Correlation cascades, ergodic properties and long memory of infinitely divisible processes

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Abstract

In this paper, we investigate the properties of the recently introduced measure of dependence called correlation cascade. We show that the correlation cascade is a promising tool for studying the dependence structure of infinitely divisible processes. We describe the ergodic properties (ergodicity, weak mixing, mixing) of stationary infinitely divisible processes in the language of the correlation cascade and establish its relationship with the codifference. Using the correlation cascade, we investigate the dependence structure of four fractional $\alpha$-stable stationary processes. We detect the property of long memory and verify the ergodic properties of the discussed processes.

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1. Introduction

Let us consider an infinitely divisible (i.d.) stochastic process $(Y_t)_{t \in \mathbb{R}}$ with the following integral representation

$$Y_t = \int_X K(t, x) N(dx).$$ (1)
Here $N$ is an independently scattered i.d. random measure on some measurable space $X$ with a control measure $m$, such that for every $m$-finite set $A \subseteq X$ (Lévy–Khinchin formula)

$$
E \exp[izN(A)] = \exp \left[ m(A) \left\{ iz\mu - \frac{1}{2} \sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx1(|x| < 1)) Q(dx) \right\} \right].
$$

The random measure $N$ is fully determined by the control measure $m$, the Lévy measure $Q$, the variance of the Gaussian part $\sigma^2$ and the drift parameter $\mu \in \mathbb{R}$. Additionally, the kernel $K(t,x)$ is assumed to take only nonnegative values.

Since, in general, the second moment and thus the correlation function for the process $Y_t$ may be infinite, the key problem is, how to describe mathematically the underlying dependence structure of $Y_t$. In the recent paper by Eliazar and Klafter [7], the authors introduce a new concept of correlation cascades, which is a promising tool for studying the properties of the Poissonian part of $Y_t$ and the dependence structure of this stochastic process. They proceed in the following way: first, they introduce a Poissonian tail-rate function $\Lambda$ of the Lévy measure $Q$

$$
\Lambda(l) = \int_{|x|>l} Q(dx), \quad l > 0,
$$

next, for $t_1, \ldots, t_n \in \mathbb{R}$ and $l > 0$, they define the function

$$
C_l(t_1, \ldots, t_n) = \int_X \Lambda \left( \frac{l}{\min\{K(t_1,x), \ldots, K(t_n,x)\}} \right) m(dx),
$$

called Correlation Cascade. As shown in [7], with the help of the function $C_l(t_1, \ldots, t_n)$ one can determine the distributional properties of the Poissonian part of $Y_t$ and describe the correlation-like structure of the process. Recall that the i.d. random measure $N$ in (1) admits the following stochastic representation (Lévy–Itô formula)

$$
N(B) = \mu \cdot m(B) + N_G(B) + \int_B \int_{|y|>1} yN_P(dx \times dy)
$$

$$
+ \int_B \int_{|y|\leq1} y(N_P(dx \times dy) - m_P(dx \times dy)),
$$

where $N_G(B)$ is a Gaussian random variable with mean zero and standard deviation equal to $\sigma m^{1/2}(B)$, while $N_P$ is the Poisson point process with the control measure $m_P = m \times Q$. Now, for $l > 0$, let us introduce the random variable

$$
\Pi_l(t) = \int_X \int_{|y|>l} 1_{[|yK(t,x)|>l]} N_P(dx \times dy).
$$

$\Pi_l(t)$ has the following interpretation: it is the number of elements of the set

$$
\{yK(t,x) : (x, y) \text{is the atom of the Poisson point process}N_P\}
$$

whose absolute value is greater than the level $l$. It is interesting to see the relationship between the random variables $\Pi_l(t)$ and the correlation cascade $C_l$. As shown in [7], the following formulas, which explain the meaning of $C_l$, hold true
In what follows, we establish the relationship between $C_l(t_1, \ldots, t_n)$ and the corresponding Lévy measure of the i.d. vector $(Y_1, \ldots, Y_n)$. The result will allow us to give a new interpretation to the function $C_l(t_1, \ldots, t_n)$ and to recognize it as an appropriate tool for characterizing the dependence structure of $Y_t$. We prove the following result.

**Proposition 1.** Let $Y_t$ be of the form (1) and let $\nu_{t_1, \ldots, t_n}$ be the Lévy measure of the i.d. random vector $(Y_1, \ldots, Y_n)$. Then, the corresponding correlation cascade $C_l$ satisfies

$$C_l(t_1, \ldots, t_n) = \nu_{t_1, \ldots, t_n} \left( \{ (x_1, \ldots, x_n) : \min(|x_1|, \ldots, |x_n|) > l \} \right).$$

**Proof.** Using the relationship between the measures $Q$ and $\nu_{t_1, \ldots, t_n}$ (see [22] for the details), we obtain

$$C_l(t_1, \ldots, t_n) = \int_X A \left( \frac{l}{\min\{K(t_1, x), \ldots, K(t_n, x)\}} \right) m(dx)$$

$$= \int_X \int_{\mathbb{R}} 1 \left( |y| > \frac{l}{\min\{K(t_1, x), \ldots, K(t_n, x)\}} \right) Q(dy)m(dx)$$

$$= \int_X \int_{\mathbb{R}} 1 \left( \min\{|yK(t_1, x)|, \ldots, |yK(t_n, x)|\} > l \right) Q(dy)m(dx)$$

$$= \int_{\mathbb{R}^n} 1 \left( \min\{|x_1|, \ldots, |x_n|\} > l \right) \nu_{t_1, \ldots, t_n}(dx_1, \ldots, dx_n)$$

$$= \nu_{t_1, \ldots, t_n} \left( \{ (x_1, \ldots, x_n) : \min(|x_1|, \ldots, |x_n|) > l \} \right). \quad \square$$

Recall that for an i.d. vector $Z = (Z_1, \ldots, Z_n)$, the independence of the coordinates $Z_1, \ldots, Z_n$ is equivalent to the fact that the Lévy measure of $Z$ is concentrated on the axes. Therefore, the above result gives the following meaning to the correlation cascade. Namely, $C_l$ indicates, how much mass of the measure $\nu_{t_1, \ldots, t_n}$ is concentrated beyond the axes and their $l$-surrounding (here by $l$-surrounding we mean the set $\{ (x_1, \ldots, x_n) : \min(|x_1|, \ldots, |x_n|) \leq l \}$). Thus, the function $C_l(t_1, \ldots, t_n)$ tells us, how dependent the coordinates of the vector $(Y_1, \ldots, Y_n)$ are. Therefore, $C_l(t_1, \ldots, t_n)$ can be considered as an appropriate measure of dependence for the Poissonian part of the i.d. process $Y_t$. In particular, the function $C_l(t_1, t_2)$ can serve as an analogue of the covariance, and the function

$$r_l(t_1, t_2) = \frac{C_l(t_1, t_2)}{\sqrt{C_l(t_1)C_l(t_2)}}$$

can play the role of the correlation function.

