergodic. Moreover, then the Boltzmann hypothesis is true; i.e., the temporal and ensemble averages coincide:*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(Y(t))dt = \mathbb{E}[f(Y(0))],$$

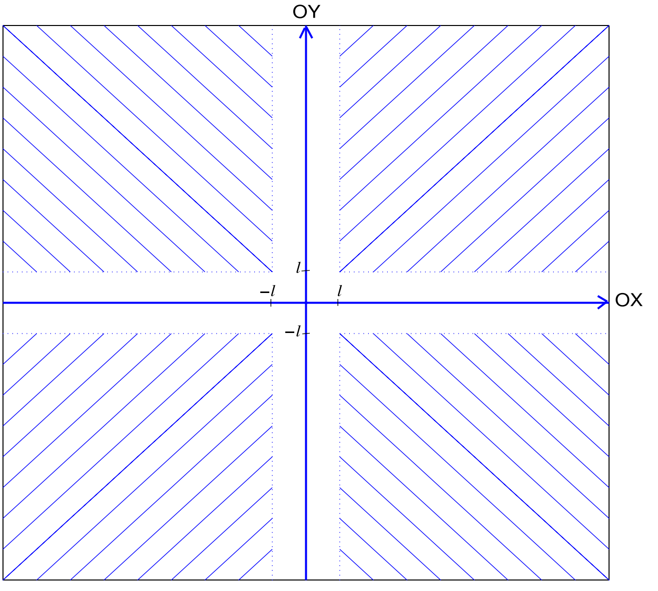


**Fig. 1.** The hashed area is the set $A = \{(x, y) : \min\{|x|, |y|\} > l\}$. The Lévy autocorrelation function $R_l(t)$ is equal (up to a constant) to $\nu_{0t}\{A\}$. This property allows us to interpret $R_l(t)$ as a measure of dependence for Lévy-driven processes. See Section 3 for the details.

*provided that* $\mathbb{E}[|f(Y(0))|] < \infty$.

To prove the above theorem, let us first note that the refined version of the classical Maruyama's mixing theorem [42,43] states that $Y(t)$ is mixing if and only if the Lévy measure $\nu_{0t}$ of the vector $(Y(0), Y(t))$ satisfies

$$\lim_{t\to\infty} \nu_{0t}\{(x, y) : |xy| > l\} = 0 \quad \text{for every } l > 0. \tag{12}$$

Thus, using (10) we get that

$$R_l(t) \xrightarrow[t\to\infty]{} 0 \quad \text{for every } l > 0$$

if and only if (12) is satisfied. Consequently, since mixing is stronger than ergodicity, we obtain that condition (11) implies ergodicity. The Boltzmann hypothesis is the immediate consequence of the Birkhoff ergodic theorem. This ends the proof.

The above considerations actually show that condition (11) is equivalent to mixing. This fact together with Theorem 1 demonstrates that $R_l(t)$ is a very powerful mathematical tool for studying ergodic properties of Lévy-driven processes. Condition (11) can be viewed as the Lévy analogue of (4) for the Gaussian case. Another great advantage of the Lévy autocorrelation function is the fact that $R_l(t)$ can be easily calculated for many relevant processes (see examples below).

In what follows, we apply the above theoretical results to some specific classes of Lévy-driven processes.

### *3.1. $\alpha$-stable processes*

The first considered example is the $\alpha$-stable processes. In this case, the process $L(x)$ in (7) is the $\alpha$-stable Lévy process, $0 < \alpha < 2$, with the corresponding Lévy measure $Q$ given by

$$Q(dx) = c\frac{1+\theta}{2}|x|^{-\alpha-1}\mathbf{1}_{\{x<0\}}dx + c\frac{1-\theta}{2}x^{-\alpha-1}\mathbf{1}_{\{x\geq 0\}}dx.$$

Here, $c > 0$ and $-1 \leq \theta \leq 1$ is the assymetry parameter. The Fourier transform of $L(x)$ is given by the well-known formula

$$\log \mathbb{E}[\exp(ikL(x))] = \begin{cases} -xc|k|^{\alpha}(1 + i\theta sgn(k)\tan(\pi\alpha/2)), & \alpha \neq 1 \\ -xc|k|(1 - 2i\theta sgn(k)\log(|k|)/\pi), & \alpha = 1. \end{cases}$$

The corresponding Lévy autocorrelation function $R_l(t)$ is given by the simple formula

$$R_l(t) = \int_{-\infty}^{\infty} \min\{|K(0,x)|, |K(t,x)|\}^{\alpha} dx.$$

The above formula shows that in the $\alpha$-stable case $R_l(t)$ does not depend on the parameter $l$ ($R_l(t)$ is resolution-free). Thus, condition (11) simplifies significantly and looks very much like (4) for the Gaussian case. It should be added that the $\alpha$-stable case is the only one for which $R_l(t)$ is resolution-free.

