PATH PROPERTIES OF SUBDIFFUSION—
A MARTINGALE APPROACH

Marcin Magdziarz

Hugo Steinhaus Center, Institute of Mathematics and Computer Science, Wroclaw
University of Technology, Wroclaw, Poland

In statistical physics, subdiffusion processes constitute one of the most relevant subclasses of the family of anomalous diffusion models. These processes are characterized by certain power-law deviations from the classical Brownian linear time dependence of the mean-squared displacement. In this article we study sample path properties of subdiffusion. We propose a martingale approach to the stochastic analysis of subdiffusion models. We verify the martingale property, Hölder continuity of the trajectories, and derive the law of large numbers. The precise asymptotic behavior of subdiffusion is obtained in the law of the iterated logarithm. The presented results may be applied to identify the type of subdiffusive dynamics in experimental data.

Keywords Inverse subordinator; Law of the iterated logarithm; Law of large numbers; Martingale; Subdiffusion.

Mathematics Subject Classification 6G17; 6G52.

1. INTRODUCTION

Recent developments in different areas of science show that mathematical models based on the classical Brownian diffusion fail to provide satisfactory description of many physical systems. The empirical analysis of various complex systems shows that some of the characteristic properties cannot be described by the Brownian diffusion models. One should mention here such properties as nonlinear in time mean-squared displacement, long-range correlations, heavy-tailed and skewed marginal distributions, lack of scale invariance, etc. Therefore, in recent years one
has observed a rapid evolution of the so-called *anomalous diffusion* models, which are introduced to capture such characteristic features\cite{8,9,19,26}.

In the large family of anomalous diffusion processes, the subclass of *subdiffusion* processes is of special importance. It is characterized by the sublinear in time mean-squared displacement $\text{Var}[X(t)] \sim ct^{\alpha}$ as $t \to \infty$ with $0 < \alpha < 1$. The last formula, compared with the linear in time mean-squared displacement of Brownian motion, explains why subdiffusion is also termed *slow diffusion*. Subdiffusion processes occur also as the scaling limits of random walks with power-law waiting times between jumps\cite{24}. These heavy-tailed waiting times are responsible for the subdiffusive character of the motion and for the appearance of fractional time derivatives in the corresponding diffusion equations\cite{26}.

The empirically confirmed list of subdiffusive physical systems is extensive. The subdiffusive regime was reported in condensed phases\cite{26}, ecology\cite{32}, biology\cite{11}, dielectric relaxation\cite{15}, and many more. Therefore, it is of great importance to develop mathematical methods that can be used to investigate sample path properties of subdiffusion processes.

The typical description of subdiffusion is in terms the fractional diffusion equation\cite{25,26}:

$$\frac{\partial w(x,t)}{\partial t} = 0 \mathcal{D}_t^{1-\alpha} \left[ \frac{1}{2} \frac{\partial^2}{\partial x^2} \right] w(x,t)$$

with the initial condition $w(x,0) = \delta(x)$. Here, the operator

$$0 \mathcal{D}_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

$0 < \alpha < 1$, $f \in C^1([0,\infty))$, is the fractional derivative of the Riemann-Liouville type\cite{30} and $\Gamma(\cdot)$ is the usual gamma function. The nonlocal fractional derivative operator introduces the memory effects to the underlying non-Markovian process. Note that for $\alpha \to 1$, equation (1.1) becomes the ordinary diffusion equation and its solution is the probability density function (PDF) of the Brownian motion. It is easy to verify\cite{26} that the corresponding mean-squared displacement is equal to $t^{\alpha}/\Gamma(\alpha+1)$, which confirms that fractional diffusion equation (1.1) can be used to model subdiffusion. For the existence of the solutions of (1.1) on bounded domains, see Ref.\cite{22}.

In equation (1.1), $w(x,t)$ denotes the PDF of a stochastic process $\{Z_\alpha(t)\}_{t \geq 0}$. The process $Z_\alpha(t)$ can be written in the form of the subordination\cite{20,21,28,33}:

$$Z_\alpha(t) = B[S_\alpha(t)],$$

(1.2)
where \( \{B(t)\}_{t \geq 0} \) is the standard Brownian motion and \( \{S_\alpha(t)\}_{t \geq 0} \) is the inverse \( \alpha \)-stable subordinator. Moreover, the processes \( S_\alpha(t) \) and \( B(t) \) are assumed to be mutually independent. The precise definition of \( S_\alpha(t) \) will be given in the next section. The PDF of \( Z_\alpha(t) \) solves the fractional equation (1.1). We call \( Z_\alpha(t) \) the subdiffusion process. It is also called the local time Brownian motion in Ref.[23], due to the relationship between inverse subordinators and local times of certain Markov processes [3].