In the next section, we describe the ergodic properties of the stationary i.d. processes of the form (1) in the language of the function $C_l$. In Section 3, we introduce the definition of long memory in the language of the correlation cascade. Next, we investigate the asymptotic dependence structure of four fractional $\alpha$-stable stationary processes. We detect the property of long memory and verify the ergodic properties of the discussed processes.
2. Ergodic properties

We begin with recalling some basic definitions. Let \((Y_t)_{t \in \mathbb{R}}\) be a stationary, i.d. stochastic process defined on the canonical space \((\mathbb{R}^\mathbb{R}, \mathcal{F}, P)\). The process \((Y_t)_{t \in \mathbb{R}}\) is said to be ergodic if

\[
\frac{1}{T} \int_0^T P(A \cap S_t'B) dt \longrightarrow P(A)P(B) \quad \text{as } T \to \infty,
\]

weakly mixing if

\[
\frac{1}{T} \int_0^T |P(A \cap S_t'B) - P(A)P(B)| dt \longrightarrow 0 \quad \text{as } T \to \infty,
\]

mixing if

\[
P(A \cap S_t'B) \longrightarrow P(A)P(B) \quad \text{as } t \to \infty,
\]

for every \(A, B \in \mathcal{F}\), where \((S_t')\) is a group of shift transformations on \(\mathbb{R}^\mathbb{R}\).

The description of the mixing property for stationary i.d. processes in terms of their Lévy characteristics dates back to the fundamental paper by Maruyama \[21\]. The independent results basing on the concept of dynamical functional were obtained in \[3, 4\], where the authors proved the equivalence of ergodicity and weak mixing in the class of i.d. stationary processes. For the purpose of this paper, we use Maruyama’s following theorem.

**Theorem** (\[21\]). An i.d. stationary process \((Y_t)_{t \in \mathbb{R}}\) is mixing if and only if the following three conditions hold

(C1) the covariance function \(r(t)\) of its Gaussian part converges to 0 as \(t \to \infty\),

(C2) \(\lim_{t \to \infty} v_{0t}(|xy| > \delta) = 0\) for every \(\delta > 0\),

(C3) \(\lim_{t \to \infty} \int_{0< x^2+y^2 \leq 1} xy v_{0t}(dx, dy) = 0\),

where \(v_{0t}\) is the Lévy measure of \((Y_0, Y_t)\).

Let us note that condition (C2) states that the Lévy measure \(v_{0t}\) is asymptotically concentrated on the axes. Since, for an i.d. process, this is equivalent to the asymptotic independence of the Poissonian parts of \(Y_0\) and \(Y_t\), the conditions (C1) and (C2) imply the asymptotic independence of \(Y_0\) and \(Y_t\), which, in view of definition (11), is the natural interpretation of mixing.

Let us prove the following lemma.

**Lemma 1.** Let \((Y_t)_{t \in \mathbb{R}}\) be an i.d. stationary process and let \(v_{0t}\) be the corresponding Lévy measure of \((Y_0, Y_t)\). Then, the following two conditions are equivalent

(i) \(\lim_{t \to \infty} v_{0t}(|xy| > \delta) = 0\) for every \(\delta > 0\),

(ii) \(\lim_{t \to \infty} v_{0t}(\min\{|x|, |y| > \delta\}) = 0\) for every \(\delta > 0\).

**Proof.** (i) \(\Rightarrow\) (ii).

We have

\[v_{0t}(\min\{|x|, |y| > \delta\}) \leq v_{0t}(|xy| > \delta^2) \longrightarrow 0\]

as \(t \to \infty\).

(ii) \(\Rightarrow\) (i).
Fix $\delta > 0$ and $\epsilon > 0$. Denote by $\nu_0$ the Lévy measure of $Y_0$. Then, there exists $n \in \mathbb{N}$, such that
\[
\nu_0(|x| > n) < \frac{\epsilon}{4}.
\]
Taking advantage of the stationarity of $Y_t$, we get
\[
\nu_0(|xy| > \delta) \leq \nu_0(\min(|x|, |y|) > \delta/n) + \nu_0(|x| > n \vee |y| > n)
\leq \nu_0(\min(|x|, |y|) > \delta/n) + \nu_0(|x| > n) + \nu_0(|y| > n)
= \nu_0(\min(|x|, |y|) > \delta/n) + 2\nu_0(|x| > n) \leq \epsilon/2 + \epsilon/2 = \epsilon
\]
for appropriately large $t$. Thus, we obtain $\nu_0(|xy| > \delta) \to 0$ as $t \to \infty$. \hfill $\square$

In what follows, we describe the ergodic properties for the i.d. stochastic processes $(Y_t)_{t \in \mathbb{R}}$ of the form (1) in the language of the function $C_l$. In what follows we assume for simplicity that the Gaussian part of $(Y_t)_{t \in \mathbb{R}}$ disappears. The next theorem is the main result of this section.

**Theorem 1.** Let $(Y_t)_{t \in \mathbb{R}}$ be a stationary i.d. process of the form (1). Then $Y_t$ is mixing iff the corresponding function $C_l$ satisfies
\[
\lim_{t \to \infty} C_l(0, t) = 0
\]
for every $l > 0$.

**Proof.** Assume first that $Y_t$ is mixing. Then, from Maruyama’s theorem, we get that $\lim_{t \to \infty} \nu_0(|xy| > l) = 0$ for every $l > 0$. In view of Lemma 1, we also have $\lim_{t \to \infty} \nu_0(\min(|x|, |y|) > l) = 0$ for every $l > 0$. Proposition 1 yields
\[
C_l(0, t) = \nu_0(\min(|x|, |y|) > l),
\]
thus, $\lim_{t \to \infty} C_l(0, t) = 0$ for every $l > 0$.

Now, assume that $\lim_{t \to \infty} C_l(0, t) = 0$ for every $l > 0$. We will show that $Y_t$ is mixing. Since the Gaussian part of $Y_t$ is assumed to disappear, its covariance function is equal to zero. Thus, it is enough to prove that conditions (C2) and (C3) in Maruyama’s theorem are satisfied. From Lemma 1 we see that $\lim_{t \to \infty} \nu_0(|xy| > l) = 0$ for every $l > 0$, which is exactly condition (C2). Therefore, it is left to show that
\[
\lim_{t \to \infty} \int_{0 < x^2 + y^2 \leq 1} xy \nu_0(dx, dy) = 0.
\]
Fix $\epsilon > 0$, put $B_\delta = \{x^2 + y^2 \leq \delta^2\}$ and $R_\delta = \{\delta^2 < x^2 + y^2 \leq 1\}$. Then, we obtain
\[
\int_{0 < x^2 + y^2 \leq 1} |xy| \nu_0(dx, dy) = \int_{B_\delta} |xy| \nu_0(dx, dy) + \int_{R_\delta} |xy| \nu_0(dx, dy) =: I_1 + I_2.
\]
We will estimate both terms $I_1$ and $I_2$ separately.