Ergodic properties of $\alpha$-stable processes in the context of the Lévy autocorrelation function were considered in detail in the recent paper [5], where it was shown that $\alpha$-stable Ornstein–Uhlenbeck processes, fractional $\alpha$-stable noises and $\alpha$-stable moving averages are ergodic and mixing. Also in [5], ergodicity breaking was reported for the family of $\alpha$-stable harmonizable processes.

### 3.2. *Tempered $\alpha$-stable processes*

The second example is the tempered $\alpha$-stable processes. Their relevance in physics stems from the fact that tempered $\alpha$-stable processes are extremely useful in the modelling of the transition from anomalous to normal diffusion [44]. Moreover, tempered stable distributions have finite moments of all orders; on the other hand they are similar to stable laws in many aspects [45]. Applications of the tempered stable distributions and processes in the context of astrophysics and relaxation can be found in [46]. For the Fokker–Planck equation describing tempered dynamics, see [46,47]. Other important applications related to finance and geophysics can be found in [48–50], respectively.

In the tempered stable case, the process $L(x)$ in representation (7) is the tempered $\alpha$-stable Lévy process, $0 < \alpha < 2$, with the corresponding Lévy measure $Q$ given by

$$Q(dx) = c\frac{1+\theta}{2}|x|^{-\alpha-1}e^{-\lambda|x|}\mathbf{1}_{\{x<0\}}dx + c\frac{1-\theta}{2}x^{-\alpha-1}e^{-\lambda x}\mathbf{1}_{\{x\geq 0\}}dx.$$

Here, $0 < \alpha \leq 2$, $c > 0$, $-1 \leq \theta \leq 1$, and $\lambda > 0$ is the tempering parameter. The Fourier transform of $L(x)$ is given by [44]

$$\log \mathbb{E}[\exp(ikL(x))] = \begin{cases} -\dfrac{xc}{2\cos(\pi\alpha/2)}[(1+\theta)(\lambda+ik)^{\alpha} + (1-\theta)(\lambda-ik)^{\alpha} - 2\lambda^{\alpha}] \\ -\dfrac{xc}{2\cos(\pi\alpha/2)}[(1+\theta)(\lambda+ik)^{\alpha} + (1-\theta)(\lambda-ik)^{\alpha} - 2\lambda^{\alpha} - 2ik\alpha\theta\lambda^{\alpha-1}] \end{cases}$$

for $0 < \alpha < 1$ and $1 < \alpha \leq 2$, respectively. The corresponding Lévy autocorrelation function $R_l(t)$ is given by

$$R_l(t) = c_l \int_{-\infty}^{\infty} \Lambda\left(l \cdot \min\{|K(0,x)|, |K(t,x)|\}^{-1}\right) dx,$$

where the tail function $\Lambda(z)$, $z > 0$, satisfies

$$\Lambda(z) = c_3 \int_{z}^{\infty} x^{-\alpha-1}e^{-\lambda x} dx$$

for some $c_3 > 0$. Clearly

$$\Lambda(z) \leq c_4 z^{-\alpha}$$

for the appropriate constant $c_4 > 0$. Therefore

$$R_l(t) \leq \int_{-\infty}^{\infty} \min\{|K(0,x)|, |K(t,x)|\}^{\alpha} dx. \tag{13}$$

Now, we are in position to discuss ergodic properties of the following specific examples of tempered $\alpha$-stable processes.