There is a surprising and unexpected relationship between \( Z_\alpha(t) \) and the process called iterated brownian motion (IBM). The IBM was first considered by Burdzy [6] in the following way. Take \( B(t) \), \( B_1(t) \), \( B_2(t) \) to be independent Brownian motions on \( \mathbb{R} \). Define a two-sided Brownian motion by \( \tilde{B}(t) = B_1(t) \) for \( t \geq 0 \) and \( B_2(-t) \) for \( t < 0 \). Then the IBM process is defined by \( X(t) = \tilde{B}[B(t)] \). Surprisingly, IBM has the same one-dimensional distributions as \( Z_\alpha(t) \) with \( \alpha = 1/2 \). Burdzy [6] showed that IBM satisfies the law of the iterated logarithm (LIL) with the normalizing function \( k(t) = t^{1/4} \log \log(1/t)^{3/4} \). A Chung-type LIL was established by Hu et al. [13] and Khoshnevisan and Lewis [17]. In this article we establish LIL for the subdiffusion process \( Z_\alpha(t) \).

In the next section, we investigate sample path properties of the subdiffusion process \( Z_\alpha(t) \). We show that it is a martingale with respect to the appropriate filtration. Next, we verify the Hölder continuity of the trajectories and derive the law of large numbers. The last result of the article is LIL for \( Z_\alpha(t) \). It is worth mentioning that a different version of LIL has been recently derived in Ref.[23] for the case \( \alpha \leq 1/2 \) using a very different method. Meerschaert et al. [23] actually considered a more general case \( B_\beta[S_\alpha(t)] \), where \( B_\beta(t) \) is the fractional Brownian motion. An earlier article [7] developed Strassen type LIL for \( B[S_\alpha(t)] \) limited to \( 0 < \alpha \leq 1/2 \).

## 2. SAMPLE PATH PROPERTIES

We begin with recalling some basic facts on subordinators and their inverses. A Lévy process \( \{U(t)\}_{t \geq 0} \) with nonnegative increments is called subordinator. The Laplace transform of \( U(t) \) has the form (Lévy-Khintchine formula)

\[
E(e^{-uU(t)}) = e^{-t \Psi(u)},
\]

where \( \Psi(u) \) is the so-called Laplace exponent

\[
\Psi(u) = \lambda u + \int_0^\infty (1 - e^{-ux}) v(dx).
\]

Here, \( \lambda \geq 0 \) is the drift parameter and \( v \) is the Lévy measure satisfying \( \int_0^\infty (1 \wedge x)v(dx) < \infty \).
For a subordinator $U(t)$, we call the process

$$S(t) = \inf\{\tau > 0 : U(\tau) > t\}$$

an inverse subordinator. It is also called the first-passage time process. Inverse subordinators have found wide applications in many areas of probability theory; in particular, their connection to local times of Markov processes is of special importance. Some applications to finance and physics can be found in Ref. and Refs. respectively. The connection between inverse subordinators and renewal theory is discussed in Refs. and Refs.

Consequently, the inverse $\alpha$-stable subordinator is defined as

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}, \quad (2.1)$$

$0 < \alpha < 1$, where $\{U_\alpha(\tau)\}_{\tau \geq 0}$ is the $\alpha$-stable subordinator with Laplace transform $E(e^{-uU_\alpha(\tau)}) = e^{-u^\alpha}$. There is an important relationship between $S_\alpha(t)$ and local times. The inverse $\alpha$-stable subordinator $S_\alpha(t)$ is the local time at zero of a symmetric stable Lévy process with index $\alpha' > 1$, where both stability parameters $\alpha$ and $\alpha'$ satisfy $\alpha = 1 - 1/\alpha'$. The trajectories of $S_\alpha(t)$ are continuous and singular with respect to the Lebesgue measure. Moreover, for every jump of $U_\alpha(\tau)$ there is a corresponding flat period of its inverse. These flat periods of $S_\alpha(t)$ are characteristic for the subdiffusive dynamics and they represent the waiting times in which the test particle gets immobilized in the trap.

Taking advantage of $1/\alpha$-self-similarity of $U_\alpha(\tau)$, we get

$$\mathbb{P}[S_\alpha(t) \leq \tau] = \mathbb{P}[U_\alpha(\tau) \geq t] = \mathbb{P}[\{t/U_\alpha(1)\}^\alpha \leq \tau]. \quad (2.2)$$

Therefore, the distribution of $S_\alpha(t)$ is equal to the distribution of the random variable $\{t/U_\alpha(1)\}^\alpha$. Moreover, the process $S_\alpha(t)$ is $\alpha$-self-similar; i.e., for every $c > 0$ we have $S_\alpha(ct) \overset{d}{=} c^\alpha S_\alpha(t)$. Computing the moments of $S_\alpha(t)$ shows

$$E[S_\alpha^n(t)] = \frac{t^{\alpha n} n!}{\Gamma(n\alpha + 1)}. \quad (2.3)$$

Consequently, the Laplace transform of $S_\alpha(t)$ equals

$$E(e^{-uS_\alpha(t)}) = E_\alpha(-ut^\alpha). \quad (2.4)$$

Here, the function $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}$ is the Mittag–Leffler function. Therefore, the distribution of $S_\alpha(t)$ is called the Mittag–Leffler distribution.
In what follows, we investigate sample path properties of the subdiffusion process \( Z_t(t) = B[S_0(t)] \). Recall that \( B(t) \) is the standard Brownian motion independent of \( S_0(t) \). Therefore, \( Z_t(t) \) is \( \alpha/2 \)-self-similar.

