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SHORT AND LONG MEMORY FRACTIONAL
ORNSTEIN–UHLENBECK $\alpha$-STABLE PROCESSES

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We show that, similar to the Gaussian case, the fractional Ornstein–Uhlenbeck $\alpha$-stable process obtained via the Lamperti transformation of the linear fractional stable motion is a different stationary process than the one defined as the solution of the Langevin equation driven by a linear fractional stable noise. We investigate the asymptotic dependence structure of the first process and prove that, in contrast to the second case, it is a short-memory process in the sense of the measure of dependence appropriate for processes with infinite second moment.

Keywords Lamperti transformation; Langevin equation; Long memory; Ornstein–Uhlenbeck process; Stable processes.

Mathematics Subject Classification Primary 60H10; Secondary 60G52.

1. INTRODUCTION

Linear fractional stable motion (LFSM) is an extension of the well-known fractional Brownian motion to the $\alpha$-stable case. It is defined in the following way\cite{5,11}: Let $0 < \alpha \leq 2$, $0 < H < 1$, $H \neq 1/\alpha$ and $a, b \in \mathbb{R}$, $|a| + |b| > 0$. Then the process

$$L_{\alpha,H}(t) = \int_{-\infty}^{\infty} \left( a \left( (t - s)^{H-1/\alpha} - (-s)^{H-1/\alpha} \right) + b \left( (t - s)^{H-1/\alpha} - (-s)^{H-1/\alpha} \right) \right) \bar{L}_{\alpha}(ds), \quad t \in \mathbb{R}, \quad (1)$$

is called LFSM. Here $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$ and $L_{\alpha}(ds)$ is the standard symmetric $\alpha$-stable random measure on $\mathbb{R}$ with control measure as Lebesgue measure (Refs.\cite{5,11}). $L_{\alpha,H}(t)$ is a self-similar, stationary-increment process. For $\alpha = 2$ it reduces to the fractional Brownian motion.

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In the Gaussian case, the prominent example of a stationary stochastic process is the celebrated Ornstein–Uhlenbeck (O–U) process. It can be derived either as a stationary solution of the Langevin equation

$$dY(t) = -\lambda Y(t)dt + dB(t), \quad \lambda > 0,$$

where $B(t)$ is the Brownian motion, or as the Lamperti transformation \cite{2,6} of $B(t)$. In Ref.\cite{3}, the authors introduce two types of fractional O–U processes using a similar analogy. The first one is defined as the solution of (2) with the noise process $B(t)$ replaced by the fractional Brownian motion $B_H(t)$, i.e.,

$$dY_1(t) = -\lambda Y_1(t)dt + dB_H(t), \quad \lambda > 0.$$ (3)

The second one is defined as the Lamperti transformation of $B_H(t)$

$$Y_2(t) = e^{-tH}B_H(e').$$ (4)

As it is shown in Ref.\cite{3}, these are two different stationary Gaussian processes, and the essential difference lies in the dependence structure. Let us recall that in the Gaussian case a stationary process $Y(t)$ has long-range dependence (LRD) \cite{1}, if its covariance function satisfies

$$\sum_{t=1}^{\infty} |Cov(Y(t), Y(0))| = \infty.$$ (5)

For $H \neq 1/2$ the covariance function of $Y_1(t)$ decays as the power function, i.e.,

$$Cov(Y_1(t), Y_1(0)) \sim ct^{2H-2}$$

as $t \to \infty$. Here $c$ is the appropriate constant. This implies that for $H \in (1/2, 1)$ we have $\sum_{n=0}^{\infty} |Cov(Y_1(n), Y_1(0))| = \infty$, thus the process exhibits LRD. The covariance function of $Y_2(t)$ decays exponentially and in this case the process does not have LRD property.

In what follows, we extend the above considerations to the $\alpha$-stable case by replacing $B_H(t)$ with the linear fractional stable motion $L_{\alpha,H}(t)$ defined in (1). However, for a stationary stable process with index of stability $\alpha < 2$ the covariance function is not defined. Instead, another measure of dependence must be used. Basing on the results in Refs.\cite{7-9,11}, we adopt the following measure of dependence for a stationary $\alpha$-stable process $Y(t)$

$$r(\theta_1; \theta_2; t) = E[\exp\{i(\theta_1 Y(t) + \theta_2 Y(0))\}]$$

$$- E[\exp\{i\theta_1 Y(t)\}]E[\exp\{i\theta_2 Y(0)\}], \quad \theta_1, \theta_2 \in \mathbb{R}.$$ (6)
Unlike the covariance function, \( r(\theta_1; \theta_2; t) \) is always well defined and is equal to zero for independent \( Y(t) \) and \( Y(0) \). Additionally, if the process has finite second moment, it is asymptotically equivalent to the covariance function. The function \( r(\theta_1; \theta_2; t) \) is closely related to the so-called “generalized codifference” (Ref.\(^{[11]}\))

\[
I(\theta_1; \theta_2; t) = -\ln E\left[ \exp\{i\theta_1 Y(t) + \theta_2 Y(0)\} \right]
+ \ln E\left[ \exp\{i\theta_1 Y(t)\} \right] + \ln E\left[ \exp\{i\theta_2 Y(0)\} \right],
\]

(7)
since \( r(\theta_1; \theta_2; t) = C(\theta_1; \theta_2) \cdot (e^{-i(\theta_1; \theta_2; t)} - 1) \) with \( C(\theta_1; \theta_2) = E[\exp\{i\theta_1 Y(0)\}] \cdot E[\exp\{i\theta_2 Y(0)\}] \). Thus, if \( I(\theta_1; \theta_2; t) \to 0 \) as \( t \to \infty \), then \( \frac{r(\theta_1, \theta_2; t)}{\exp(i\theta_1 Y(0))} \to -C(\theta_1; \theta_2) \) as \( t \to \infty \), which implies that \( r(\cdot) \) and \( I(\cdot) \) are asymptotically equivalent. In the special case, when \( \theta_1 = 1 \), \( \theta_2 = -1 \), the generalized codifference \( I(1, -1, t) \) becomes (up to a constant) the standard codifference (Ref.\(^{[11]}\)). As shown in Ref.\(^{[10]}\), the asymptotic behaviour of the codifference determines the chaotic properties (ergodicity, weak mixing, mixing) of the stationary \( z \)-stable processes. Thus, the asymptotic equivalence of \( r(\cdot) \) and \( I(\cdot) \) indicates that \( r(\cdot) \) can also be helpful in detecting the ergodic properties of the \( z \)-stable processes. The presence of the parameters \( \theta_1 \) and \( \theta_2 \) in the definition of \( r(\cdot) \) has the following advantage: consider two stationary \( z \)-stable stochastic processes \( Y \) and \( Y' \). In order to show that the two processes are different, we examine the asymptotic behaviour of the corresponding measures of dependence \( r_y(\theta_1; \theta_2; t) \) and \( r_y(\theta'_1; \theta'_2; t) \). If the measures are not asymptotically equivalent at least for one specific choice of \( \theta_1 \) and \( \theta_2 \), then the processes \( Y \) and \( Y' \) must be different.