*Tempered $\alpha$-stable Ornstein–Uhlenbeck process.* This process is defined as

$$Y_1(t) = \sigma \int_{-\infty}^{t} e^{-\kappa(t-x)} dL(x), \quad \kappa, \sigma > 0.$$

Recall that here $L(x)$ is the tempered $\alpha$-stable Lévy process. Using (13) with $K(t,x) = \sigma e^{-\kappa(t-x)} \mathbf{1}_{\{x<t\}}$, we get that the Lévy autocorrelation function corresponding to $Y_1(t)$ satisfies

$$R_l(t) \le ce^{-\alpha\kappa t}$$

for some appropriately large $c > 0$. Thus, condition (11) is satisfied. Therefore, $Y_1(t)$ is mixing and ergodic.

*Tempered $\alpha$-stable noise*. The process of increments of the tempered $\alpha$-stable Lévy process, defined as

$$l(t) = L(t+1) - L(t), \quad t \in \mathbb{N},$$

is a stationary sequence of independent and identically distributed random variables. $l(t)$ is called the tempered $\alpha$-stable noise. It can be represented as

$$l(t) = L(t+1) - L(t) = \int_{t}^{t+1} dL(x) = \int_{-\infty}^{\infty} \mathbf{1}_{\{t<x<t+1\}} dL(x).$$

Therefore, by (13), the Lévy autocorrelation function of $l(t)$ is equal to zero. This corresponds to the well known property that the autocorrelation of independent random variables is equal to zero. Since condition (11) is satisfied, the tempered $\alpha$-stable noise $l(t)$ is ergodic and mixing.

*Fractional tempered $\alpha$-stable noise*. Let $0 < \alpha \le 2$, $0 < H < 1$, $H > 1/\alpha$. Then, the process

$$L_H(t) = \int_{-\infty}^{\infty} \left[ (t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha} \right] dL(x) \tag{14}$$

is called the fractional tempered $\alpha$-stable motion. Here $x_+ = \max\{x, 0\}$. The stationary process of increments of $L_H(t)$ defined as

$$l_H(t) = L_H(t+1) - L_H(t) = \int_{-\infty}^{\infty} \left[ (t+1-x)_+^{H-1/\alpha} - (t-x)_+^{H-1/\alpha} \right] dL(x),$$

$t \in \mathbb{N}$, is called the fractional tempered $\alpha$-stable noise. Contrary to the above considered noise $l(t)$, the dependence between even very distant time points of $l_H(t)$ is very strong. Therefore, the fractional tempered $\alpha$-stable noise is often used to model phenomena displaying long memory [51,52]. Using (13) we get that

$$R_l(t) \le c \left( \frac{1}{t} \right)^{\alpha(1-H)}$$

for the appropriate constant $c > 0$. Therefore, the Lévy autocorrelation function of $l_H(t)$ yields

$$\lim_{t\to\infty} R_l(t) = 0.$$

This implies that the fractional tempered $\alpha$-stable noise is ergodic and mixing.

## 4. Ergodic properties of fractional Ornstein–Uhlenbeck processes

The classical O–U process is one of the most fundamental processes in statistical physics. It gives the foundation of linear nonequilibrium thermodynamics and describes the velocity in the celebrated Klein–Kramers model. The O–U process is defined as a stationary solution of the Langevin equation for the overdamped oscillator

$$dZ(t) = -\lambda Z(t)dt + dB(t), \quad \lambda > 0, \tag{15}$$

with $B(t)$ being the standard Brownian motion. The corresponding Fokker–Planck equation, describing the probability density function of $Z(t)$, has the form

$$\frac{\partial f(x,t)}{\partial t} = \left[ \frac{\partial}{\partial x} \lambda x + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right] f(x,t). \tag{16}$$

It can be easily verified that the autocorrelation function of $Z(t)$ satisfies $r(t) = \exp\{-\lambda t\}$, therefore the O–U process is ergodic and mixing.

Interestingly enough, the O–U process can be equivalently defined via the so-called Lamperti transformation [53,54] from the Brownian motion $B(t)$, i.e.

$$Z(t) = e^{-\lambda t} B(e^{2\lambda t}). \tag{17}$$

This curious fact shows that by the appropriate stretching of time and shrinking of space, self-similar Brownian motion can be transformed into the stationary O–U process.