Let us introduce the filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \), where

\[
\mathcal{F}_t = \bigcap_{u \geq t} \{ \sigma[B(y) : 0 \leq y \leq u] \cup \sigma[S_0(y) : y \geq 0] \}. \tag{2.5}
\]

Here, by \( \mathcal{E}_1 \cup \mathcal{E}_2 \) we denote the \( \sigma \)-field generated by the union \( \mathcal{E}_1 \cup \mathcal{E}_2 \) of \( \sigma \)-fields \( \mathcal{E}_1, \mathcal{E}_2 \). We begin with two results verifying the martingale property.

**Theorem 2.1.** The subdiffusion process \( Z_0(t) = B[S_0(t)] \) is a martingale with respect to the filtration \( \{ \mathcal{F}_S(t) \}_{t \geq 0} \), where \( \{ \mathcal{F}_t \} \) is given in (2.5). The quadratic variation of \( Z_0(t) \) satisfies \( \langle Z_0(t), Z_0(t) \rangle = S_0(t) \). Moreover, the stochastic exponential of \( Z_0(t) \) defined as

\[
Y(t) = \exp \left\{ \lambda Z_0(t) - \frac{\lambda^2}{2} \langle Z_0(t), Z_0(t) \rangle \right\}, \quad \lambda > 0,
\]

is also a \( \{ \mathcal{F}_S(t) \} \)-martingale.

**Proof.** Note first that the filtration \( \{ \mathcal{F}_t \} \) is right-continuous, \( B(t) \) is a \( \{ \mathcal{F}_t \} \)-martingale, and for every fixed \( t_0 > 0 \) the random variable \( S_0(t_0) \) is a stopping time with respect to the filtration \( \{ \mathcal{F}_t \} \). Put \( \mathcal{G}_t = \mathcal{F}_S(t) \). Let us introduce the sequence of \( \{ \mathcal{F}_t \} \)-stopping times

\[
T_n = \inf\{ \tau > 0 : |B(\tau)| = n \}.
\]

Clearly, \( T_n \not\to \infty \) as \( n \to \infty \). Moreover, the stopped process \( B(T_n \wedge t) \) is a bounded martingale, since \( |B(T_n \wedge t)| \leq n \). Thus, from Doob’s optional sampling theorem we have for \( s < t \)

\[
E[B(T_n \wedge S_0(t)) | \mathcal{G}_s] = B[T_n \wedge S_0(s)].
\]

The right side of the above equation converges to \( B[S_0(s)] \) as \( n \to \infty \). As for the left side, we have that

\[
|B[T_n \wedge S_0(t)]| \leq \sup_{0 \leq u \leq t} |B[S_0(u)]|.
\]

Now, conditioning on \( \sigma[S_0(y) : y \geq 0] \) and taking advantage of Doob’s maximal inequality, we get that \( E[\sup_{0 \leq u \leq t} |B[S_0(u)]|] < \infty \). Consequently, from the dominated convergence theorem

\[
E[B(T_n \wedge S_0(t)) | \mathcal{G}_s] \to E[B[S_0(t)] | \mathcal{G}_s].
\]
as \(n \to \infty\). Finally, we obtain \(E\{B[S_n(t)] | \mathcal{G}_t\} = B[S_n(s)]\), thus \(Z_x(t)\) is a \(\{\mathcal{G}_t\}\)-martingale.

Now, let us determine the quadratic variation of \(Z_x(t)\). We know that the process \(Z_x^2(t) - \langle Z_x(t), Z_x(t) \rangle\) is a martingale. Moreover, using the fact that the process \(B^2(t) - t\) is a martingale and repeating arguments from the first part of the proof, we get that the process \(Z_x^2(t) - S_x(t)\) is also a martingale. Finally, from the uniqueness of the Doob–Meyer decomposition we obtain

\[
\langle Z_x(t), Z_x(t) \rangle = S_x(t).
\]

Let us now show that the exponential process \(Y(t)\) is a martingale. From Proposition 3.4, Chapter 4 in Ref.\cite{29} we get that \(Y(t)\) is a local martingale. To prove the martingale property, one only needs to verify integrability of the random variable \(\sup_{0 \leq u \leq t} Y(u)\), with \(t > 0\). We have

\[
\sup_{0 \leq u \leq t} Y(u) \leq \sup_{0 \leq u \leq t} \exp[\lambda Y(S_x(u))].
\]

Moreover, the process \(\exp[\lambda Y(S_x(t))]\) is a positive submartingale. Thus, from Doob’s maximal inequality, we get

\[
E \left( \sup_{0 \leq u \leq t} \exp[\lambda Y(S_x(u))] \right)^2 \leq 4E(\exp[2\lambda Y(S_x(t))]).
\]

Now, using the fact that for \(X \sim N(0, \sigma^2)\) we have \(E(\exp[X]) = \exp[\sigma^2/2]\), and conditioning on \(\sigma(S_x(y) : y \geq 0)\), we obtain

\[
E(\exp[2\lambda Y(S_x(t))]) = E(\exp[2\lambda^2 S_x(t)])
= \sum_{n=0}^{\infty} \frac{(2\lambda^2)^n E[S_x^n(t)]}{n!} = \sum_{n=0}^{\infty} \frac{(t^2 2\lambda^2)^n}{\Gamma(n \gamma + 1)}.
\]

The last series is convergent by the Stirling’s formula. Consequently, \(E[\sup_{0 \leq u \leq t} Y(u)] < \infty\) and \(Y(t)\) is a martingale.