Taking advantage of stationarity of $\nu_0$, we get for the first term
\[
I_1 \leq \frac{1}{2} \int_{B_\delta} x^2 \nu_0(dx, dy) + \frac{1}{2} \int_{B_\delta} y^2 \nu_0(dx, dy)
\leq \frac{1}{2} \int_{\{x^2 \leq \delta^2\}} x^2 \nu_0(dx) + \frac{1}{2} \int_{\{y^2 \leq \delta^2\}} y^2 \nu_0(dy) = \int_{|x| \leq \delta} x^2 \nu_0(dx).
\]
Thus, for some appropriately small $\delta_0$ we have
\[
I_1 = \int_{B_{\delta_0}} |xy| \nu_0(dx, dy) \leq \epsilon/2. \tag{12}
\]

For the next term, put $l_0 = \min\{\frac{\delta_0}{2}, \frac{\epsilon}{8q}\}$, with $q = v_0(|x| > \frac{\delta_0}{2}) < \infty$. Then, for $C = R_{\delta_0} \cap \{|x| \wedge |y| > l_0\}$ we obtain
\[
I_2 = \int_{C} |xy| \nu_0(dx, dy) + \int_{R_{\delta_0} \setminus C} |xy| \nu_0(dx, dy) \leq v_0(C) + \int_{R_{\delta_0} \setminus C} \frac{\epsilon}{8q} \nu_0(dx, dy)
\]
\[
\leq v_0(|x| \wedge |y| > l_0) + \frac{\epsilon}{8q} v_0(R_{\delta_0} \setminus C)
\]
\[
\leq v_0(|x| \wedge |y| > l_0) + \frac{\epsilon}{8q} v_0\left(\left\{|x| > \frac{\delta_0}{2}\right\} \cup \left\{|y| > \frac{\delta_0}{2}\right\}\right)
\]
\[
\leq v_0(|x| \wedge |y| > l_0) + \frac{\epsilon}{8q} v_0\left(|x| > \frac{\delta_0}{2}\right) + \frac{\epsilon}{8q} v_0\left(|y| > \frac{\delta_0}{2}\right)
\]
\[
= v_0(|x| \wedge |y| > l_0) + \frac{\epsilon}{4q} v_0\left(|x| > \frac{\delta_0}{2}\right) = v_0(|x| \wedge |y| > l_0) + \frac{\epsilon}{4}.
\]

Using the fact that $\lim_{t \to \infty} C_I(0,t) = 0$, for large enough $t$ we have $v_0(|x| \wedge |y| > l_0) < \frac{\epsilon}{4}$, and therefore
\[
I_2 = \int_{R_{\delta_0}} |xy| \nu_0(dx, dy) < \frac{\epsilon}{2}. \tag{13}
\]

Finally, combining (12) and (13), and letting $\epsilon \downarrow 0$, we obtain the desired result. $\Box$

**Remark.** It follows from the proof of Theorem 1 that condition (C2) in Maruyama’s result implies condition (C3). This is an interesting refinement of Maruyama’s theorem; see also [16]. One should also mention Corollary 3 of [24], which is closely related to this problem and can be applied to derive the same refinement.

The following function
\[
\rho(t) = \log E e^{i(Y_t - Y_0)} - \log E e^{iY_t} - \log E e^{iY_0}, \tag{14}
\]
called the codifference of the stationary process $(Y_t)_{t \in \mathbb{R}}$, is an alternative measure of dependence for i.d. processes. As shown in [24] (see also [9,25]), it carries enough information to detect ergodic properties of $(Y_t)_{t \in \mathbb{R}}$. Codifference is closely related to the dynamical functional used in [3,4,12] to investigate the chaotic behavior of i.d. processes. The next result establishes the relationship between the asymptotic behavior of $\rho(t)$ and $C_I(0,t)$.

**Theorem 2.** Let $(Y_t)_{t \in \mathbb{R}}$ be a stationary i.d. process of the form (1). If the Lévy measure $\nu_0$ of $Y_0$ has no atoms in $2\pi \mathbb{Z}$, then the following two conditions are equivalent
(i) $\lim_{t \to \infty} C_I(0,t) = 0$ for every $l > 0$,
(ii) $\lim_{t \to \infty} \rho(t) = 0$.

**Proof.** Theorem 1 yields the equivalence of (i) and mixing. From [24], Theorem 1, we get that condition (ii) is equivalent to mixing in case when the Lévy measure $\nu_0$ of $Y_0$ has no atoms in $2\pi \mathbb{Z}$. Thus, conditions (i) and (ii) must be equivalent. $\Box$
In what follows, we show, how to modify the obtained results in order to characterize ergodicity and weak mixing. We recall that for the class of i.d. stationary processes these two properties are equivalent, [4].

As already discussed in [24], Maruyama’s theorem can be applied to the case of weak mixing if one replaces the convergence on the whole set $\mathbb{R}$ to the convergence on a subset of density one. We recall that a set $D \subset \mathbb{R}_+$ is of density one if $\lim_{C \to \infty} \lambda(D \cap [0, C])/C = 1$. Here $\lambda$ denotes the Lebesgue measure. Thus, the version of Maruyama’s theorem for weak mixing has the form

**Corollary 1.** An i.d. stationary process $(Y_t)_{t \in \mathbb{R}}$ is weakly mixing (ergodic) if and only if for some set $D$ of density one the following three conditions hold

(C1) the covariance function $r(t)$ of its Gaussian part converges to 0 as $t \to \infty$, $t \in D$,
(C2) $\lim_{t \to \infty, t \in D} \nu_0(|xy| > \delta) = 0$ for every $\delta > 0$,
(C3) $\lim_{t \to \infty, t \in D} \int_{0 < x^2 + y^2 \leq 1} xy \nu_0(dx, dy) = 0$,

where $\nu_0$ is the Lévy measure of $(Y_0, Y_t)$.

Since the intersection of a finite number of sets of density one is still the set of density one, we can repeat the arguments of Lemma 1 and Theorem 1 restricted to a set of density one. Hence, we obtain:

**Theorem 3.** Let $(Y_t)_{t \in \mathbb{R}}$ be a stationary i.d. process of the form (1) with no Gaussian part. Then $Y_t$ is weakly mixing (ergodic) iff for some set $D$ of density one, the corresponding function $C_l$ satisfies

$$\lim_{t \to \infty, t \in D} C_l(0, t) = 0$$

for every $l > 0$.

Since, for a measurable, nonnegative and bounded function $f : \mathbb{R}_+ \to \mathbb{R}$ and for a set $D$ of density one, the condition

$$\lim_{t \to \infty, t \in D} f(t) = 0$$

is equivalent to the following one

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u)du = 0;$$

hence, we obtain the following corollary.

**Corollary 2.** Let $(Y_t)_{t \in \mathbb{R}}$ be a stationary i.d. process of the form (1) with no Gaussian part. Then $Y_t$ is weakly mixing (ergodic) iff for the corresponding function $C_l$ satisfies

$$\lim_{t \to \infty} \frac{1}{T} \int_0^T C_l(0, t)dt = 0$$

for every $l > 0$.

Thus, the results of this section give a full description of the ergodic properties of i.d. processes of the form (1) in terms of function $C_l$. 
3. Long memory and the correlation cascade

The property of long memory (or long-range dependence) refers to a phenomenon in which the events that are arbitrarily distant still influence each other. This concept dates back to a series of papers by Mandelbrot et al. [18–20] that explained and proposed the appropriate mathematical model for the unusual behavior of the water levels in the Nile river. Since then the property of long memory has become particularly important in a wide range of applications starting with hydrology, ending with network traffic and finance. The typical way of defining long memory in the time domain is in terms of the rate of decay of the correlation function [1,6]. We say that a stationary process \( (X(t))_{t \in \mathbb{R}} \) with finite second moment has long memory if the following condition holds

\[
\sum_{n=0}^{\infty} |\text{Corr}(n)| = \infty. \tag{15}
\]

Here \( \text{Corr}(n) = \frac{E[X(n)X(0)] - E[X(n)]E[X(0)]}{\sqrt{\text{Var}[X(n)]\text{Var}[X(0)]}} \) is the correlation function. Conversely, the process \( X(t) \) is said to have short memory if the series (15) is convergent. Thus, the long-range dependence can be fully characterized by the asymptotic behavior of the correlation function. The typical examples of long-memory processes are the fractional Gaussian noise and FARIMA time series [1].