The above mentioned properties of \( r(\theta_1; \theta_2; t) \) indicate that it can be regarded as an appropriate tool to examine the dependence structure of the \( z \)-stable processes.

Now, let us introduce the following definition of LRD in the \( z \)-stable case. We say that a stationary \( z \)-stable process has long memory, if the corresponding measure of dependence \( r(\theta_1; \theta_2; t) \) satisfies

\[
\sum_{i=1}^{\infty} |r(\theta_1; \theta_2; t)| = \infty.
\]

(8)
The above definition of long memory is a straightforward extension of the one used in the Gaussian case, (5), and has found a widespread acceptance in the \( z \)-stable case (Refs.\(^{[7-9]}\)). For \( z = 2 \) the definitions (5) and (8) are equivalent. The terms “long memory” and “long-range dependence” are used interchangeably (Refs.\(^{[11]}\)), and define the phenomena in which the measure of dependence decays slowly. Let us emphasize that the concept of long memory in the non-Gaussian world is still not well-formulated and
is a subject of extensive research. Therefore, the presented definition of LRD and the obtained results should be viewed as one of the possible approaches to long memory for \( \alpha \)-stable processes.

The fractional Langevin equation with linear fractional stable noise is defined as

\[
d Z_1(t) = -\lambda Z_1(t) \, dt + dL_{\alpha,H}(t), \quad \lambda > 0. \tag{9}
\]

As it is shown in a recent paper by Maejima and Yamamoto\[^8\], for \( \alpha \in (1, 2) \) and \( H > 1/\alpha \), the unique stationary solution of (9) has the form

\[
Z_1(t) = L_{\alpha,H}(t) - \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} L_{\alpha,H}(s) \, ds. \tag{10}
\]

Moreover, the authors prove that for the process \( Z_1(t) \) the measure of dependence satisfies

\[
r(\theta_1; \theta_2; t) \sim c |t|^{\alpha(H-1)} \quad \text{as} \quad t \to \infty,
\]

where \( c \) is the appropriate constant, thus for \( H > 1/\alpha \) the process has long memory, which is the analogous result to the Gaussian case. \( Z_1(t) \) is called the long memory \( \alpha \)-stable fractional O–U Process.

In the next section, we investigate the dependence structure of another stationary \( \alpha \)-stable process, being the extension of (4) to the \( \alpha \)-stable case, and defined as the following Lamperti transformation

\[
Z_2(t) = e^{-H} L_{\alpha,H}(e^t). \tag{11}
\]

We show that, similarly to the Gaussian counterpart (4), the measure of dependence \( r(\cdot) \) of \( Z_2(t) \) decays exponentially and therefore we can call \( Z_2(t) \) the short memory \( \alpha \)-stable fractional O–U process.

\section*{2. ASYMPTOTIC DEPENDENCE STRUCTURE}

In the next three theorems we give the precise formulas for the asymptotic behaviour of the measure of dependence \( r(\cdot) \) for \( Z_2(t) \). We exclude the two cases \( \theta_1 \theta_2 = 0 \) and \( a = b = 0 \), since then \( r(\theta_1; \theta_2; t) = 0 \).

In the proofs we frequently use the following property (Ref.\[^{[11]}\], p. 122)

\[
E\left[ \exp \left\{ i\theta \int_B f(x) L_\alpha(dx) \right\} \right] = \exp \left\{ -\left| \theta \right|^\alpha \int_B |f(x)|^\alpha dx \right\} \tag{12}
\]

with \( B \subset \mathbb{R} \) and \( f \in L^\infty(B, dx) \). We also take advantage of the two key inequalities (Ref.\[^{[8]}\]): For \( r, s \in \mathbb{R} \)

\[
\|r + s\|^\alpha - |r|^\alpha - |s|^\alpha \leq \begin{cases} 2|r|^\alpha & \text{if } 0 < \alpha \leq 1 \\ (\alpha + 1)|r|^\alpha + \alpha |r||s|^{\alpha-1} & \text{if } 1 < \alpha \leq 2. \end{cases} \tag{13}
\]
Theorem 2.1. Let $0 < \alpha < 1$ and $0 < H < 1$. Then the measure of dependence of $Z_2(t)$ satisfies

$$r(\theta_1; \theta_2; t) \sim C_\alpha(\theta_1; \theta_2) A_\alpha(\theta_1; \theta_2) e^{-\alpha H(1-H)}$$

as $t \to \infty$, where

$$C_\alpha(\theta_1; \theta_2) = \exp \left\{ - \sum_{i=1}^{2} |\theta_i|^2 \int_{-\infty}^{\infty} \left[ a \left( (1-s)^{H-1/2} - (s)^{H-1/2} \right) + b \left( (1-s)^{H-1/2} - (s)^{H-1/2} \right) \right] ds \right\}$$

and

$$A_\alpha(\theta_1; \theta_2) = \int_{0}^{\infty} \left\{ -\theta_1 a s^{H-1/2} + \theta_2 a (H-1/2) s^{H-1/2-1} \right\}^2 ds$$

$$+ \int_{0}^{\infty} \left\{ -\theta_1 b s^{H-1/2} + \theta_2 b (1/\alpha - H) s^{H-1/2-1} \right\}^2 ds.$$ (14)

Proof. We have $Z_2(t) = e^{-\alpha H} L_{\alpha H}(e^t) = \int_{-\infty}^{\infty} f(s, t) L_{\alpha}(ds)$ with

$$f(s, t) = e^{-\alpha H} a \left[ (e^t - s)^{H-1/2} - (-s)^{H-1/2} \right] + e^{-\alpha H} b \left[ (e^t - s)^{-H-1/2} - (-s)^{-H-1/2} \right].$$

For the generalized codifference $I(\theta_1; \theta_2; t)$, the following relation holds true $r(\theta_1; \theta_2; t) = C_\alpha(\theta_1; \theta_2) \cdot (e^{-i(\theta_1; \theta_2; t)} - 1)$, where $C_\alpha(\theta_1; \theta_2) = E[\exp(i \theta_1 Y(0))] E[\exp(i \theta_2 Y(0))]$. Note that by formula (12), the constant $C_\alpha(\cdot)$ is given by (14). Therefore, it is enough to investigate the asymptotic behaviour of $I(\cdot)$. Taking advantage of (7) and (12) we obtain

$$I(\theta_1; \theta_2; t) = \int_{-\infty}^{\infty} \left[ \theta_1 f(s, t) + \theta_2 f(s, 0) \right]^2 ds$$

$$= \int_{-\infty}^{0} \ldots ds + \int_{0}^{1} \ldots ds + \int_{1}^{e^t} \ldots ds + \int_{e^t}^{\infty} \ldots ds$$

$$=: I_1(t) + I_2(t) + I_3(t) + I_4(t).$$ (16)