Theorem 2.1 implies that \(Z_x(t)\) is also a martingale with respect to the possibly smaller natural filtration \(\{\mathcal{N}_t\}_{t \geq 0}, \mathcal{N}_t = \sigma[Z_x(s) : 0 \leq s \leq t]\).

The next result verifies regularity of the sample paths and the law of large numbers for subdiffusion \(Z_x(t)\). Recall that a real-valued function \(f\) is locally Hölder continuous of order \(\gamma\) if, for every \(M > 0\),

\[
\sup \left\{ \frac{|f(t) - f(s)|}{|t - s|^\gamma} : |t|, |s| \leq M, t \neq s \right\} < \infty.
\]
Theorem 2.2. The trajectories of the subdiffusion process $Z_\alpha(t)$ are a.s. locally Hölder continuous of order $\gamma$ for every $\gamma \in (0, \alpha/2)$. Moreover,

$$\lim_{t \to \infty} \frac{Z_\alpha(t)}{t} = 0 \quad \text{a.s.}$$

Proof. The fact that $B(S_\alpha(t))$ is locally Hölder continuous of any order less than $\alpha/2$ follows immediately upon combining the Hölder regularity of $B$ and $S_\alpha$ (the trajectories of $B$ are locally Hölder continuous of any order less than $1/2^{[20]}$ and the trajectories of $S_\alpha$ are locally Hölder continuous of any order less than $\alpha^{[5]}$).

The proof of the law of large numbers is an application of the Borel–Cantelli lemma. Fix $\epsilon > 0$ and let

$$A_n = \left\{ \sup_{2^n \leq t \leq 2^{n+1}} \left| \frac{Z_\alpha(t)}{t} \right| > \epsilon \right\}.$$

From the Markov’s inequality

$$\mathbb{P}(A_n) \leq \frac{E\left( \sup_{2^n \leq t \leq 2^{n+1}} \left| \frac{Z_\alpha(t)}{t} \right| \right)^2}{\epsilon^2} \leq \frac{E\left[ \sup_{2^n \leq t \leq 2^{n+1}} |Z_\alpha(t)| \right]^2}{2^n \epsilon^2}.$$

Since $|Z_\alpha(t)|$ is a nonnegative submartingale, from Doob’s maximal inequality we get

$$E\left( \sup_{2^n \leq t \leq 2^{n+1}} |Z_\alpha(t)| \right)^2 \leq 4E[Z_\alpha^2(2^{n+1})] = \frac{4 \cdot 2^{(n+1)\alpha}}{\Gamma(x + 1)}.$$

Finally, we get

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq \sum_{n=1}^{\infty} \frac{4 \cdot 2^{(n+1)\alpha}}{2^n \epsilon^2 \Gamma(x + 1)} < \infty.$$

Now, the Borel–Cantelli lemma ends the proof.

To prove the next theorem, we need the following technical lemma.

Lemma 2.1. Let $\{\tilde{U}_\theta(t)\}_{t \geq 0}$ be the stable subordinator with Laplace transform $E(e^{-z \tilde{U}_\theta(t)}) = e^{-\psi(z)}$. Put $0 < \epsilon < 1$, $\theta > 1$ and

$$g(t) = t^\theta [\log |\log(1/t)|]^{1-\epsilon},$$

$$\tilde{h}(t) = \sqrt{g(t) \log \log [1/g(t)]},$$

$$x_n = e^{-\omega^\theta}.$$

(2.6)
Then, the following conditions are satisfied:

(i) The series

\[ \sum_{n=1}^{\infty} \mathbb{P}[\hat{U}_\beta(x_{n+1}) > \varepsilon \hat{h}^{-1}(x_n)] \]

converges.

(ii) The series

\[ \sum_{n=1}^{\infty} \mathbb{P}[\hat{U}_\beta(x_n) - \hat{U}_\beta(x_{n+1}) \leq (1 + \varepsilon_1)\beta(1 - \beta) \frac{1}{\tau} \hat{h}^{-1}(x_n)] \]

diverges. Here, \( \varepsilon_1 > 0 \) is chosen in such a way that \( (1 + \varepsilon_1)\frac{\beta}{\tau} = \theta^{-3/2} \).

**Proof.** We will use the following properties\(^{[37]}\):

\[
\begin{align*}
\mathbb{P}[\hat{U}_\beta(1) < x] &\sim A_\beta x^{\beta/[2(1-\beta)]} \exp\{-B_\beta x^{-\beta/(1-\beta)}\} \quad \text{as} \quad x \searrow 0 \\
\mathbb{P}[\hat{U}_\beta(1) > x] &\sim C_\beta x^{-\beta} \quad \text{as} \quad x \nearrow \infty
\end{align*}
\]

Here, \( A_\beta \) and \( C_\beta \) are the appropriate positive constant, and \( B_\beta = (1 - \beta)\beta^{\beta/(1-\beta)} \).