The fractional (subdiffusive) O–U process is defined by the celebrated fractional Fokker–Planck equation [36]

$$\frac{\partial w(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial}{\partial x}\lambda x + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right] w(x,t), \tag{18}$$

which is a fractional extension of (16). Here, the operator ${}_0D_t^{1-\alpha}$, $0 < \alpha < 1$ is the fractional Riemann–Liouville derivative. The above equation was derived in the framework of continuous-time random walk (CTRW) with heavy-tailed waiting times. The Langevin-type process corresponding to (18) has the form [55–58]

$$Z_1(t) = Z(S_\alpha(t)), \tag{19}$$

where $Z(t)$ is given by (15) and $S_\alpha(t)$ is the inverse $\alpha$-stable subordinator independent of $Z(t)$. $S_\alpha(t)$ is defined in the following way:

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}, \tag{20}$$

where $U_\alpha(t)$ is the $\alpha$-stable subordinator [22,37] with Laplace transform given by $E\left(e^{-uU_\alpha(t)}\right) = e^{-tu^\alpha}$, $0 < \alpha < 1$.

On the other hand, following the same lines as in (17) we can define the fractional O–U process via the Lamperti transormation. Since the force-free subdiffusion process $B(S_\alpha(t))$ is $\alpha/2$-self-similar, the Lamperti transformation from $B(S_\alpha(t))$ leads to the following fractional O–U process:

$$Z_2(t) = e^{-\alpha\lambda t} B(S_\alpha(e^{2\lambda t})). \tag{21}$$

In the classical case, the Fokker–Planck equation (16) and the Lamperti transformation (17) lead to the same O–U process $Z(t)$. However, this is no longer true in the fractional case. It appears that $Z_1(t)$ and $Z_2(t)$ are two different fractional O–U processes, both originating from subdiffusive dynamics. As shown in [59], the process $Z_1(t)$ displays ergodicity breaking, since its time average does not vanish for long times. However, our next result shows that the second fractional O–U process $Z_2(t)$ preserves the mixing property. Therefore, it is also ergodic and the Boltzmann hypothesis holds. This assures that $Z_1(t)$ and $Z_2(t)$ are two very different fractional O–U processes.

**Theorem 2.** *The fractional O–U process $Z_2(t)$ defined in (21) is mixing.*

**Proof.** See Appendix. □

## 5. Conclusions

In this paper we have analyzed ergodic properties of some classes of anomalous diffusion processes. Within the family of Gaussian processes, we have shown that both the Langevin equation with fractional Gaussian noise and the fractional Langevin equation are ergodic and mixing.

Based on the concept of Lévy Correlation Cascades, we have developed a rigorous mathematical approach to analyze ergodic properties of the large class of Lévy-driven processes. We have extended the classical Khinchin ergodic theorem to the family of Lévy-driven processes. As an example, we have studied the subclass of tempered $\alpha$-stable processes in some detail.

Finally, we have verified the chaotic properties of two different fractional Ornstein–Uhlenbeck processes, both originating from subdiffusive dynamics. We have shown that only one of them is ergodic and mixing.

It should be added that the ergodic properties of infinitely divisible processes can be analyzed using different tools. The approach based on the so-called dynamical functional can be found in [22,62,63], the relationship between codifference and ergodicity is discussed in detail in [64,65].

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## Appendix

**Proof of Theorem 2.** Let us put for simplicity $\lambda = 1/2$ (the proof for arbitrary $\lambda > 0$ is analogous, only the notation is more complicated). In order to prove that $Z_2(t)$ is mixing it suffices to show that [42]

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{m} z_k Z_2(s_k) + i\sum_{k=m+1}^{n} z_k Z_2(s_k+t)\right)\right] \xrightarrow{t\to\infty} \mathbb{E}\left[\exp\left(i\sum_{k=1}^{m} z_k Z_2(s_k)\right)\right]\mathbb{E}\left[\exp\left(i\sum_{k=m+1}^{n} z_k Z_2(s_k)\right)\right], \tag{22}$$

for every $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \le m \le n$, $z_1, \ldots, z_n \in \mathbb{R}$ and $0 < s_1 < \cdots < s_n$.

In what follows, we will show a more general result, namely

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{m} z_k e^{-s_k\alpha/2} B(S_\alpha(e^{s_k}-x)) + i\sum_{k=m+1}^{n} z_k e^{-(s_k+t)\alpha/2} B(S_\alpha(e^{s_k+t}-x))\right)\right] \xrightarrow{t\to\infty} \mathbb{E}\left[\exp\left(i\sum_{k=1}^{m} z_k e^{-s_k\alpha/2} B(S_\alpha(e^{s_k}-x))\right)\right]\mathbb{E}\left[\exp\left(i\sum_{k=m+1}^{n} z_k e^{-s_k\alpha/2} B(S_\alpha(e^{s_k}))\right)\right], \tag{23}$$

where $x$ is an arbitrary real number satisfying $0 \le x \le e^{s_1}$. Note that for $x = 0$ we obtain (22).