However, the situation becomes more complicated when considering processes with infinite variance, in particular, processes with \( \alpha \)-stable marginal distributions, \( 0 < \alpha < 2 \) (see [11,29]). In the \( \alpha \)-stable case, the correlations can no longer be calculated and the definition of long memory has to be reformulated. Since there are no correlations to look at, one has to look at the substitute measure of dependence. The first idea is to replace the correlation function in (15) by the codifference \( \rho(\cdot) \). This approach was discussed in [8,29]. It turns out that even in the case of fractional stable noise, different limiting behavior of \( \rho(\cdot) \) are observed according to the relative values of \( \alpha \) and self-similarity index \( H \). Thus, the full characterization of long memory only in terms of the codifference is impossible.

A different definition of long memory for stable processes was proposed in [27,28]. It was based on the integral representation of stationary stable processes derived by Rosinski [23]. Using the so-called Hopf decomposition of a \( \sigma \)-finite measure space, Rosinski has shown that each stationary stable process is generated by a conservative or by a dissipative flow. Now, the boundary between dissipative and conservative flows results in the boundary between stationary stable processes with short and long memory. Since stable random measures assign independent values to disjoint sets, the processes generated by conservative flows have longer memory than those generated by dissipative flows. Indeed, for a process generated by a conservative flow, the same values of the random measure contribute to the value of process many times, giving rise to the property of long memory.

One should also mention another approach to long-range dependence for self-similar noises presented in [10]. It explores the asymptotic behavior of the sample Allen variance. This approach is particularly useful for self-similar fractional stable noise [29] and leads to the property of long memory for the case \( H > 1/\alpha \).

In this paper we propose a different definition of long memory for processes with possibly infinite variance. As already discussed in the introduction, the function \( r_l(t_1, t_2) = \frac{C_l(t_1, t_2)}{\sqrt{C_l(t_1)C_l(t_2)}} \) defined in (8) can serve as the analogue of the classical correlation function for the cases, where the second moment is infinite. This function is particularly attractive for \( \alpha \)-stable processes of
the form (1). In such case, the Lévy measure of \( N \) has the form

\[
Q(dx) = \frac{c_1}{x^{1+\alpha}}I_{(0,\infty)}(x)dx + \frac{c_2}{|x|^{1+\alpha}}I_{(-\infty,0)}(x)dx,
\]

where \( c_1 \) and \( c_2 \) are the appropriate constants. Consequently, the tail function is given by

\[
A(l) = C \cdot l^{-\alpha}
\]

and for the correlation cascade we get

\[
C_l(t_1, \ldots, t_n) = C \cdot l^{-\alpha} \int_X \min\{K(t_1, x), \ldots, K(t_n, x)\}^{\alpha} m(dx),
\]

where \( C \) is an appropriate constant. From the last formula we get that the correlation-like function \( r_l(t_1, t_2) \) does not depend on the parameter \( l \). The advantage of \( r_l(t_1, t_2) \) over the codifference is that the first one is easier to calculate for \( \alpha \)-stable processes. Additionally, if the process \( Y_t \) is stationary, then the function \( r_l(\tau, \tau + t) \) does not depend on \( \tau \). Therefore, the function

\[
r(t) := r_l(\tau, \tau + t) = \frac{C_l(0, t)}{\sqrt{C_l(0)C_l(0)}} = \frac{\int_X \min\{K(t, x), K(0, x)\}^{\alpha} m(dx)}{\int_X K(0, x)^{\alpha} m(dx)}
\]

(16)

can be considered a correlation-like measure of dependence for stationary \( \alpha \)-stable process \( Y_t \). The immediate consequence is the following, alternative definition of long memory in the \( \alpha \)-stable case.

**Definition 1.** A stationary \( \alpha \)-stable process \( (Y_t)_{t \in \mathbb{R}} \) is said to have long memory in terms of the correlation cascade if the following condition holds

\[
\sum_{n=0}^{\infty} |r(n)| = \infty,
\]

(17)

where \( r(\cdot) \) is given by (16).

Note that, in view of (16), in order to verify the long-memory property of \( Y_t \), it is enough to examine the asymptotic behavior of the correlation cascade \( C_l(0, t) \). The correlation cascade corresponding to the fractional stable noise satisfies \( C_l(0, t) \sim t^{-\alpha(1-H)} \) as \( t \to \infty \) [7]. Therefore, for \( H > 1/\alpha \), fractional stable noise has long memory in terms of the correlation cascade. Here, by \( f(t) \sim g(t) \) we mean \( \lim_{t \to \infty} f(t)/g(t) = 1 \).

Comparison of the different approaches to long memory is a difficult task. The relationship between the correlation cascade and the codifference obtained in Theorem 2 does not verify the rate of convergence of these functions. Comparing the results of Sections 3.1–3.4 with those obtained in [14,15,17], we get that the rates of decay of \( C_l(0, t) \) and \( \rho(t) \) computed for the same stationary processes can be different. Therefore, it is hard to compare two definitions of long memory based on the correlation cascade and the codifference. Also, it is important to note that the fractional stable noise and the process \( Z_0(t) \) which will be introduced in Section 3.1, are generated by dissipative flows. Therefore, they have short memory in the sense of the definition introduced by Samorodnitsky in [27,28]. On the other hand, these two processes have long memory in the sense of the rate of decay of the correlation cascade and the codifference. Thus, we emphasize that the definition of long memory introduced in this paper should be viewed as one of the possible approaches to long memory for infinitely divisible processes. The concept
of long memory in the non-Gaussian case is still not fully formulated and is a subject of many extensive research.

The classical $\alpha$-stable Ornstein–Uhlenbeck process $(Z(t))_{t \in \mathbb{R}}$ can be defined as the following stochastic integral

$$Z(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} L_\alpha(ds), \quad \lambda > 0.$$ \hfill (18)

Here, $L_\alpha(ds)$, $0 < \alpha \leq 2$, is the symmetric $\alpha$-stable random measure with control measure as the Lebesgue measure. Since $Z(t)$ is a Markov process, it does not have long memory. It is also straightforward to verify that the corresponding correlation cascade decays exponentially.

In what follows, we apply the introduced measure of dependence $C_l(\cdot)$ in order to investigate the property of long memory for four ‘fractional’ generalizations of the process $Z(t)$. The results obtained for the asymptotic behavior of the correlation cascade will also allow us to verify the mixing property for the discussed models.

3.1. Fractional stable noise

The first considered process $Z_0(t)$ is a modified version of the fractional stable noise. It is defined as [17]

$$Z_0(t) = \frac{1}{\Gamma(\kappa)} \int_{-\infty}^{t} (t - s + 1)^{\kappa - 1} L_\alpha(ds), \quad t \in \mathbb{R}. \hfill (19)$$

The process $Z_0(t)$ is stationary, since it is a moving-average process. It is well defined for $\kappa < 1 - 1/\alpha$. Because of its dependence structure similar to the $\alpha$-stable FARIMA time series, $Z_0(t)$ is also called the Continuous-time FARIMA process [17].