(i) Using \( 1/\beta \) self-similarity of \( \hat{U}_\beta \), we have

\[ \mathbb{P}[\hat{U}_\beta(x_{n+1}) > \varepsilon \hat{h}^{-1}(x_n)] = \mathbb{P}[\hat{U}_\beta(1) > \varepsilon x_n^{-1/\beta} \hat{h}^{-1}(x_n)]. \]

Note that \( \hat{h}^{-1}(t) \) is asymptotically equal to \( t^{1/\beta} [\log \log(1/t)]^{1-1/\beta} \). Thus, by (2.7) we get

\[ \mathbb{P}[\hat{U}_\beta(x_{n+1}) > \varepsilon \hat{h}^{-1}(x_n)] \sim C_\beta \varepsilon^{-\beta} x_n^{-\beta/\beta} [\log \log(1/x_n)]^{-\beta+1} \]

\[ = C_\beta \varepsilon^{-\beta} e^{-h^0(n+1)/\log(n^0)} [\log(n^0)]^{-\beta+1} \]

\[ \leq C_\beta \varepsilon^{-\beta} e^{-h^0(n+1)/\log(n^0)} [\log(n^0)]^{-\beta+1} \]

as \( n \to \infty \). Thus, the series \( \sum_{n=1}^{\infty} \mathbb{P}[\hat{U}_\beta(x_{n+1}) > \varepsilon \hat{h}^{-1}(x_n)] \) converges.

(ii) We have

\[ \mathbb{P}[\hat{U}_\beta(x_n) - \hat{U}_\beta(x_{n+1}) \leq (1 + \varepsilon_1)\beta(1 - \beta) \frac{1}{\tau} \hat{h}^{-1}(x_n)] \]

\[ \geq \mathbb{P}[\hat{U}_\beta(x_n) \leq (1 + \varepsilon_1)\beta(1 - \beta) \frac{1}{\tau} \hat{h}^{-1}(x_n)] \]

\[ = \mathbb{P}[\hat{U}_\beta(1) \leq (1 + \varepsilon_1)x_n^{-1/\beta} \beta(1 - \beta) \frac{1}{\tau} \hat{h}^{-1}(x_n)]. \]
Therefore, applying (2.7) and using relation $(1 + \epsilon_1)^{\frac{\beta}{\alpha (1 - \beta)}} = 0^{-3/2}$, we obtain

$$
\mathbb{P}[\hat{U}_\beta(1) \leq (1 + \epsilon_1)x_n^{-1/\beta}(1 - \beta)^{1/\beta} \hat{h}^{-1}(x_n)]
$$

$$
\sim A_\beta\left\{(1 + \epsilon_1)\beta(1 - \beta)^{1/\beta}[\log(n^\theta)]^{1-1/\beta}\right\}^{\beta/(1-\beta)}
\times \exp\left\{-(1 + \epsilon_1)^{\beta(1-\beta)} \log(n^\theta)\right\}
$$

$$
= A_\beta\left\{(1 + \epsilon_1)\beta(1 - \beta)^{1/\beta}[\log(n^\theta)]^{1-1/\beta}\right\}^{\beta/(1-\beta)} \frac{1}{n^{\theta(1+\epsilon_1)-\beta(1-\beta)}}
$$

$$
= A_\beta\left\{(1 + \epsilon_1)\beta(1 - \beta)^{1/\beta}[\log(n^\theta)]^{1-1/\beta}\right\}^{\beta/(1-\beta)} \frac{1}{n^{\theta-1/2}}
$$

as $n \to \infty$. Thus, the series

$$
\mathbb{P}[\hat{U}_\beta(x_n) - \hat{U}_\beta(x_{n+1}) \leq (1 + \epsilon_1)\beta(1 - \beta)^{1/\beta} \hat{h}^{-1}(x_n)]
$$

diverges.

Now, we turn to the LIL. Put $c_\alpha = \gamma^{-2}(1 - z)^{-(1-z)}$ and $g(t)$ as in (2.6). For the inverse subordinator $S_\alpha(t)$ we have \cite{2.10}

$$
\limsup_{t \to \infty} \frac{S_\alpha(t)}{g(t)} = c_\alpha.
$$

Thus, by the Khintchine theorem, for every $\epsilon > 0$ and large enough $t > 0$ we get that $B[S_\alpha(t)] \leq (1 + \epsilon_1)\sqrt{2c_\alpha g(t) \log \log[1/g(t)]}$. Consequently, we obtain

$$
\limsup_{t \to \infty} \frac{B[S_\alpha(t)]}{\sqrt{g(t) \log \log[g(t)]}} \leq \sqrt{2c_\alpha}.
$$

Theorem 2.3 below shows that the above result is not sharp. The explanation is that in its derivation we have assumed the worst case in which the large increments of $B$ coincide with those of $S_\alpha$. Therefore, the constant on the right-hand side of the above formula must be replaced by a smaller one. The next theorem determines precisely the asymptotic behavior of the trajectories of $Z_\alpha(t) = B[S_\alpha(t)]$.