We will prove formula (23) by induction on the parameter $n$.

Step I. Let $n = 1$. We consider two cases:

(i) First let $m = 0$. Then, we have

$$\mathbb{E}\left[\exp\left(iz_1 e^{-(s_1+t)\alpha/2} B(S_\alpha(e^{s_1+t}-x))\right)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{2}z_1^2 e^{-(s_1+t)\alpha} S_\alpha(e^{s_1+t}-x)\right)\right]$$
$$= E_\alpha\left(-\frac{1}{2}z_1^2 e^{-(s_1+t)\alpha}(e^{s_1+t}-x)^\alpha\right) \xrightarrow{t\to\infty} E_\alpha\left(-\frac{1}{2}z_1^2\right) = \mathbb{E}\left[\exp\left(iz_1 e^{-s_1\alpha/2} B(S_\alpha(e^{s_1}))\right)\right],$$

which proves expression (23). Here, we have used the fact that the Laplace transform of $S_\alpha(t)$ is given by $\mathbb{E}[\exp(-zS_\alpha(t))] = E_\alpha(-zt^\alpha)$, with $E_\alpha(\cdot)$ being the Mittag-Leffler function [60].

(ii) For $m = 1$, formula (23) is trivially fulfilled.

Step II. In the second step of induction, let us assume that formula (23) holds for some $n - 1$. We will show that (23) also holds for $n$.

We will use the following elementary formula for Brownian motion:

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{n} a_k B(\tau_k)\right)\right] = \exp\left(-\frac{1}{2}\sum_{k=1}^{n}\tau_k\left(a_k^2 + 2a_k\sum_{j=k+1}^{n} a_j\right)\right), \tag{24}$$

where $n \ge 1$, $0 \le \tau_1 \le \cdots \le \tau_n$, $a_1, \ldots, a_n \in \mathbb{R}$. We will also use the following recurrence relation [61]:

$$\mathbb{E}\left[\exp\left(\sum_{k=1}^{n}\theta_k S_\alpha(\tau_k)\right)\right] = \mathbb{E}\left[\exp\left(\sum_{k=2}^{n}\theta_k S_\alpha(\tau_k)\right)\right] + \frac{\theta_1}{\sum_{k=1}^{n}\theta_k}\int_0^{\tau_1}\mathbb{E}\left[\exp\left(\sum_{k=2}^{n}\theta_k S_\alpha(\tau_k - x)\right)\right] d_x\mathbb{E}\left[\exp\left(S_\alpha(x)\sum_{k=1}^{n}\theta_k\right)\right], \tag{25}$$

where $n \ge 2$, $0 \le \tau_1 \le \cdots \le \tau_n$, $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

In order to complete the second step of induction, we consider two cases:
(i) First, let $m = 0$. Set

$$a_k = z_k e^{-(s_k+t)\alpha/2}, \qquad \widetilde{a}_k = z_k e^{-s_k\alpha/2},$$
$$c_k = \left(a_k^2 + 2a_k \sum_{j=k+1}^{n} a_j\right), \qquad \widetilde{c}_k = \left(\widetilde{a}_k^2 + 2\widetilde{a}_k \sum_{j=k+1}^{n} \widetilde{a}_j\right)$$

for $k = 1, \ldots, n$. Then, applying (24) and (25), we obtain that the left side of expression (23) is equal to

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=1}^{n} c_k S_\alpha(e^{s_k+t} - x)\right)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{n} c_k S_\alpha(e^{s_k+t} - x)\right)\right]$$
$$+ \frac{-\frac{1}{2}c_1}{-\frac{1}{2}\sum\limits_{k=1}^{n} c_k}\int_0^{e^{s_1+t}-x} \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{n} c_k S_\alpha(e^{s_k+t} - x - y)\right)\right] d_y \mathbb{E}\left[\exp\left(-\frac{1}{2}S_\alpha(y)\sum_{k=1}^{n} c_k\right)\right].$$