We will show that $Z_0(t)$ is a long-memory process. To find the asymptotic behavior of its correlation cascade, let us discuss the following, more general situation. Consider the $\alpha$-stable moving-average process

$$Y_t = \int_{-\infty}^{t} f(t - x)L_\alpha(dx).$$

Here $f$ is assumed to be nonnegative, monotonically decreasing function and $\alpha$-integrable. In this case the function $C_l$ has the simple form

$$C_l(0, t) = \text{const} \cdot l^{-\alpha} \int_{l}^{\infty} |f(y)|^\alpha dy. \hfill (20)$$

Since $f$ is $\alpha$-integrable, we get that $\lim_{t \to \infty} C_l(0, t) = 0$ for every $l > 0$. By Theorem 1, it implies that every $\alpha$-stable moving average is mixing. In particular, we have:

**Corollary 3.** The process $Z_0(t)$ is mixing.

Moreover, using (20), we prove the following result.

**Theorem 4.** Let $\kappa < 1 - 1/\alpha$ and $0 < \alpha < 2$. Then the correlation cascade of $Z_0(t)$ satisfies

$$C_l(0, t) \sim K \cdot t^{\alpha(\kappa-1)+1} \quad \text{as} \quad t \to \infty. \hfill (21)$$

Here $K$ is the appropriate positive constant dependent only on the parameters $\alpha$ and $\kappa$. 

Proof. Since the function \( f(s) = (s + 1)^{\kappa - 1} 1_{\{s > 0\}} \) is non-increasing and positive, we get from formula (20)
\[
C_1(0, t) = c \int_t^\infty (s + 1)^{\alpha(\kappa - 1)} \, ds,
\]
where \( c \) is the appropriate positive constant. Therefore, we immediately obtain
\[
C_1(0, t) \sim K \cdot t^{\alpha(\kappa - 1) + 1} \quad \text{as} \quad t \to \infty
\]
for appropriate positive constant \( K \). □

We get the following corollary:

**Corollary 4.** For \( 1 - \frac{2}{\alpha} \leq \kappa < 1 - \frac{1}{\alpha} \) the process \( Z_0(t) \) has long memory in the sense of (17).

Proof. Since the correlation cascade of \( Z_0(t) \) satisfies \( C_1(0, t) \sim K \cdot t^{\alpha(\kappa - 1) + 1} \) as \( t \to \infty \), the series (17) is divergent for \( 1 - \frac{2}{\alpha} \leq \kappa < 1 - \frac{1}{\alpha} \). □

The process \( Z_0(t) \) has long memory also in the sense of the codifference [17]. On the other hand, it is generated by a dissipative flow. Thus, it has short memory in the sense of definition introduced in [27,28].

### 3.2. Type I fractional \( \alpha \)-stable Ornstein–Uhlenbeck process

Linear fractional stable motion (LFSM) is an extension of the well-known fractional Brownian motion to the \( \alpha \)-stable case. It is defined in the following way [29]: Let \( 0 < \alpha \leq 2 \), \( 0 < H < 1 \), \( H \neq 1/\alpha \) and \( a, b \in \mathbb{R}, |a| + |b| > 0 \). Then the process
\[
L_{\alpha,H}(t) = \int_{-\infty}^\infty \left[ a \left( (t - s)^{H-1/\alpha}_+ - (-s)^{H-1/\alpha}_+ \right) + b \left( (t - s)^{H-1/\alpha}_- - (-s)^{H-1/\alpha}_- \right) \right] \, L_\alpha(ds), \quad t \in \mathbb{R},
\]
is called LFSM. Here \( x_+ = \max\{x, 0\} \), \( x_- = \max\{-x, 0\} \) and \( L_\alpha(ds) \) is the standard symmetric \( \alpha \)-stable random measure on \( \mathbb{R} \) with control measure as the Lebesgue measure, [11,29]. \( L_{\alpha,H}(t) \) is a self-similar, stationary-increment process. For \( \alpha = 2 \) it reduces to the fractional Brownian motion.

Now, we consider the **Type I fractional \( \alpha \)-stable Ornstein–Uhlenbeck process** \( (Z_1(t))_{t \in \mathbb{R}} \) defined as the following Lamperti transformation [2,13] from the LFSM (cf. [5] for the Gaussian case):
\[
Z_1(t) = e^{-tH} L_{\alpha,H}(e^t).
\]
By the Lamperti result, \( Z_1(t) \) is stationary. Recall that the standard Ornstein–Uhlenbeck process can also be obtained as the Lamperti transformation from Brownian motion. The process \( Z_1(t) \) was first defined in [15], where it was shown that the corresponding codifference decays exponentially. It follows that \( Z_1(t) \) has short memory in the sense of the rate of decay of the codifference.

In the next theorem we give a precise formula for the asymptotic behavior of the corresponding correlation cascade. Next, we show that \( Z_1(t) \) is mixing and has short memory.
Theorem 5. Let $0 < \alpha < 2$, $0 < H < 1$, $a \geq 0$, $a + b \leq 0$, and $H - 1/\alpha > 0$. Then the correlation cascade of $Z_1(t)$ satisfies

$$C_1(0, t) \sim K \cdot e^{-t \frac{a(1-H)}{\alpha(H-1/\alpha)+1}}$$

as $t \to \infty$. (24)

Here $K$ is the appropriate positive constant dependent only on the parameters $\alpha$ and $H$.

Proof. We have

$$C_1(0, t) = \int_{-\infty}^{\infty} \min\{f(s, t); f(s, 0)\}^\alpha ds,$$

where

$$f(s, t) = e^{-tH} a \left[ (e^t - s)^{-1/\alpha} - (-s)^{-1/\alpha} \right] + e^{-tH} b \left[ (e^t - s)^{-1/\alpha} - (-s)^{-1/\alpha} \right].$$

For simplicity, we omit the constant before the integral in the definition of $C_1$. Let us introduce the following decomposition

$$C_1(0, t) = \int_{-\infty}^{\infty} \min\{f(s, t); f(s, 0)\}^\alpha ds = \int_{-\infty}^{0} \ldots ds + \int_{0}^{1} \ldots ds + \int_{1}^{\infty} \ldots ds + \int_{e^t}^{\infty} \ldots ds =: I_1(t) + I_2(t) + I_3(t) + I_4(t).$$

In what follows, we estimate the rate of convergence of every $I_j(t)$, $j = 1, \ldots, 4$, separately. Let us begin with $I_3(t)$. We have

$$I_3(t) = \int_{1}^{e^t} \min\{e^{-tH}[a(e^t - s)^{-1/\alpha} - bs^{-1/\alpha}]; b[(s-1)^{-1/\alpha} - s^{-1/\alpha}]\}^\alpha ds.$$

Set

$$k(t) := e^{\frac{-t}{\alpha(H-1/\alpha)+1}} \left( \frac{a}{(-b)(H-1/\alpha)} \right)^{\frac{1}{\alpha(H-1/\alpha)}}.$$