In what follows, we estimate every $I_j(t)$, $j = 1, \ldots, 4$, separately.
Let us begin with $I_1(t)$. After some standard calculations and by the change of variables $s \to -e^{\alpha t} s$, we get
\[ I_1(t) = e^{2\alpha H t} \int_0^\infty \left[ |p(s,t) + q(s,t)|^2 - |p(s,t)|^2 - |q(s,t)|^2 \right] ds, \]
where
\[ p(s,t) = e^{-\alpha t} \partial_s [ (e^{(1-H)} + s)^{H-1/2} - s^{H-1/2} ] \quad \text{and} \quad \tag{17} \]
\[ q(s,t) = \partial_s (e^{-\alpha t} + s)^{H-1/2} - s^{H-1/2}. \tag{18} \]
For fixed $s \in (0, \infty)$ we see that
\[ e^{\alpha t} p(s,t) \to -\partial_s s^{H-1/2} =: \rho(s) \]
as $t \to \infty$. Using the mean-value theorem
\[ f(r + s) - f(r) = s \int_0^1 f'(r + us) du, \tag{19} \]
where $f$ is accordingly smooth, and the dominated convergence theorem (DCT), we obtain
\[ e^{\alpha t} q(s,t) = \partial_s (H - 1/2) \int_0^1 (s + ue^{-\alpha t})^{H-1/2-1} du =: q(s) \]
as $t \to \infty$. Consequently, for fixed $s \in (0, \infty)$
\[ e^{2\alpha H} \left[ |p(s,t) + q(s,t)|^2 - |p(s,t)|^2 - |q(s,t)|^2 \right] \]
\[ \to \left[ |\rho(s) + q(s)|^2 - |\rho(s)|^2 - |q(s)|^2 \right] \tag{20} \]
as $t \to \infty$. To apply DCT, we use inequality (13) together with the mean-value theorem and get
\[ \sup_{t \geq 1} e^{2\alpha H} |p(s,t) + q(s,t)|^2 - |p(s,t)|^2 - |q(s,t)|^2 | \]
\[ \leq \sup_{t \geq 1} 1_{(0,1)}(s) e^{2\alpha H} \left[ |p(s,t) + q(s,t)|^2 - |p(s,t)|^2 - |q(s,t)|^2 \right] \]
\[ + \sup_{t \geq 1} 1_{(1,\infty)}(s) e^{2\alpha H} \left[ |p(s,t) + q(s,t)|^2 - |p(s,t)|^2 - |q(s,t)|^2 \right] \]
\[ \leq \sup_{t \geq 1} 1_{(0,1)}(s) e^{2\alpha H} 2 |p(s,t)|^2 + \sup_{t \geq 1} 1_{(1,\infty)}(s) e^{2\alpha H} 2 |q(s,t)|^2 \]
\[ \leq 1_{(0,1)}(s) e^{2H-1} + 1_{(1,\infty)}(s) e^{2H-1-\alpha}, \]
which is integrable on $(0, \infty)$. Here $c_1$ and $c_2$ are the appropriate constants independent of $s$ and $t$. Thus, from DCT we get

$$I_1(t) \sim e^{i \lambda H} e^{-i \lambda H} \int_0^\infty \{|p_\infty(s) + q_\infty(s)|^2 - |p_\infty(s)|^2 - |q_\infty(s)|^2\} ds \quad (21)$$

as $t \to \infty$.

For the next term, we use inequality (13) together with DCT and show in a similar manner that $I_2(t) = O(e^{-i \lambda H})$. Thus, the contribution of $I_2$ is negligible. The same statement applies to $I_4(t) = O(e^{-i \lambda H})$.

We continue our estimations for $I_3(t)$. After the change of variables $s \to e^{i \lambda H} s$, we have

$$I_3(t) = e^{i \lambda H} \int_0^\infty \{|w(s, t) + z(s, t)|^2 - |w(s, t)|^2 - |z(s, t)|^2\} ds,$$

where

$$w(s, t) = e^{-i \lambda H} \left[a(e^{i (1-H)} - s)^{H-1/2} - bs^{H-1/2}\right] \cdot 1_{(e^{-i \lambda H}, e^{i (1-H)})}(s) \quad (22)$$

and

$$z(s, t) = \theta_2 b \left[(s - e^{-i \lambda H})^{H-1/2} - s^{H-1/2}\right] \cdot 1_{(e^{-i \lambda H}, e^{i (1-H)})}(s). \quad (23)$$

In a similar manner as for $I_1(t)$, we get that for fixed $s \in (0, \infty)$

$$e^{i \lambda H} w(s, t) \to -\theta_1 b s^{H-1/2} =: w_\infty(s)$$

and also

$$e^{i \lambda H} z(s, t) \to \theta_2 b (1/\alpha - H)s^{H-1/2-1} =: z_\infty(s)$$

as $t \to \infty$. Consequently, DCT yields

$$I_3(t) \sim e^{i \lambda H} e^{-i \lambda H} \int_0^\infty \{|w_\infty(s) + z_\infty(s)|^2 - |w_\infty(s)|^2 - |z_\infty(s)|^2\} ds \quad (24)$$

as $t \to \infty$. Finally, putting together formulas (21) and (24), we get the desired result.

We pass on to the case $\alpha = 1$. We recall the fact that the two cases $\theta_1 \theta_2 = 0$ and $a = b = 0$ are excluded, since then $r(\theta_1; \theta_2; t) = 0$. 
Theorem 2.2. Let \( \alpha = 1 \) and \( 0 < H < 1 \). Then the measure of dependence of \( Z_2(t) \) satisfies

(i) If \( b = 0 \) and \( \theta_1, \theta_2 > 0 \) then \( r(\theta_1; \theta_2; t) = 0 \),

(ii) If \( a = 0 \) and \( \theta_1, \theta_2 < 0 \) then

\[
\begin{align*}
  r(\theta_1; \theta_2; t) &\sim 2|b|C_4(\theta_1; \theta_2) \left( \frac{|\theta_1|}{H} e^{-H \cdot 1_{(0, 1/2]}(H)} + |\theta_2| e^{-H \cdot 1_{[1/2, 1]}(H)} \right), \\
  &\text{as } t \to \infty, \\
\end{align*}
\]

from the triangle inequality we see that

\[
\begin{align*}
  \text{otherwise} \quad r(\theta_1; \theta_2; t) &\sim -C_4(\theta_1; \theta_2)A_1(\theta_1; \theta_2)e^{-H(1-H)} \\
  &\text{as } t \to \infty, \text{ where } C_4 \text{ and } A_1 \text{ are given in (14) and (15), respectively.}
\end{align*}
\]

Proof. First, we determine, in which case the constant \( A_1(\theta_1; \theta_2) = 0 \). From (15) we get

\[
A_1(\theta_1; \theta_2) = \int_0^\infty \ldots ds + \int_0^\infty \ldots ds =: A_{11}(\theta_1; \theta_2) + A_{12}(\theta_1; \theta_2).
\]

From the triangle inequality we see that \( A_1(\theta_1; \theta_2) = 0 \Leftrightarrow \{ A_{11}(\theta_1; \theta_2) = 0 \text{ and } A_{12}(\theta_1; \theta_2) = 0 \} \). Additionally, we have \( A_{11}(\theta_1; \theta_2) = 0 \Leftrightarrow \{ a = 0 \text{ or } \theta_1, \theta_2 > 0 \} \) as well as \( A_{12}(\theta_1; \theta_2) = 0 \Leftrightarrow \{ b = 0 \text{ or } \theta_1, \theta_2 < 0 \} \). Since the cases \( \theta_1, \theta_2 = 0 \text{ or } a = b = 0 \) are excluded, we obtain

\[
A_1(\theta_1; \theta_2) = 0 \Leftrightarrow \{ a = 0 \text{ and } \theta_1, \theta_2 < 0 \} \text{ or } \{ b = 0 \text{ and } \theta_1, \theta_2 > 0 \}.
\]

The case \( \{ b = 0 \text{ and } \theta_1, \theta_2 > 0 \} \) is trivial, since then it is easy to verify that for every term in formula (16) we have \( I_i(t) = 0, \ i = 1, \ldots, 4 \). Thus, we obtain part (i) of the theorem.