**Theorem 2.3** (Law of the Iterated Logarithm). The trajectories of sub-diffusion process $Z_\alpha(t)$ satisfy

(i) \quad $\limsup_{t \to \infty} \frac{Z_\alpha(t)}{\sqrt{g(t) \log \log[1/g(t)]}} = d_\alpha$
Path Properties of Subdiffusion

(ii) \[
\liminf_{t \downarrow 0} \frac{Z_s(t)}{\sqrt{g(t) \log \log[1/g(t)]}} = -d_u
\]

(iii) \[
\limsup_{t \uparrow \infty} \frac{Z_s(t)}{\sqrt{g(t) \log \log[1/g(t)]}} = d_u
\]

(iv) \[
\liminf_{t \uparrow \infty} \frac{Z_s(t)}{\sqrt{g(t) \log \log[g(t)]}} = -d_u
\]
a.s., where \(d_u = 2^{(\alpha - 1)/2 \alpha - \alpha/2} (1 - \alpha/2)^{\alpha/2 - 1}\) and \(g(t)\) is given by (2.6).

**Proof.** (i) Let us put \(h(t) = d_u \sqrt{g(t) \log \log[1/g(t)]}\). First, we will show that

\[
\limsup_{t \downarrow 0} \frac{Z_s(t)}{h(t)} \leq 1.
\]

Choose an arbitrary \(\eta > 1\) and \(\theta < 1\). We will apply the Borel–Cantelli lemma for the sets

\[
A_n = \left\{ \sup_{n+1 < s < n} \frac{|B[S_n(s)]|}{h(s)} > \eta \right\}.
\]

Using the fact that \(h(\theta^n) \leq \theta^{-x/2} h(s)\) for \(s \in (\theta^{n+1}, \theta^n)\), we get

\[
\mathbb{P}(A_n) \leq \mathbb{P}\left( \sup_{n+1 < s < n} \frac{|B[S_n(s)]|}{\theta^{x/2} h(\theta^n)} > \eta \right) \leq \mathbb{P}\left( \sup_{0 < s < \theta^n} |B[S_n(s)]| > \eta h(\theta^n) \right) = \mathbb{P}(B^*[S_n(\theta^n)] > \eta h(\theta^n)).
\] (2.8)

Here, \(B^*[t] = \sup_{0 < s < t} |B(s)|\). It follows from Ref.\[16\] (Problem 2.8.2) that the PDF \(p_{B^*}(y, t)\) of \(B^*[t]\) is bounded by

\[
p_{B^*}(y, t) \leq \frac{4}{\sqrt{2\pi t}} \exp\{-y^2/2t\}, \quad y > 0.
\]

Therefore (Ref.\[16\], Problem 2.9.22)

\[
\mathbb{P}(B^*[t] > a) \leq 4 \int_a^\infty \frac{1}{\sqrt{2\pi t}} \exp\{-y^2/2t\} dy \leq \frac{4\sqrt{t}}{a\sqrt{2\pi}} \exp\{-a^2/2t\}.
\]
Combining the above result with (2.8), we obtain

\[ P(A_n) \leq \int_0^\infty \frac{4\sqrt{y}}{a_n \sqrt{2\pi}} \exp\left\{-a_n^2/2y\right\} p_{S_n(\theta^n)}(y) dy, \tag{2.9} \]

where \( p_{S_n(t)}(y) \) is the PDF of \( S_n(t) \) and \( a_n = \eta h(\theta^n) \). Differentiating (2.2) we get

\[ p_{S_n(t)}(y) = \frac{t}{y^{1+1/\alpha}} u_y \left( \frac{t}{y^{1/\alpha}} \right), \]

where \( u_y(y) \) is the PDF of the random variable \( U_n(1) \) (recall that \( U_n(t) \) is the stable subordinator). It is known\[^{37}\] that \( u_y(y), y > 0, \) is a smooth function with one maximum. Moreover, it satisfies

\[ u_y(y) \sim C_1 y^{(\alpha-2)/(2-\alpha)} \exp\left\{-(1-\alpha) \alpha^x/(1-\alpha) y^{\alpha/(\alpha-1)}\right\} \text{ as } y \to 0, \]
\[ u_y(y) \sim C_2 y^{-\alpha-1} \text{ as } y \to \infty, \tag{2.10} \]

where \( C_1 \) and \( C_2 \) are the appropriate positive constants dependent only on \( \alpha \). Consequently, by (2.9) and some standard calculations

\[ P(A_n) \leq \int_0^\infty \frac{4\sqrt{y}}{a_n \sqrt{2\pi}} \exp\left\{-a_n^2/2y\right\} p_{S_n(\theta^n)}(y) dy \]
\[ = \int_0^\infty \frac{4\theta^n \sqrt{y}}{a_n y^{1+1/\alpha} \alpha \sqrt{2\pi}} \exp\left\{-a_n^2/2y\right\} u_y(\theta^n/y^{1/\alpha}) dy \]
\[ = \frac{4}{a_n \sqrt{2\pi}} \int_0^\infty \theta^{n/2} y^{3/2} \exp\left\{-a_n^2 y^\alpha/2\theta^{n^2}\right\} u_y(y) dy \]
\[ = \frac{4}{a_n \sqrt{2\pi}} \int_{y_0}^{y_1} \ldots dy + \frac{4}{a_n \sqrt{2\pi}} \int_{y_1}^{y_2} \ldots dy \]
\[ + \frac{4}{a_n \sqrt{2\pi}} \int_{y_2}^{\infty} \ldots dy \]
\[ =: I_1 + I_2 + I_3. \tag{2.11} \]