Next, introduce the decomposition

$$I_3(t) = \int_{1}^{e^t} \min\{\ldots\}^{\alpha} ds = \int_{1}^{k(t)} \min\{\ldots\}^{\alpha} ds + \int_{k(t)}^{e^t} \min\{\ldots\}^{\alpha} ds =: I_{31}(t) + I_{32}(t).$$

We will estimate both components separately. We start with $I_{31}(t)$. For $s \in (1, k(t))$ we have

$$e^{-tH}[a(e^t - s)^{-1/\alpha} - bs^{-1/\alpha}] \leq (a - b(k(t)e^{-t})^{-1/\alpha}) e^{-t/\alpha} \leq \frac{(a - b(k(t)e^{-t})^{-1/\alpha})}{a} (-b)(H-1/\alpha) \int_{0}^{1} (s-u)^{-1/(\alpha-1)} du.$$

Therefore, since $k(t)e^{-t} \to 0$, we get

$$I_{31}(t) \sim \int_{1}^{k(t)} e^{-tH\alpha}[a(e^t - s)^{-1/\alpha} - bs^{-1/\alpha}]^\alpha ds$$

as $t \to \infty$. After substituting $s \to k(t)s$, we obtain

$$\int_{1}^{k(t)} e^{-tH\alpha}[a(e^t - s)^{-1/\alpha} - bs^{-1/\alpha}]^\alpha ds = e^{-tH\alpha} k(t)^{H\alpha} \int_{1}^{1/g(t)} g(s, t) ds.$$
with \( g(s,t) = \left[ a \left( \frac{e^t - s}{k(t)} \right)^{H-1/\alpha} - bs^{H-1/\alpha} \right]^{\alpha} \). Since for fixed \( s \in (0,1) \) we have \( g(s,t) \sim a^\alpha \left( \frac{e^t}{k(t)} \right)^{Ha-1} \) as \( t \to \infty \), the dominated convergence theorem yields

\[
I_{31}(t) \sim c_1 e^{-t} k(t) \sim c_2 e^{-t \frac{\alpha(1-H)}{\alpha(1-H)+t}}
\]  

as \( t \to \infty \). Here \( c_1 \) and \( c_2 \) are the appropriate positive constants independent of \( t \).

Next, we estimate \( I_{32}(t) \). For \( s \in (k(t), e^t) \) we have

\[
e^{-tH[a(e^t-s)^{H-1/\alpha} - bs^{H-1/\alpha}]} \geq (-b)(H - 1/\alpha)k(t)^{H-1/\alpha-1}.
\]

Consequently,

\[
I_{32}(t) \sim (-b)^\alpha \int_{k(t)}^{e^t} [s^{H-1/\alpha} - (s - 1)^{H-1/\alpha}]^\alpha ds
= (-b)^\alpha \int_{k(t)}^{\infty} [s^{H-1/\alpha} - (s - 1)^{H-1/\alpha}]^\alpha ds
- \int_{e^t}^{\infty} (-b)^\alpha [s^{H-1/\alpha} - (s - 1)^{H-1/\alpha}]^\alpha ds
\]

as \( t \to \infty \). The first integral in the above formula is equal to \((-b)^\alpha k(t)^{H\alpha} \int_1^{\infty} [s^{H-1/\alpha} - (s - 1/k(t))^{H-1/\alpha}]^\alpha ds \). Moreover,

\[
s^{H-1/\alpha} - (s - 1/k(t))^{H-1/\alpha} = (H - 1/\alpha) \frac{1}{k(t)} \int_0^1 (s - u/k(t))^{H-1/\alpha-1} du.
\]

Thus, for fixed \( s \in (1, \infty) \) we get

\[
\frac{[s^{H-1/\alpha} - (s - 1/k(t))^{H-1/\alpha}]^\alpha}{k(t)^{-\alpha}} \to (H - 1/\alpha)^\alpha s^{Ha-1-\alpha}
\]

as \( t \to \infty \), which is integrable on \((1, \infty)\). Consequently, the dominated convergence theorem yields \( \int_1^{\infty} [s^{H-1/\alpha} - (s - 1/k(t))^{H-1/\alpha}]^\alpha ds \sim c_3 k(t)^{-\alpha} \), where \( c_3 \) is the appropriate positive constant. Therefore,

\[
(-b)^\alpha \int_{k(t)}^{\infty} [s^{H-1/\alpha} - (s - 1)^{H-1/\alpha}]^\alpha ds \sim \widehat{c}_3 e^{-t \frac{\alpha(1-H)}{\alpha(1-H)+t}}
\]

as \( t \to \infty \). Here \( \widehat{c}_3 \) is the appropriate positive constant.

The second integral in (28) is equal to \((-b)^\alpha e^{tH\alpha} \int_1^{\infty} [s^{H-1/\alpha} - (s - e^{-t})^{H-1/\alpha}]^\alpha ds \). For fixed \( s \in (1, \infty) \) we have

\[
s^{H-1/\alpha} - (s - e^{-t})^{H-1/\alpha} = (H - 1/\alpha)e^{-t} \int_0^1 (s - ue^{-t})^{H-1/\alpha-1} du.
\]

Therefore, from the dominated convergence theorem, we obtain

\[
\int_1^{\infty} [s^{H-1/\alpha} - (s - e^{-t})^{H-1/\alpha}]^\alpha ds \sim c_4 e^{-t\alpha}
\]

as \( t \to \infty \). Here \( c_4 \) is the appropriate positive constant. Consequently

\[
\int_{e^t}^{\infty} (-b)^\alpha [s^{H-1/\alpha} - (s - 1)^{H-1/\alpha}]^\alpha ds \sim \widehat{c}_4 e^{-t\alpha(1-H)}
\]

(29)
as $t \to \infty$ and its contribution is negligible. Finally

$$I_{32}(t) \sim \hat{c}_3 e^{-\frac{t}{\alpha(1-H)}}.$$  

Combining the above result with (27) we obtain

$$I_3(t) \sim c_5 e^{-\frac{t}{\alpha(1-H)}}\alpha(1-H) + 1,$$  
(30)

as $t \to \infty$, where $c_5$ is the appropriate positive constant independent of $t$.

For the term $I_2(t)$ in decomposition (25) we have

$$I_2(t) \leq e^{-tH} \int_0^1 [a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}]^a ds \leq e^{-tH} \alpha(e^{t(H-1/\alpha)} - b)^\alpha.$$  

Therefore its contribution is negligible.

For the term $I_4(t)$ in decomposition (25), we get from (29) that its contribution is also negligible.

For the term $I_1(t)$ we have

$$I_1(t) = \int_0^\infty \min\{h(-s, t) ; h(-s, 0)\}^a ds$$
$$= \int_0^1 \min\{\cdots\}^a ds + \int_{e^t}^\infty \min\{\cdots\}^a ds + \int_{e^t}^\infty \min\{\cdots\}^a ds,$$

where $h(s, t) = ae^{-tH} [(e^t - s)^{H-1/\alpha} - (-s)^{H-1/\alpha}]$. Thus, by similar arguments as for $I_2(t) - I_4(t)$ we obtain

$$I_1(t) \sim c_6 e^{-\frac{t}{\alpha(1-H)}}\alpha(1-H) + 1$$

as $t \to \infty$.