We pass on to the second possibility \( \{ a = 0 \text{ and } \theta_1, \theta_2 < 0 \} \). In this case only \( I_1(t) \) and \( I_2(t) \) from (16) disappear, therefore we need to find the asymptotic behaviour of \( I_2(t) \) and \( I_4(t) \). Using some standard arguments and DCT we show that

\[
I_2(t) \sim -2\frac{|\theta_1b|}{H} e^{-Ht} \quad (25)
\]

and

\[
I_4(t) \sim -2|\theta_2b| e^{-t(1-H)} \quad (26)
\]

as \( t \to \infty \).
Now, from (25) and (26) we get that for $H < 1/2$ we obtain $I(\theta_1; \theta_2; t) \sim I_2(t)$, for $H > 1/2$ we obtain $I(\theta_1; \theta_2; t) \sim I_4(t)$ and for $H = 1/2$ we get $I(\theta_1; \theta_2; t) \sim I_2(t) + I_4(t)$ as $t \to \infty$. Thus, we have proved part (ii) of the theorem.

In any other case, i.e., when $A_1(\theta_1; \theta_2)$ is a non-zero constant, the proof of Theorem 2.1 applies and we get $r(\theta_1; \theta_2; t) \sim -C_1(\theta_1; \theta_2)A_1(\theta_1; \theta_2)e^{-H/(1-H)}$ as $t \to \infty$.

The next theorem determines the asymptotic dependence structure of $Z_\alpha(t)$ when the index of stability is such that $1 < \alpha < 2$.

**Theorem 2.3.** Let $1 < \alpha < 2$, $0 < H < 1$ and $H \neq 1/\alpha$. Then the measure of dependence of $Z_\alpha(t)$ satisfies

(i) If $1 - \frac{1}{\alpha} < H < \frac{1}{\alpha}$ then

$$ r(\theta_1; \theta_2; t) \sim -C_\alpha(\theta_1; \theta_2)A_\alpha(\theta_1; \theta_2)e^{-i\alpha H(1-H)} $$

(ii) If $H < 1 - \frac{1}{\alpha}$ then

$$ r(\theta_1; \theta_2; t) \sim -C_\alpha(\theta_1; \theta_2)B_\alpha(\theta_1; \theta_2)e^{-iH} $$

(iii) If $H > \frac{1}{\alpha}$ then

$$ r(\theta_1; \theta_2; t) \sim -C_\alpha(\theta_1; \theta_2)D_\alpha(\theta_1; \theta_2)e^{-iH(1-H)} $$

as $t \to \infty$. The constants $C_\alpha$ and $A_\alpha$ are given in (14) and (15), respectively, whereas

$$ B_\alpha(\theta_1; \theta_2) = \int_{0}^{\infty} a \theta_1 \text{sgn}\{\theta_2\} |\theta_2|^{\alpha - 1} |a|^\alpha (1 + s)^{H-1/2} - s^{H-1/2} s^{H-1/2} \, ds $$

$$ - \int_{0}^{1} a \theta_1 b s^{H-1/2} |\theta_2| a(1 - s)^{H-1/2} - b s^{H-1/2} |^{\alpha - 1} $$

$$ \times \text{sgn}\{\theta_2\} a(1 - s)^{H-1/2} - b s^{H-1/2} | \, ds $$

$$ - \int_{1}^{\infty} a \theta_1 \text{sgn}\{\theta_2\} b s^{H-1/2} |\theta_2| b((s - 1)^{H-1/2} - s^{H-1/2}) |^{\alpha - 1} \, ds, $$

and

$$ D_\alpha(\theta_1; \theta_2) = |a|^\alpha \int_{0}^{\infty} a(H - 1/\alpha) \theta_2 |\theta_2|^{\alpha - 1} \text{sgn}\{\theta_1\} s^{H-1/2} $$

$$ \times (s + 1)^{H-1/2} - s^{H-1/2} |^{\alpha - 1} \, ds $$
+ \int_0^1 z|\theta_1|^{p-1}\theta_2 b(1/\chi - H)s^{H-1/2-1}|a(1-s)^{H-1/2} - b_s^{H-1/2}|^{p-1}
\times \text{sgn}\{\theta_1[a(1-s)^{H-1/2} - b_s^{H-1/2}]\}ds
+ \int_1^\infty z\theta_2 b(H - 1/\chi)\text{sgn}\{\theta_1 b|\theta_1 b|^{p-1}
\times |(s - 1)^{H-1/2} - s^{H-1/2-1}|s^{H-1/2-1}ds.

\textbf{Proof.} (i) Recall the decomposition in (16). We have \(I_1(t) = e^{zH^2} \int_0^\infty \{[p(s, t) + q(s, t)]^2 - |p(s, t)|^2 - |q(s, t)|^2\}ds\), where \(p(s, t)\) and \(q(s, t)\) are given in (17) and (18), respectively. For fixed \(s \in (0, \infty)\) we get that (20) holds, since \(H - 1/\chi < 0\). Next, we apply DCT and the second part of inequality (13) in combination with the mean-value theorem (19). We obtain

\[I_1(t) \sim e^{zH^2} e^{-i\chi H} \int_0^\infty \{|p_\infty(s) + q_\infty(s)|^2 - |p_\infty(s)|^2 - |q_\infty(s)|^2\}ds \quad (27)\]
as \(t \to \infty\).

For the next term \(I_2(t)\) we use inequality (13) together with DCT and show in a standard way that \(I_2(t) = O(e^{-\alpha t})\), \(t \to \infty\). Thus, since \(z(1 - H) < 1\), the integral \(I_2(t)\) decays faster than \(I_1(t)\).