Here,

\[ y_0 = \left( \frac{a_n^2}{2\theta^{n^2} \alpha^x/(1-\alpha)} \right)^{(1-\alpha)/(\alpha x-2)}. \]

It is elementary to check that the only maximum of the function

\[ s(y) = \exp\left\{-a_n^2 y^\alpha/2\theta^{n^2} - (1-\alpha) \alpha^x/(1-\alpha) y^{\alpha/(\alpha-1)}\right\}, y > 0, \]
is attained exactly at \( y_0 \). Moreover,

\[
    s(y_0) = \exp\left\{ -(1 - z/2)x^{z/(2 - z)} \right\} \quad \exp\left\{ -\left(1 - \frac{z}{2}\right)z^{(1-z)/(2-z)}a_n^{2/(2-z)} \right\} \\
    \sim C_5 \left( \frac{1}{n} \right) \quad \text{as } n \to \infty. \tag{2.12}
\]

Here \( C_5 \) is the appropriate positive constant. The constants \( k_1 \leq k_2 \) in (2.11) depend only on \( z \) and are chosen in such a way that

\[
    \exp\left\{ -(1 - z)x^{z/(1-z)}(k_1 y_0)^{z/(z-1)} \right\} = s(y_0),
\]

and

\[
    \exp\left\{ -a_n^2 (k_2 y_0)^{z/2} \right\} = s(y_0).
\]

We will estimate all the components \( I_1, \ldots, I_3 \) in decomposition (2.11) separately. Note that

\[
    y_0 \sim C_4 \left[ \log \log \left( \frac{1}{\theta^n} \right) \right]^{(z-1)/2} \quad \text{as } n \to \infty, \tag{2.13}
\]

where \( C_4 \) is the appropriate positive constant. Using (2.10), for large enough \( n \) we have

\[
    I_1 = C_6 \frac{\theta_n z}{a_n} \int_{y_0}^{k_1 y_0} \frac{1}{y^{2/3}} \exp\left\{ -a_n^2 y^{2}/2 \theta_n z \right\} u_{y}(y) \, dy \\
    \leq C_7 \frac{\theta_n z}{a_n} \int_{y_0}^{k_1 y_0} y^{-2/3} \exp\left\{ -(1 - z)x^{z/(1-z)}y^{z/(z-1)} \right\} \, dy \\
    = C_7 \frac{\theta_n z}{a_n} (k_1 y_0)^{z/(2-2z)} \int_{y_0}^{k_1 y_0} y^{1/(z-1)} \exp\left\{ -(1 - z)x^{z/(1-z)}y^{z/(z-1)} \right\} \, dy \\
    \leq C_7 \frac{\theta_n z}{a_n} (k_1 y_0)^{z/(2-2z)} \int_{y_0}^{k_1 y_0} y^{1/(z-1)} \exp\left\{ -(1 - z)x^{z/(1-z)}y^{z/(z-1)} \right\} \, dy \\
    = C_7 \frac{\theta_n z}{a_n} (k_1 y_0)^{z/(2-2z)} s(y_0). \tag{2.14}
\]

Similarly, for the second term, we have

\[
    I_2 = C_6 \frac{\theta_n z}{a_n} \int_{y_0}^{k_2 y_0} \frac{1}{y^{2/3}} \exp\left\{ -a_n^2 y^{2}/2 \theta_n z \right\} u_{y}(y) \, dy \\
    \leq C_6 \frac{\theta_n z}{a_n} y_0 (k_2 - k_1) y_0^{(z^2 - 2)/(2-2z)} s(y_0). \tag{2.15}
\]
For the last term
\[
I_5 = C_{10} \frac{\theta^{n^2/2}}{a_n} \int_{k_n \rho_0}^{\infty} \frac{1}{y^{n/2}} \exp\left\{-a_n^2 y^2 / 2 \theta^{n^2}\right\} u_n(y) \, dy
\]
\[
\leq C_{11} \left( \frac{\theta^{n^2/2}}{a_n} \right)^3 (k_n \rho_0)^{-s/2 - 2s} \eta_0.
\] (2.16)

Here, \(C_i, i = 5, \ldots, 11\) are the appropriate positive constants. Now, we put the estimates (2.14)–(2.16) into (2.11). Thus, by the property (2.12) and by the fact that \(\eta > 1\), we obtain that the series \(\sum_{n=1}^{\infty} P(A_n)\) converges (note that except for \(s(\eta_0)\), the other terms in estimates (2.14)–(2.16) display logarithmic behavior). The Borel–Cantelli lemma ensures that only a finite number of events
\[
\left\{ \sup_{\eta^{n+1} < t < \eta^n} \left| \frac{Z_\eta(s)}{h(s)} \right| > \eta^{-s/2} \right\}
\]
occur. It follows that
\[
\limsup_{t \searrow 0} \frac{Z_\eta(t)}{h(t)} \leq \eta^{-s/2}.
\]

Finally, letting \(\theta \nearrow 1\) and \(\eta \searrow 1\) we get
\[
\limsup_{t \searrow 0} \frac{Z_\eta(t)}{h(t)} \leq 1.
\]