Now, applying the fact that $\mathbb{E}[\exp(-zS_\alpha(t))] = E_\alpha(-zt^\alpha)$ and by the change of variables $y \to u(\sum_{k=1}^{n} c_k)^{-1/\alpha}$, we get that the above formula is equal to

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{n} c_k S_\alpha(e^{s_k+t} - x)\right)\right] + \frac{-\frac{1}{2}c_1}{-\frac{1}{2}\sum\limits_{k=1}^{n} c_k}$$
$$\times \int_0^{(e^{s_1+t}-x)(\sum\limits_{k=1}^{n} c_k)^{1/\alpha}} \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{n} c_k S_\alpha\left(e^{s_k+t} - x - u\left(\sum_{k=1}^{n} c_k\right)^{-1/\alpha}\right)\right)\right] d_u E_\alpha\left(-\frac{1}{2}u^\alpha\right).$$

Consequently, applying the induction assumption and the dominated convergence theorem, we get that for $t \to \infty$ the above formula converges to

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{n} \widetilde{c}_k S_\alpha(e^{s_k})\right)\right]$$
$$+ \frac{-\frac{1}{2}\widetilde{c}_1}{-\frac{1}{2}\sum\limits_{k=1}^{n} \widetilde{c}_k}\int_0^{e^{s_1}(\sum\limits_{k=1}^{n} \widetilde{c}_k)^{1/\alpha}} \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{n} \widetilde{c}_k S_\alpha\left(e^{s_k} - u\left(\sum_{k=1}^{n} \widetilde{c}_k\right)^{-1/\alpha}\right)\right)\right] d_u E_\alpha\left(-\frac{1}{2}u^\alpha\right)$$
$$= \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{n} \widetilde{c}_k S_\alpha(e^{s_k})\right)\right]$$
$$+ \frac{-\frac{1}{2}\widetilde{c}_1}{-\frac{1}{2}\sum\limits_{k=1}^{n} \widetilde{c}_k}\int_0^{e^{s_1}} \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{n} \widetilde{c}_k S_\alpha\left(e^{s_k} - y\right)\right)\right] d_y \mathbb{E}\left[\exp\left(-\frac{1}{2}S_\alpha(y)\sum_{k=1}^{n} \widetilde{c}_k\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=1}^{n} \widetilde{c}_k S_\alpha(e^{s_k})\right)\right] = \mathbb{E}\left[\exp\left(i\sum_{k=1}^{n} z_k e^{-s_k\alpha/2} B(S_\alpha(e^{s_k}))\right)\right].$$

This ends the proof for the case $m = 0$.
(ii) The second case is $m \geq 1$. Put

$$a_k = z_k e^{-s_k\alpha/2} \quad \text{for } k = 1, \ldots, m,$$
$$a_k = z_k e^{-(s_k+t)\alpha/2} \quad \text{for } k = m+1, \ldots, n,$$
$$c_k = \left(a_k^2 + 2a_k \sum_{j=k+1}^{n} a_j\right) \quad \text{for } k = 1, \ldots, n.$$

Then, applying (24) and (25), we get that the left side of expression (23) equals

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=1}^{m} c_k S_\alpha(e^{s_k}-x) - \frac{1}{2}\sum_{k=m+1}^{n} c_k S_\alpha(e^{s_k+t}-x)\right)\right]$$
$$= \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{m} c_k S_\alpha(e^{s_k}-x) - \frac{1}{2}\sum_{k=m+1}^{n} c_k S_\alpha(e^{s_k+t}-x)\right)\right]$$
$$+ \frac{-\frac{1}{2}c_1}{-\frac{1}{2}\sum\limits_{k=1}^{n} c_k}\int_0^{e^{s_1}-x} \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{m} c_k S_\alpha(e^{s_k}-x-y)\right.\right.$$
$$\left.\left.-\frac{1}{2}\sum_{k=m+1}^{n} c_k S_\alpha(e^{s_k+t}-x-y)\right)\right] d_y \mathbb{E}\left[\exp\left(-\frac{1}{2}S_\alpha(y)\sum_{k=1}^{n} c_k\right)\right]. \tag{26}$$