In case of \(I_3(t)\) we cannot use DCT directly, we need more delicate estimations. We have \(I_3(t) = e^{zH^2} \int_0^\infty \{|w(s, t) + z(s, t)|^2 - |w(s, t)|^2 - |z(s, t)|^2|ds\), where \(w(s, t)\) and \(z(s, t)\) are given in (22) and (23), respectively. Set

\[G(s, t) := |w(s, t) + z(s, t)|^2 - |w(s, t)|^2 - |z(s, t)|^2,\]
\[G_\infty(s) := |w_\infty(s) + z_\infty(s)|^2 - |w_\infty(s)|^2 - |z_\infty(s)|^2\]

with \(w_\infty(s) = -\theta_1 bs^{H-1/2}\) and \(z_\infty(s) = \theta_2 b(1/\chi - H)s^{H-1/2-1}\). We will show that

\[\left|\int_0^\infty [e^{zH} G(s, t) - G_\infty(s)]ds\right| \to 0\]
as \(t \to \infty\). Fix \(\epsilon > 0\) and put

\[\int_0^\infty [e^{zH} G(s, t) - G_\infty(s)]ds = \int_0^{e^{-\alpha t}} \ldots ds + \int_{e^{-\alpha t}}^{e^{-\alpha t} + \epsilon} \ldots ds + \int_{e^{-\alpha t} + \epsilon}^{e^{(1-H) - \epsilon}} \ldots ds
+ \int_{e^{(1-H) - \epsilon}}^{e^{(1-H)}} \ldots ds + \int_{e^{(1-H)}}^\infty \ldots ds
= : f_1(t) + f_2(t) + f_3(t) + f_4(t) + f_5(t).\]
Since $G_\infty(s)$ is integrable on $(0, \infty)$, we obtain $|f_i(t)| \to 0$ and $|f_i(t)| \to 0$ as $t \to \infty$. For the second term we have from (13) that

$$
|f_2(t)| \leq \frac{\epsilon}{\alpha} \int_{-\infty}^{t} e^{\alpha H} \left| L(s, t) - G_\infty(s) \right| ds \leq (\epsilon + 1) \int_{-\infty}^{t} e^{2\alpha H} |w(s, t)|^2 ds
$$

Moreover,

$$
\int_{-\infty}^{t} e^{2\alpha H} |w(s, t)|^2 ds \to 0
$$

Next, since $H - 1/\alpha < 0$, we have

$$
J_{21}(t) \leq d_1 \int_{-\infty}^{t} \left( (1 - s)^{\alpha H - 1} + s^{\alpha H - 1} \right) ds
$$

Additionally, since $H - 1/\alpha < 0$, we have

$$
J_{21}(t) \leq d_1 \left[ (1 - s)^{\alpha H - 1} + s^{\alpha H - 1} \right] ds
$$

$$
= -\frac{d_1}{\alpha H} \left[ (1 - s)^{\alpha H - 1} - (1 - e^{-\alpha H})^{\alpha H} \right]
$$

$$
+ \frac{d_1}{\alpha H} \left[ (e^{-\alpha H} + \epsilon)^{\alpha H} - (e^{-\alpha H})^{\alpha H} \right].
$$

Next, since $H - 1/\alpha < 0$, we get for $s \in (e^{-\alpha H}, e^{-\alpha H+\epsilon})$

$$
|(s - e^{-\alpha H})^{\alpha H - 1/2} - s^{\alpha H - 1/2}| \leq (1/\alpha - H) e^{-\alpha H}(s - e^{-\alpha H})^{\alpha H - 1/2 - 1},
$$

and consequently

$$
J_{22}(t) \leq d_2 \int_{-\infty}^{t} \left[ (1 - s)^{\alpha H - 1} + s^{\alpha H - 1} \right] (s - e^{-\alpha H})^{\alpha H - 1/2 - 1} ds
$$

$$
\leq d_2 (1 - e^{-\alpha H} - \epsilon)^{\alpha H - 1/2} e^{2\alpha H - 1/2 - 1} + d_4 e^{2\alpha H - 1/2 - 1}.
$$

Moreover,

$$
J_{23}(t) = d_3 \left[ (e^{-\alpha H} + \epsilon)^{\alpha H} - (e^{-\alpha H})^{\alpha H} \right],
$$

$$
J_{24}(t) = d_4 \left[ (e^{-\alpha H} + \epsilon)^{\alpha H - 1/2} - (e^{-\alpha H})^{\alpha H - 1/2} \right],
$$

where $d_i, i = 1, \ldots, 6$ are the appropriate constants independent of $t, s$, and $\epsilon$. Using the fact that $1 - 1/\epsilon < H < 1/\epsilon$, we obtain $\lim_{\epsilon \to 0} \lim_{t \to \infty} J_{2i}(t) = 0$ for every $i = 1, \ldots, 4$, which implies $\lim_{\epsilon \to 0} \lim_{t \to \infty} J_2(t) = 0$.

We pass on to $J_3(t)$. For fixed $s \in (0, \infty)$ we get from the proof of Theorem 2.1

$$
e^{\alpha H} G(s, t)1_{(e^{-\alpha H}, e^{-\alpha H+\epsilon})}(s) \to G_\infty(s)1_{(0, \infty)}(s).
$$
as $t \to \infty$. Additionally,

$$
\sup_{t > \frac{1}{\gamma}} e^{2\alpha_1} |G(s, t)\mathbf{I}_{(e^{\alpha_1 + \varepsilon, (1 - H)^{-\varepsilon)}}(s) - G_{\infty}(s)| \leq \sup_{t > \frac{1}{\gamma}} e^{2\alpha_1} (s + 1)|w(s, t)|^2
$$

$$
+ \sup_{t > \frac{1}{\gamma}} e^{2\alpha_1} |s| |w(s, t)| |z(s, t)|^{2-1}
$$

$$
+ \sup_{t > \frac{1}{\gamma}} e^{2\alpha_1} (s + 1)|z(s, t)|^2
$$

$$
+ \sup_{t > \frac{1}{\gamma}} e^{2\alpha_1} |z(s, t)| |w(s, t)|^{2-1}
$$

which is integrable on $(0, \infty)$. Thus, from DCT we get $J_3(t) \to 0$ as $t \to \infty$. For $J_4(t)$ we have

$$
|J_4(t)| \leq \int_{e^{\alpha_1 - \varepsilon}}^{e^{\alpha_1}} |e^{2\alpha_1} G(s, t) - G_{\infty}(s)| \, ds \leq (s + 1) \int_{e^{\alpha_1 - \varepsilon}}^{e^{\alpha_1}} e^{2\alpha_1} |z(s, t)|^{2-1} \, ds
$$

$$
+ \varepsilon \int_{e^{\alpha_1 - \varepsilon}}^{e^{\alpha_1}} e^{2\alpha_1} |w(s, t)|^{2-1} \, ds + (s + 1) \int_{e^{\alpha_1 - \varepsilon}}^{e^{\alpha_1}} |z_{\infty}(s)|^2 \, ds
$$

$$
+ \varepsilon \int_{e^{\alpha_1 - \varepsilon}}^{e^{\alpha_1}} |z_{\infty}(s)| |w_{\infty}(s)|^{2-1} \, ds =: J_{41}(t) + J_{42}(t) + J_{43}(t) + J_{44}(t),
$$

and one shows similarly, as for $J_3(t)$ that $\lim_{\varepsilon \to 0} \lim_{t \to \infty} J_4(t) = 0$. Finally, we have proved

$$
\lim_{\varepsilon \to 0} \lim_{t \to \infty} J_i(t) = 0
$$

for $i = 1, \ldots, 5$. Thus,

$$
\left| \int_0^\infty e^{2\alpha_1} G(s, t) - G_{\infty}(s) \, ds \right| \to 0,
$$
which implies for the term $I_5(t)$ in (16) that

$$I_5(t) \sim e^{-t\alpha(1-H)} \int_0^\infty G_\alpha(s) \, ds$$

as $t \to \infty$.