Let us now show that
\[
\limsup_{t \searrow 0} \frac{Z_\eta(t)}{h(t)} \geq 1.
\]

Choose \(0 < \epsilon_0 < 1, \theta > 1\) and put \(\hat{h}(t) = \sqrt{g(t) \log \log[1/g(t)]}\). Plainly, \(h(t) = d_x \hat{h}(t)\). Introduce the following notation
\[
Z^*_\eta(t) = \sup_{0 < s < t} Z_\eta(s),
\]
\[
x_n = e^{-n^\theta}, \quad y_n = \hat{h}^{-1}(x_n), \quad z_n = (1 + \epsilon_0) d_x^{-2/2} y_n.
\]

Let us consider the events
\[
B_n = \left\{ Z^*_\eta(z_n) \geq \hat{h}[z_n d_x^{2/2}(1 + \epsilon_0)^{-1}] \right\}.
\] (2.17)
Clearly,

\[ B_n = \{ Z^\ast \left[ (1 + \epsilon_0) d_x^{-2/\alpha} y_n \right] \geq \hat{h}(y_n) \} = \{ Z^\ast \left[ (1 + \epsilon_0) d_x^{-2/\alpha} \hat{h}^{-1}(x_n) \right] \geq x_n \}. \]

Consequently, using the fact that \( Z^\ast(t) = \sup_{0 < s < X(t)} B(s) \) together with Proposition 2.8.5 in Ref.\cite{16} and Theorem 30.1 in Ref.\cite{31}, we obtain

\[ B_n = \left\{ \hat{U}_{\beta/2}(x_n) \leq (1 + \epsilon_0) \beta(1 - \beta)^{-1/\beta} \hat{h}^{-1}(x_n) \right\}. \quad (2.18) \]

Here, \( \{ \hat{U}_{\beta/2}(t) \}_{t \geq 0} \) is the stable subordinator with Laplace transform \( E(e^{-i \hat{U}_{\beta/2}(t)}) = e^{-t^{\beta/2}} \). Now, by the application of Borel–Cantelli lemma together with part (ii) of Lemma 2.1 for \( \beta = \lambda/2 \), we obtain that the events

\[ \left\{ \hat{U}_\beta(x_n) - \hat{U}_\beta(x_{n+1}) \leq (1 + \epsilon_1) \beta(1 - \beta)^{-1/\beta} \hat{h}^{-1}(x_n) \right\} \]

occur infinitely often (i.o.). Consequently, by the property (i) in Lemma 2.1, the events

\[ \left\{ \frac{\hat{U}_\beta(x_n)}{\hat{h}^{-1}(x_n)} \leq (1 + \epsilon_1) \beta(1 - \beta)^{-1/\beta} + \epsilon \right\} \]

also occur i.o. Setting \( (1 + \epsilon_0) \beta(1 - \beta)^{-1/\beta} \hat{h}^{-1}(x_n) \leq (1 + \epsilon_1) \beta(1 - \beta)^{-1/\beta} + \epsilon \), we get that the events

\[ \left\{ \frac{\hat{U}_\beta(x_n)}{\hat{h}^{-1}(x_n)} \leq (1 + \epsilon_0) \beta(1 - \beta)^{-1/\beta} \right\} = B_n \]

occur i.o. This and (2.17) imply that the events

\[ \left\{ \frac{Z^\ast(z_n)}{d_x \hat{h}(z_n)} \geq (1 + \epsilon_0)^{-1/\beta} \right\} \]

also occur i.o. Since \( \epsilon_0 \) can be chosen arbitrarily close to 0 and since \( \hat{h} \) is monotonically increasing, we immediately conclude that

\[ \limsup_{t \to 0} \frac{Z_x(t)}{d_x \hat{h}(t)} \geq 1, \]

which ends the proof of part (i) of the theorem.

The proof of part (iii) of the theorem is analogous to (i). Parts (ii) and (iv) follow immediately from (i) and (iii), respectively, by the fact that the process \(-B(t)\) is also Brownian motion. \( \square \)
Remark 2.1. From the above theorem one can obtain the law of large numbers derived previously in Theorem 2.2. However, we leave the proof of Theorem 2.2, since it is elementary and exploits the martingale property.

Remark 2.2. Another fundamental model of subdiffusion is the fractional Brownian motion \( \{B_H(t)\}_{t \geq 0} \) with self-similarity parameter \( 0 < H < 1/2 \). Recall that \( B_H(t) \) is the centered Gaussian process with the covariance function \( E[B_H(s)B_H(t)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) \). The process \( B_H(t) \) is not a martingale. It is even not a semi-martingale, since its quadratic variation diverges. Moreover, \( B_H(t) \) satisfies LIL with the normalizing function \( k(t) = t^H \log \log (1/t)^{1/2} \). Therefore, results of Theorems 2.1 and 2.3 in the article can be applied to compare the asymptotic behavior of single trajectories as well as the quadratic variations for both models of subdiffusion. This may be useful in solving the timely problem of identification of the anomaly type from experimental data \([27,35]\).

ACKNOWLEDGMENTS

The author thanks the anonymous referees for their constructive comments leading to improvement of the presentation. The author was partially supported by the Foundation for Polish Science through the Domestic Grant for Young Scientists (2009).

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Path Properties of Subdiffusion