Put $\widetilde{c}_k = \left(a_k^2 + 2a_k \sum_{j=k+1}^{m} a_j\right)$, $k = 1, \ldots, m$. Now, applying the induction assumption and the dominated convergence theorem, we get that the right side of Eq. (26) converges to (as $t \to \infty$)

$$\mathbb{E}\left[\exp\left(i\sum_{k=m+1}^{n} z_k e^{-s_k\alpha/2} B(S_\alpha(e^{s_k}))\right)\right]\left\{\mathbb{E}\left[\exp\left(i\sum_{k=2}^{m} z_k e^{-s_k\alpha/2} B(S_\alpha(e^{s_k}-x))\right)\right]\right.$$
$$\left.+ \frac{-\frac{1}{2}\widetilde{c}_1}{-\frac{1}{2}\sum\limits_{k=1}^{n}\widetilde{c}_k}\int_0^{e^{s_1}-x} \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=2}^{m}\widetilde{c}_k S_\alpha(e^{s_k}-x-y)\right)\right] d_y \mathbb{E}\left[\exp\left(-\frac{1}{2}S_\alpha(y)\sum_{k=1}^{m}\widetilde{c}_k\right)\right]\right\}$$
$$= \mathbb{E}\left[\exp\left(i\sum_{k=m+1}^{n} z_k e^{-s_k\alpha/2} B(S_\alpha(e^{s_k}))\right)\right]\mathbb{E}\left[\exp\left(i\sum_{k=1}^{m} z_k e^{-s_k\alpha/2} B(S_\alpha(e^{s_k}-x))\right)\right].$$

This yields (23) and ends the proof of the theorem. □

## References


[1] G. Gallavotti, Statistical Mechanics, Springer-Verlag, New York, 1999.
[2] M. Badino, Found. Sci. 11 (2006) 323–347.
[3] L. Sklar, Physics and Chance, Cambridge University Press, Cambridge, 1993.
[4] A.I. Khinchin, Mathematical Foundations of Statistical Mechanics, Dover, New York, 1949.
[5] A. Weron, M. Magdziarz, Phys. Rev. Lett. 105 (2010) 260603.
[6] X. Brokmann, J.P. Hermier, G. Messin, P. Desbiolles, J.P. Bouchaud, M. Dahan, Phys. Rev. Lett. 90 (2003) 120601.
[7] G. Margolin, E. Barkai, Phys. Rev. Lett. 94 (2005) 080601.
[8] G. Bel, E. Barkai, Phys. Rev. Lett. 94 (2005) 240602.
[9] A. Rebenshtok, E. Barkai, Phys. Rev. Lett. 99 (2007) 210601.
[10] S. Burov, R. Metzler, E. Barkai, Proc. Natl. Acad. Sci. USA 107 (2010) 13228.
[11] B. Dybiec, J. Stat. Mech. (2009) P08025.
[12] J.D. Bao, P. Hänggi, Y.Z. Zhuo, Phys. Rev. E 72 (2005) 061107.
[13] L.C. Lapas, R. Morgado, M.H. Vainstein, J.M. Rubí, F.A. Oliveira, Phys. Rev. Lett. 101 (2008) 230602.
[14] M.H. Lee, Phys. Rev. Lett. 87 (2001) 250601.
[15] I.V.L. Costa, R. Morgado, M.V.B.T. Lima, F.A. Oliveira, Europhys. Lett. 63 (2003) 173–179.
[16] I.V.L. Costa, et al., Physica A 371 (2006) 130–134.
[17] A. Dhar, K. Wagh, Europhys. Lett. 79 (2007) 60003.
[18] M.H. Lee, Phys. Rev. Lett. 98 (2007) 190601.
[19] W. Deng, E. Barkai, Phys. Rev. E 79 (2009) 011112.
[20] J.H. Jeon, R. Metzler, Phys. Rev. E 81 (2010) 021103.
[21] A. Lasota, M.C. Mackey, Chaos, Fractals, and Noise. Stochastic Aspects of Dynamics, Springer-Verlag, New York, 1994.
[22] A. Janicki, A. Weron, Simulation and Chaotic Behaviour of $\alpha$-Stable Stochastic Processes, Marcel Dekker, New York, 1994.
[23] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag, New York, 1988.
[24] G. Maruyama, Mem. Fac. Sci. Kyushu Univ. 4 (1949) 45–106.
[25] U. Grenander, Ark. Mat. 1 (1950) 195–277.
[26] K. Itô, Proc. Imp. Acad. 20 (1944) 54–55.
[27] H. Totoki, Mem. Fac. Sci. Kyushu Univ. 18 (1964) 136–139.