For the last term we show in a standard manner that $I_4(t) = O(e^{-t(1-H)})$ and, since $\alpha H < 1$, its contribution is negligible.

Finally, we get that $I_2(t)$ and $I_4(t)$ decay faster than $I_1(t)$ and $I_5(t)$. Therefore, $I(\theta_1; \theta_2; t) \sim I_1(t) + I_5(t)$ as $t \to \infty$, which completes the proof of part (i).

(ii) Recall the decomposition (16). For the first term we have

$$I_1(t) = e^{t\alpha H^2} \int_0^\infty \left\{ |\tilde{p}(s, t) + q(s, t)|^2 - |p(s, t)|^2 - |q(s, t)|^2 \right\} \, ds,$$

$$= e^{t\alpha H^2} \int_0^1 \ldots \, ds + e^{t\alpha H^2} \int_1^\infty \ldots \, ds =: I_{11}(t) + I_{12}(t),$$

where $p(s, t)$ and $q(s, t)$ are given in (17) and (18), respectively. For $I_{12}(t)$ one shows similarly as in the part (i) of the proof that

$$I_{12}(t) \sim e^{-t\alpha H(1-H)} \int_0^\infty |p_\infty(s) + q_\infty(s)|^2 - |p_\infty(s)|^2 - |q_\infty(s)|^2 \, ds,$$

as $t \to \infty$. Here $p_\infty(s) = -\theta_1 a s^{H-1/2}$ and $q_\infty(s) = \theta_2 a (H-1/2) s^{H-1/2-1}$. For $I_{11}(t)$, after the change of variables $s \to e^{-\alpha H} s$, we get

$$I_{11}(t) = e^{-t\alpha H} \int_0^\infty \left\{ |\tilde{p}(s, t) + \tilde{q}(s)|^2 - |\tilde{p}(s, t)|^2 - |\tilde{q}(s)|^2 \right\} \, ds,$$

where $\tilde{p}(s, t) = e^{-t\alpha H} \theta_1 a \left[ (e^t + s)^{H-1/2} - s^{H-1/2} \right]$ and $\tilde{q}(s) = \theta_2 a (1 + s)^{H-1/2-1}.\{ (1 + s)^{H-1/2} - s^{H-1/2}\}$. For fixed $s \in (0, \infty)$ we have that $e^{t\alpha H} \tilde{p}(s, t) 1_{(0, \alpha^t s)}(s) \to -\theta_1 a s^{H-1/2}$ as $t \to \infty$, and from the mean-value theorem we obtain

$$e^{t\alpha H} \left\{ |\tilde{p}(s, t) + \tilde{q}(s)|^2 - |\tilde{q}(s)|^2 \right\} 1_{(0, \alpha^t s)}(s)$$

$$\to \infty \quad \text{sgn}(\theta_1) |\theta_2| |a|^{\alpha^t} \left[ (1 + s)^{H-1/2} - s^{H-1/2} \right]^{\alpha^t} \text{ s}^{H-1/2} =: H_\infty(s),$$

since $\frac{d}{dx}|x|^2 = x|x|^{\alpha^t} \text{sgn}(x)$ for $x \neq 0$. Using inequality (13) and DCT we show in a standard way that $I_{11}(t) \sim e^{-t\alpha H} \int_0^\infty H_\infty(s) \, ds$ as $t \to \infty$. Since $\alpha(1-H) > 1$, we see that $I_{12}(t)$ decays faster than $I_{11}(t)$, and we finally obtain

$$I_1(t) \sim e^{-t\alpha H} \int_0^\infty H_\infty(s) \, ds$$

as $t \to \infty$. 


Next, we have $I_2(t) = \int_0^1 |v(s, t) + u(s)|^2 - |v(s, t)|^2 - |u(s)|^2 | ds$, with
\[ v(s, t) = e^{-\alpha t} \theta_1 \left[ a(e^t - s)^{H-1/2} - bs^{H-1/2} \right] \] (28)
and
\[ u(s) = \theta_2 \left[ a(1 - s)^{H-1/2} - bs^{H-1/2} \right]. \] (29)

For fixed $s \in (0, 1)$ we obtain $e^{\alpha t} v(s, t) \to -\theta_1 bs^{H-1/2}$ as $t \to \infty$, and from the mean value theorem we get
\[ e^{\alpha t} \{ |u(s) + v(s, t)|^2 - |v(s, t)|^2 \} \]
\[ \to -\alpha \theta_1 bs^{H-1/2} \left[ \theta_1 \left[ a(1 - s)^{H-1/2} - bs^{H-1/2} \right] \right]^{s-1} \times \text{sgn} \left\{ \theta_2 \left[ a(1 - s)^{H-1/2} - bs^{H-1/2} \right] \right\} \]
\[ =: M_\infty(s). \]

Since for the appropriate constants $m_1$ and $m_2$ we get
\[ \sup_{t > 1} e^{\alpha t} \| v(s, t) + u(s) \|^2 - |v(s, t)|^2 - |u(s)|^2 | \]
\[ \leq \sup_{t > 1} e^{\alpha t} (x + 1) |v(s, t)|^2 + \sup_{t > 1} e^{\alpha t} x |v(s, t)| \| u(s) \|^2 - 1 \]
\[ \leq m_1 (1 - s)^{H_2 - 1} + m_2 s^{H_2 - 1}, \]
which is integrable on $(0, 1)$, DCT yields
\[ I_2(t) \to e^{-\alpha t} \int_0^1 M_\infty(s) \, ds. \]

For the next component we need some more delicate estimations. We have $I_3(t) = \int_0^1 \{ |d(s, t) + f(s)|^2 - |d(s, t)|^2 - |f(s)|^2 \} \, ds$ with $d(s, t) = e^{-\alpha t} \theta_1 \left[ a (e^t - s)^{H-1/2} - bs^{H-1/2} \right]$ and $f(s) = \theta_2 b \left[ (s - 1)^{H-1/2} - s^{H-1/2} \right]$. For fixed $s \in (1, \infty)$ we get
\[ e^{\alpha t} d(s, t) \to -\theta_1 bs^{H-1/2} \]
and from the mean-value theorem
\[ e^{\alpha t} \{ |d(s, t) + f(s)|^2 - |f(s)|^2 \} \]
\[ \to -\alpha \theta_1 bs^{H-1/2} \left[ \theta_2 b \left[ (s - 1)^{H-1/2} - s^{H-1/2} \right] \right]^{s-1} \text{sgn} \left\{ \theta_2 b \right\} \]
\[ =: N_\infty(s). \]
Note that $N_\infty(s)$ is integrable on $(1, \infty)$. Set

$$N(s, t) := |d(s, t) + f(s)|^2 - |d(s, t)|^2 - |f(s)|^2.$$ 

Then, we have

$$I_3(t) = \int_1^2 N(s, t) \, ds + \int_2^\infty N(s, t) \, ds := I_{31}(t) + I_{32}(t).$$