Finally, putting together all the above results, we obtain

$$C_l(0, t) \sim K \cdot e^{-\frac{t}{\alpha(1-H)}}\alpha(1-H) + 1 \quad \text{as} \quad t \to \infty,$$

where $K$ is the appropriate positive constant dependent only on the parameters $\alpha$ and $H$. \hfill $\square$

From the above theorem we get the following conclusions.

**Corollary 5.** The process $Z_1(t)$ does not have long memory in the sense of (17).

**Proof.** Since the correlation cascade $C_l(0, t)$ decays exponentially, the series (17) converges. \hfill $\square$

**Corollary 6.** The process $Z_1(t)$ is mixing.

**Proof.** Since $C_l(0, t) \to 0$ as $t \to \infty$, from Theorem 1 we get that the process must be mixing. \hfill $\square$

### 3.3. Type II fractional $\alpha$-stable Ornstein–Uhlenbeck process

In this section we consider the second generalization $(Z_2(t))_{t \in \mathbb{R}}$ of the standard $\alpha$-stable Ornstein–Uhlenbeck process. First, let us introduce the finite-memory fractional $\alpha$-stable motion
\[ \{ \tilde{L}_{\alpha, H}(t), t \geq 0 \} \text{ defined as the following stochastic integral} \]
\[
\tilde{L}_{\alpha, H}(t) = \int_{0}^{t} (t - s)^{H - 1/\alpha} dL_{\alpha}(s), \quad t \geq 0, 
\]
where \( H > 0, \alpha \in (0, 2], \Gamma(\cdot) \) is the Gamma function and \( L_{\alpha}(s) \) is the symmetric \( \alpha \)-stable random measure with the Lebesgue measure as control measure. Observe that \((t - s)^{H - 1/\alpha} \) is \( \alpha \)-integrable on \((0, t)\) for every \( t \geq 0 \), thus \( \tilde{L}_{\alpha, H}(t) \) is a well-defined \( \alpha \)-stable process. Additionally, for \( H = 1/\alpha \) we get the standard symmetric \( \alpha \)-stable motion. The process \( \tilde{L}_{\alpha, H}(at) \) is \( H \)-self-similar, but unlike the linear fractional stable motion \( L_{\alpha, H}(t) \) defined in (22), it does not have stationary increments.

Now, we consider the Type II fractional \( \alpha \)-stable Ornstein–Uhlenbeck process \( Z_{\delta}(t) \), which is defined as the Lamperti transformation of \( \tilde{L}_{\alpha, H}(t) \)
\[
Z_{\delta}(t) = e^{-TH}\tilde{L}_{\alpha, H}(e^{t})
\]
\[
= e^{-TH} \int_{0}^{e^{t}} (e^{t} - s)^{H - 1/\alpha} dL_{\alpha}(s), \quad t \in \mathbb{R}. 
\]
By the Lamperti result, \( Z_{\delta}(t) \) is stationary. For \( H = 1/\alpha \) we get the standard \( \alpha \)-stable Ornstein–Uhlenbeck process with short memory. The process \( Z_{\delta}(t) \) was first studied in [14]. Its codifference decays exponentially, thus \( Z_{\delta}(t) \) has short memory in the sense of the rate of decay of \( \rho(t) \).

In the next theorem we give a precise formula for the asymptotic behavior of the correlation cascade corresponding to \( Z_{\delta}(t) \). We prove that \( Z_{\delta}(t) \) has short memory and is mixing.

**Theorem 6.** Let \( H > 0 \) and \( 0 < \alpha < 2 \). Then the correlation cascade of \( Z_{\delta}(t) \) satisfies

(i) if \( H = 1/\alpha \) then
\[
C_{l}(0, t) \sim C_{l}^{-\alpha} e^{-H_{\alpha}t} \quad \text{as } t \to \infty. 
\]

(ii) if \( H \neq 1/\alpha \) then
\[
C_{l}(0, t) \sim C_{l}^{-\alpha} e^{-t} \quad \text{as } t \to \infty. 
\]

**Proof.** We have
\[
C_{l}(0, t) = C_{l}^{-\alpha} \int_{-\infty}^{\infty} \min\{e^{-H}(e^{t} - s)^{H - 1/\alpha} \mathbf{1}_{[0,e^{t}]}(s); (1 - s)^{H - 1/\alpha} \mathbf{1}_{(0,1)}(s)\}^{\alpha} ds.
\]
Thus, for \( H = 1/\alpha \) we immediately obtain (33).

Consider first the case \( H - 1/\alpha < 0 \). Then, we obtain
\[
C_{l}(0, t) = C_{l}^{-\alpha} e^{-H_{\alpha}t} \int_{0}^{1} (e^{t} - s)^{H_{\alpha} - 1} ds.
\]
Since for fixed \( s \in (0, 1) \) we have \( (e^{t} - s)^{H_{\alpha} - 1} \to 1 \) as \( t \to \infty \), from the dominated convergence theorem we obtain \( \int_{0}^{1} (e^{t} - s)^{H_{\alpha} - 1} ds \sim e^{t(H_{\alpha} - 1)} \). Consequently,
\[
C_{l}(0, t) \sim C_{l}^{-\alpha} e^{-t} \quad \text{as } t \to \infty.
\]
We pass to the second case $H - 1/\alpha > 0$. For $s \in (0, 1)$ we get
\[ e^{-tH} (e^t - s)^{H-1/\alpha} < (1 - s)^{H-1/\alpha} \iff s < \frac{e^{-t/H\alpha} - 1}{e^{-t/H\alpha} - 1}. \]

Set $k(t) := \frac{e^{-t/H\alpha} - 1}{e^{-t/H\alpha} - 1}$. Then, we obtain
\[ C(t, 0, t) = CI^{-\alpha} \left( \int_0^{k(t)} e^{-tH\alpha} (e^t - s)^{H\alpha-1} ds + \int_{k(t)}^{1} (1 - s)^{H\alpha-1} ds \right) \]
\[ =: CI^{-\alpha} (I_1(t) + I_2(t)). \]

For the first term $I_1(t)$, after some standard calculations, we get
\[ I_1(t) = e^{-tH\alpha} k(t) \int_0^{1} (e^t - k(t)s)^{H\alpha-1} ds. \]

Since $k(t) \rightarrow 1$ and for fixed $s \in (0, 1)$ we have $\frac{(e^t - k(t)s)^{H\alpha-1}}{e^{t(H\alpha-1)}} \rightarrow 1$ as $t \rightarrow \infty$, the dominated convergence theorem yields
\[ I_1(t) \sim e^{-t} \quad \text{as } t \rightarrow \infty. \]

For the second term $I_2(t)$, we obtain $I_2(t) = \frac{(1-k(t))^{H\alpha}}{e^{t(H\alpha-1)}} \sim \frac{e^{-t/H\alpha}}{H\alpha}$ as $t \rightarrow \infty$. Thus, $I_2(t)$ decays faster than $I_1(t)$ and its contribution is negligible. Finally, we get
\[ C(t, 0, t) \sim CI^{-\alpha} e^{-t} \quad \text{as } t \rightarrow \infty. \]

We get the following conclusions.