[28] P. Cheridito, H. Kawaguchi, M. Maejima, Electron. J. Probab. 8 (2003) 1–14.
[29] O.Y. Sliusarenko, V.Y. Gonchar, A.V. Chechkin, I.M. Sokolov, R. Metzler, Phys. Rev. E 81 (2010) 041119.
[30] H. Mori, Progr. Theoret. Phys. 33 (1965) 423–455.
[31] R. Kubo, Rep. Progr. Phys. 29 (1966) 255–284.
[32] S.C. Kou, Ann. Appl. Stat. 2 (2008) 501–535.
[33] S. Burov, E. Barkai, Phys. Rev. Lett. 100 (2008) 070601.
[34] V.M. Zolotarev, One-Dimensional Stable Distributions, Amer. Math. Soc., Providence, 1986.
[35] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, Chapman and Hall, New York, 1994.
[36] R. Metzler, J. Klafter, Phys. Rep. 339 (2000) 1–77.
[37] K.-I. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge, 1999.
[38] I. Eliazar, J. Klafter, Physica A 376 (2007) 1–26.
[39] M. Magdziarz, Stochastic Process. Appl. 119 (2009) 3416–3434.
[40] I. Eliazar, J. Klafter, J. Phys. A 40 (2007) F307–F314.
[41] I. Eliazar, J. Klafter, Phys. Rev. E 75 (2007) 031108.
[42] G. Maruyama, Theory Probab. Appl. 15 (1970) 1–22.
[43] M. Magdziarz, Theory Probab. Appl. 54 (2009) 407–409.
[44] Á. Cartea, D. del-Castillo-Negrete, Phys. Rev. E 76 (2007) 041105.
[45] J. Rosinski, Stochastic Process. Appl. 117 (2007) 677–707.
[46] A. Stanislavsky, A. Weron, K. Weron, Phys. Rev. E 78 (2008) 051106.
[47] J. Gajda, M. Magdziarz, Phys. Rev. E 82 (2010) 011117.
[48] O.E. Barndorff-Nielsen, Scandinavian J. Statist. 24 (1997) 1–13.
[49] A. Matacz, Int. J. Theor. Appl. Finance 3 (2000) 143–160.
[50] M.M. Meerschaert, Y. Zhang, B. Baeumer, Geophys. Res. Lett. 35 (2008) L17403.
[51] A. Weron, K. Burnecki, S. Mercik, K. Weron, Phys. Rev. E 71 (2005) 016113.
[52] T. Marquardt, Bernoulli 12 (2006) 1009–1126.
[53] J.W. Lamperti, Trans. Amer. Math. Soc. 104 (1962) 62–78.
[54] K. Burnecki, M. Maejima, A. Weron, Yokohama Math. J. 44 (1997) 25–42.
[55] I.M. Sokolov, Phys. Rev. E 66 (2002) 041101.
[56] M.M. Meerschaert, D.A. Benson, H.P. Scheffer, B. Baeumer, Phys. Rev. E 65 (2002) 1103–1106.
[57] M. Magdziarz, A. Weron, K. Weron, Phys. Rev. E 75 (2007) 016708.
[58] B. Dybiec, E. Gudowska-Nowak, Chaos 20 (2010) 043129.
[59] L. Turgeman, S. Carmi, E. Barkai, Phys. Rev. Lett. 103 (2009) 190201.
[60] S.G. Samko, A.A. Kilbas, D.I. Maritchev, Integrals and Derivatives of the Fractional Order and Some of Their Applications, Gordon and Breach, Amsterdam, 1993.
[61] I. Kaj, A. Martin-Löf, Scaling limit results for the sum of many inverse Lévy subordinators, Preprint, Institut Mittag-Leffler, 2005.
[62] S. Cambanis, C.D. Hardin Jr., A. Weron, Stochastic Process. Appl. 24 (1987) 1–18.
[63] S. Cambanis, K. Podgorski, A. Weron, Studia Math. 115 (1995) 109–127.
[64] J. Rosinski, T. Zak, Stochastic Process. Appl. 61 (1996) 277–288.
[65] J. Rosinski, T. Zak, J. Theoret. Probab. 10 (1997) 73–86.