We will find the rate of convergence for every $I_{3i}, i = 1, 2$, separately. For $I_{31}(t)$, fix $s \in (1, 2)$, then we get $e^{Ht} N(s, t) \to N_\infty(s)$ as $t \to \infty$. Additionally,

$$\sup_{t > 2} e^{Ht} |N(s, t)| \leq \sup_{t > 2} e^{Ht} \left( (s + 1) |d(s, t)|^2 + \sup_{t > 2} e^{Ht} 2 |d(s, t)||f(s)|^2 \right)$$

$$\leq p_1 [ (s - 1)^{H_2 - 1} + s^{H_2 - 1} ]$$

$$+ p_2 [ (s - 1)^{H - 1/2} + s^{H - 1/2} ] \left[ (s - 1)^{(H - 1/2)(s - 1)} + s^{(H - 1/2)(s - 1)} \right],$$

which is integrable on $(1, 2)$. Here $p_1$ and $p_2$ are the appropriate constants independent of $s$ and $t$. Hence,

$$I_{31}(t) \sim e^{-Ht} \int_1^2 N_\infty(s) \, ds.$$ 

For $I_{32}(t)$ we need more subtle estimations. In what follows we show that

$$\left| \int_2^\infty \left[ e^{Ht} N(s, t) \mathbf{1}_{(2, \infty)}(s) - N_\infty(s) \right] ds \right| \to_\infty 0$$

Fix $\varepsilon > 0$ appropriately small and put

$$\int_2^\infty \left[ e^{Ht} N(s, t) \mathbf{1}_{(2, \varepsilon)}(s) - N_\infty(s) \right] ds$$

$$= \int_2^{\varepsilon} \ldots ds + \int_2^{\varepsilon - \varepsilon H - \varepsilon} \ldots ds + \int_2^{\varepsilon - \varepsilon H} \ldots ds + \int_2^{\varepsilon - \varepsilon} \ldots ds + \int_2^\infty \ldots ds$$

$$=: f_1(t) + f_2(t) + f_3(t) + f_4(t) + f_5(t).$$

Let us begin with $f_1(t)$. For fixed $s \in (2, \infty)$ we obtain

$$e^{Ht} N(s, t) \mathbf{1}_{(2, \varepsilon H)}(s) \to_\infty N_\infty(s).$$

Additionally, we use the fact that $s \in (2, e^{Ht})$, which implies that there exist $t_0$ such that for every $t > t_0$ and every $s \in (2, e^{Ht})$ we have $e^t - s > s$. 
Hence,
\[
\sup_{t > 0} e^{\mathcal{H}t} |N(s, t)| 1_{(2, \alpha)}(s) \\
\leq \sup_{t > b} e^{\mathcal{H}t} (z + 1)|d(s, t)| 1_{(2, \alpha)}(s) + \sup_{t > b} e^{\mathcal{H}t} \alpha|f(s)|^{q-1} 1_{(2, \alpha)}(s) \\
\leq \sup_{t > b} q_1 e^{\mathcal{H}t} e^{-z \mathcal{H}t} s^H 1_{(2, \alpha)}(s) + \sup_{t > b} q_2 s^{H-1/2} (s - 1)^{(H-1/2)(z-1)} 1_{(2, \alpha)}(s) \\
\leq q_1 s^{H/2-z} + q_2 s^{H-1/2} (s - 1)^{(H-1/2)(z-1)},
\]
which is integrable on \((2, \infty)\). Here \(q_1\) and \(q_2\) are the appropriate constants independent of \(s\) and \(t\). Hence, DCT yields \(f_1(t) \to 0\) as \(t \to \infty\).

For the next component we have
\[
|J_2(t)| \leq \int_{\mathcal{H}t}^{e^{\mathcal{H}t}} e^{\mathcal{H}t} |N(s, t)| ds + \int_{\mathcal{H}t}^{e^{\mathcal{H}t}} |N_\infty(s)| ds =: f_{21}(t) + f_{22}(t).
\]
Since \(N_\infty(s)\) is integrable on \((1, \infty)\), we get \(f_{22}(t) \to 0\) as \(t \to \infty\).

Furthermore,
\[
f_{21}(t) \leq \int_{\mathcal{H}t}^{e^{\mathcal{H}t}} e^{\mathcal{H}t} (z + 1)|f(s)|^2 ds + \int_{\mathcal{H}t}^{e^{\mathcal{H}t}} e^{\mathcal{H}t} \alpha|f(s)| d(s, t)|^{q-1} ds \\
\leq u_1 e^{\mathcal{H}t} \int_{\mathcal{H}t}^{e^{\mathcal{H}t}} (s - 1)^{H-1/2} ds \\
\quad + u_2 e^{\mathcal{H}t} \int_{\mathcal{H}t}^{e^{\mathcal{H}t}} (s - 1)^{(H-1/2)(z-1)} ds \\
\quad + u_3 e^{\mathcal{H}t} \int_{\mathcal{H}t}^{e^{\mathcal{H}t}} (s - 1)^{(H-1/2)(z-1)} ds \to_\infty 0,
\]
where \(u_i, i = 1, 2, 3\), are the appropriate constants independent of \(s, t,\) and \(e\). Hence, \(|J_2(t)| \to 0\) as \(t \to \infty\).

Next, for the third term we get
\[
|J_3(t)| \leq \int_{e^{\mathcal{H}t}-\epsilon}^{e^{\mathcal{H}t}} e^{\mathcal{H}t} |N(s, t)| ds + \int_{e^{\mathcal{H}t}-\epsilon}^{e^{\mathcal{H}t}} |N_\infty(s)| ds =: f_{31}(t) + f_{32}(t).
\]
Since \(N_\infty(s)\) is integrable on \((1, \infty)\), we obtain \(f_{32}(t) \to 0\) as \(t \to \infty\).

Additionally,
\[
f_{31}(t) \leq \int_{e^{\mathcal{H}t}-\epsilon}^{e^{\mathcal{H}t}} e^{\mathcal{H}t} (z + 1)|f(s)|^2 ds + \int_{e^{\mathcal{H}t}-\epsilon}^{e^{\mathcal{H}t}} e^{\mathcal{H}t} \alpha|f(s)| d(s, t)|^{q-1} ds
\]
The last component where $\gamma$ is integrable on $\mathbb{R}$ and $\epsilon > 0$, therefore, $|J_3(t)| \to 0$ as $t \to \infty$.