**Corollary 7.** The process $Z_3(t)$ does not have long memory in the sense of (17).

**Proof.** The correlation cascade $C(t, 0, t)$ decays exponentially, thus the series (17) converges. □

**Corollary 8.** The process $Z_3(t)$ is mixing.

**Proof.** Since $C(t, 0, t) \rightarrow 0$ as $t \rightarrow \infty$, from Theorem 1 we get that the process is mixing. □

### 3.4. Type III fractional $\alpha$-stable Ornstein–Uhlenbeck process

The **Type III fractional $\alpha$-stable Ornstein–Uhlenbeck process** $(Z_3(t))_{t \in \mathbb{R}}$ is defined as the following stochastic integral
\[ Z_3(t) = \frac{1}{\Gamma(\kappa)} \int_{-\infty}^{t} (t - s)^{\kappa-1} e^{-\lambda(t-s)} L_\alpha(ds), \quad \kappa > 0, \lambda > 0, \quad (35) \]

where $L_\alpha(s)$ is the symmetric $\alpha$-stable random measure with the Lebesgue control measure. For $\kappa > 1 - 1/\alpha$ the kernel in (35) belongs to the Lebesgue space $L^\alpha((-\infty, t), ds)$ and the stochastic integral is well defined. $Z_3(t)$ was first introduced in [30] in the context of Telecom process. It can be shown [17] that $Z_3(t)$ is the solution of the following fractional Langevin equation
\[ \left( \lambda I + \frac{d}{dt} \right)^\kappa Z_3(t) = l_\alpha(t). \quad (36) \]
Here, the operator $\left( \frac{d}{dt} + \lambda \right)^\kappa$ is the so-called modified Bessel derivative [26] and $l_\alpha(t)$ is the $\alpha$-stable noise, formally $l_\alpha(t) = dL_\alpha(t)/dt$. Note that for $\kappa = 1$ the above equation becomes the standard $\alpha$-stable Langevin equation and its stationary solution is the $\alpha$-stable Ornstein–Uhlenbeck process with short memory.

As shown in [17], the codifference of $Z_3(t)$ decays exponentially. Thus, it has short memory in the sense of the rate of decay of $\rho(t)$.

Now, we verify the asymptotic behavior of the corresponding correlation cascade and show that $Z_3(t)$ does not have long memory in the sense of (17) and that it is mixing.

**Theorem 7.** Let $\kappa > 1 - 1/\alpha$ and $0 < \alpha < 2$. Then the correlation cascade of $Z_3(t)$ satisfies

$$C_1(0, t) \sim \frac{C l^{-\alpha}(\Gamma(\kappa))^{-\alpha}}{\lambda \alpha} t^{\alpha(\kappa - 1)} e^{-\lambda \alpha t} \quad \text{as } t \to \infty. \quad (37)$$

**Proof.** Consider first the case $\kappa \leq 1$. We have

$$C_1(0, t) = Cl^{-\alpha}(\Gamma(\kappa))^{-\alpha} \int_{-\infty}^{\infty} \min\{e^{-\lambda(t-s)}(t-s)^{\kappa-1} 1_{[s < t]} ; e^{\lambda s} (-s)^{\kappa-1} 1_{[s < 0]} \}^\alpha ds$$

$$= Cl^{-\alpha}(\Gamma(\kappa))^{-\alpha} \int_{-\infty}^{0} e^{-\lambda \alpha (t-s)} (t-s)^{\alpha(\kappa-1)} ds$$

$$= Cl^{-\alpha}(\Gamma(\kappa))^{-\alpha} e^{-\lambda \alpha t} \int_{0}^{\infty} e^{-\lambda \alpha s} (t+s)^{\alpha(\kappa-1)} ds.$$ 

For fixed $s \in (0, \infty)$ we have $e^{\lambda \alpha (t+s)^{\alpha(\kappa-1)}} \to e^{-\lambda \alpha s}$ as $t \to \infty$. Additionally, $\frac{e^{-\lambda \alpha (t+s)^{\alpha(\kappa-1)}}}{\nu^{\alpha(\kappa-1)}} \leq c_1 e^{-\lambda \alpha s}$, which is integrable on $(0, \infty)$. Here $c_1$ is the appropriate positive constant. Thus, from the dominated convergence theorem we get

$$C_1(0, t) \sim Cl^{-\alpha}(\Gamma(\kappa))^{-\alpha} t^{\alpha(\kappa-1)} e^{-\lambda \alpha t}$$

as $t \to \infty$.

We pass to the case $\kappa > 1$. For $s < 0$ we have

$$e^{-\lambda t} (t-s)^{\kappa-1} < (-s)^{\kappa-1} \iff s < -\frac{t}{e^{\lambda t/(\kappa-1) - 1}}.$$ 

Set $k(t) := -\frac{t}{e^{\lambda t/(\kappa-1) - 1}}$. Then, we have

$$C_1(0, t) = Cl^{-\alpha}(\Gamma(\kappa))^{-\alpha} \int_{-\infty}^{\infty} \min\{e^{-\lambda(t-s)} (t-s)^{\kappa-1} 1_{[s < t]} ; e^{\lambda s} (-s)^{\kappa-1} 1_{[s < 0]} \}^\alpha ds$$

$$= Cl^{-\alpha}(\Gamma(\kappa))^{-\alpha} \left( \int_{-\infty}^{k(t)} e^{-\lambda \alpha (t-s)} (t-s)^{\alpha(\kappa-1)} ds + \int_{k(t)}^{0} e^{\lambda \alpha s} (-s)^{\alpha(\kappa-1)} ds \right)$$

$$=: Cl^{-\alpha}(\Gamma(\kappa))^{-\alpha} (I_1(t) + I_2(t)).$$

For the first term, after some standard calculations, we get $I_1(t) = e^{-\lambda \alpha t} \int_{0}^{\infty} h(s, t) ds$, where $h(s, t) = e^{-\lambda \alpha (k(t)+s)} (t-k(t)+s)^{\alpha(\kappa-1)}$. Additionally, for fixed $s \in (0, \infty)$, we obtain $h(s, t) \to e^{-\lambda \alpha t}$ as $t \to \infty$. Consequently, from the dominated convergence theorem, we obtain

$$I_1(t) \sim \frac{1}{\lambda \alpha} t^{\alpha(\kappa-1)} e^{-\lambda \alpha t}$$

as $t \to \infty$. 


For the second term we get $I_2(t) \leq -k(t)(-k(t))^{\alpha(\kappa-1)} = (-k(t))^{\alpha(\kappa-1)+1}$. Thus, $I_2(t)$ decays faster than $I_1(t)$ and its contribution is negligible. Finally, we obtain

$$C_I(0, t) \sim C \lambda^{-\alpha}(\Gamma(\kappa))^{-\alpha} I_1(t) \sim \frac{C \lambda^{-\alpha}(\Gamma(\kappa))^{-\alpha}}{\lambda \alpha} t^{\alpha(\kappa-1)} e^{-\lambda \alpha t}$$

as $t \to \infty$. □

We get the following conclusions.

**Corollary 9.** The process $Z_3(t)$ does not have long memory in the sense of (17).

**Proof.** Since the correlation cascade $C_I(0, t)$ decays exponentially, the series (17) is convergent. □

**Corollary 10.** The process $Z_3(t)$ is mixing.

**Proof.** Since $C_I(0, t) \to 0$ as $t \to \infty$, from Theorem 1 we obtain that the process must be mixing. □

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**References**