For the fourth part we put

$$|J_4(t)| \leq \int_{t-e}^{t} e^{Ah} |N(s, t)| ds + \int_{t-e}^{t} |N_\infty(s)| ds =: J_{11}(t) + J_{12}(t).$$

Since $N_\infty(s)$ is integrable on $[1, \infty)$, we get $J_{12}(t) \to 0$ as $t \to \infty$. Further,

$$J_{11}(t) \leq \int_{t-e}^{t} e^{Ah} (s + 1)|d(s, t)|^2 ds + \int_{t-e}^{t} e^{Ah} |d(s, t)||f(s)|^{-1} ds$$

$$\leq z_1 e^{Ah} e^{-\beta H} \int_{t-e}^{t} [(e^t - s)^{H_2 - 1} + s^{H_2 - 1}] ds$$

$$+ z_2 e^{Ah} \int_{t-e}^{t} [(e^t - s)^{H_2 - 1} + s^{H_2 - 1}](s - 1)^{(H-1/2)(z-1))} ds$$

$$\leq z_3 e^{H_2} + z_4 e + z_5 e^{H_2 - 1/z},$$

where $z_i, i = 1, \ldots, 5$, are the appropriate constants independent of $s, t,$ and $\epsilon$. Thus, we obtain

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} |J_4(t)| = 0.$$

The last component $J_5(t) = \int_{t}^{\infty} N_\infty(s) ds \to 0$ as $t \to \infty$, since $N_\infty(s)$ is integrable on $(1, \infty)$. Finally, combining the results for $J_i(t)$, $i = 1, \ldots, 5$, we obtain

$$\int_{t}^{\infty} e^{Ah} N(s, t) \mathbf{1}_{(2, e]}(s) - N_\infty(s) ds \to 0.$$

Hence, $L_{32}(t) \sim e^{-\beta H} \int_{t}^{\infty} N_\infty(s) ds$, which implies

$$I_5(t) = I_{31}(t) + I_{32}(t) \sim e^{-\beta H} \int_{t}^{\infty} N_\infty(s) ds$$

as $t \to \infty$. 


From part (i) of the proof we have \( I_i(t) = O(e^{-n(1-H)}) \). Additionally, the assumption \( 2(1 - H) > 1 \) implies that \( H < 1/2 \). Thus, the contribution of \( I_i(t) \) is negligible.

Finally, putting together the results for \( I_i(t), j = 1, \ldots, 4 \), we get

\[
I(0; \theta_2; t) \sim e^{-ht} \left\{ \int_0^\infty H_\infty(s) ds + \int_0^1 M_\infty(s) ds + \int_1^\infty N_\infty(s) ds \right\}
\]

as \( t \to \infty \), which ends the proof of part (ii).

(iii) We begin with showing the following key inequality.

**Lemma 2.1.**

\[
\|r\|^2 + |s|^2 - |r - s|^2 \leq (\alpha + 1)|r||s|^{\alpha-1}, \quad \forall r, s \in \mathbb{R}, \quad 1 < \alpha \leq 2. \tag{30}
\]

**Proof.** Inequality (30) follows directly from the two following inequalities

(a) \(|r^2 + s^2 - |r - s|^2| = r^2 + s^2 - |r - s|^2 \leq (\alpha + 1)rs^{\alpha-1},
(b) \(|r^2 + s^2 - |r + s|^2| = (r + s)^2 - r^2 - s^2 \leq (\alpha + 1)rs^{\alpha-1},

where \( r, s > 0, 1 < \alpha \leq 2 \).

Proof of inequality (a) Let \( r \geq s \). Define \( f_1(r) := r^2 + s^2 - |r - s|^2 -(\alpha + 1)rs^{\alpha-1}. \) We have \( f_1(0) = 0 \) and

\[
f_1'(r) = 2r - 2(r - s) - (\alpha + 1)s^{\alpha-1} = 2r - 2(r - s) - (\alpha + 1)s^{\alpha-1} \leq 0.
\]

Thus \( f_1(r) \leq 0 \).

Let \( r < s \). Using the mean-value theorem we get

\[
r^2 + s^2 - (s - r)^2 \leq rs^{\alpha-1} + zr \int_0^1 [(s - r) + ru]^{\alpha-1} du
\leq rs^{\alpha-1} + zrs^{\alpha-1} = (\alpha + 1)rs^{\alpha-1},
\]

which proves (a).

Proof of inequality (b) We put \( h_i(r) = r^2 + s^2 + (\alpha + 1)rs^{\alpha-1} - (r + s)^2. \) Since \( h_i(0) = 0 \) and \( h_i'(r) = 2r - (\alpha + 1)s^{\alpha-1} - (r + s)^2 \geq 0 \), we get \( h_i(r) \geq 0 \).

Now, using the above results we determine the rate of convergence for every \( I_i(t), j = 1, \ldots, 4 \), from decomposition (16).

For \( I_1(t) \), after some standard calculations, we have

\[
I_1(t) = e^{\alpha H|a|^2} \int_0^\infty \left\{ |\tilde{p}(s, t) + \tilde{q}(s, t)|^2 - |\tilde{p}(s, t)|^2 - |\tilde{q}(s, t)|^2 \right\} ds
\]
with \( \bar{p}(s, t) = e^{-\alpha H} \theta_1 (s + 1)^{H-1/2} - s^{H-1/2} \) and \( \bar{q}(s, t) = \theta_2 (s + e^{-\alpha}H^{-1/2} - s^{H-1/2}) \). For fixed \( s \in (0, \infty) \) we get
\[
e^{\alpha \bar{q}(s, t)} \rightarrow_{s \to \infty} (H - 1/2) \theta_2 s^{H-1/2-1}
\]
and
\[
e^{\pm \alpha H} \left[ |\bar{p}(s, t) + \bar{q}(s, t)|^2 - |\bar{p}(s, t)|^2 \right] \rightarrow_{s \to \infty} \alpha(H - 1/2) \theta_2 \theta_1 \sin\left\{ \theta_1 \right\} s^{H-1/2-1}|(s + 1)^{H-1/2} - s^{H-1/2}|^2 - 1 =: P_\infty(s).
\]

Note that \( P_\infty(s) \) is integrable on \((0, \infty)\). Now, applying inequality (30) and DCT, we get by some standard arguments
\[
I_1(t) \sim e^{-\alpha H} |a|^2 \int_0^\infty P_\infty(s) ds
\]
as \( t \to \infty \).

For the next term we have \( I_2(t) = \int_0^t |v(s, t) + u(s)|^2 - |v(s, t)|^2 - |u(s)|^2| ds \), with \( v(s, t) \) and \( u(s) \) given in (28) and (29), respectively. Additionally, for fixed \( s \in (0, 1) \), we obtain \( e^{\alpha/2} v(s, t) \to a \theta_1 \) as \( t \to \infty \), and by the mean value theorem
\[
e^{\alpha/2} \left| |u(s) + v(s, t)|^2 - |u(s)|^2 \right| \rightarrow_{s \to \infty} \alpha a \theta_1 |\theta_2|^{H-1/2} |a(1 - s)^{H-1/2} - b s^{H-1/2}|^2 \sin\left\{ \theta_2 \left(a(1-s)^{H-1/2} - b s^{H-1/2}\right) \right\}.
\]

Note that the limit function is integrable on \((0, 1)\). Now, applying DCT in a standard manner, we see that \( I_2(t) = O(e^{-\alpha}) \), thus its contribution is negligible, since \( H > 1/2 \) and \( 1 < \alpha \leq 2 \) imply \( 1/2 > 1 - 1/\alpha > 1 - H \).

For the next component, after some standard calculations, we have
\[
I_3(t) = e^{\alpha H} \int_{-\infty}^t \left\{ |\tilde{w}(s, t) + \tilde{z}(s, t)|^2 - |\tilde{w}(s, t)|^2 - |\tilde{z}(s, t)|^2 \right\} ds,
\]
where
\[
\tilde{w}(s, t) = \theta_1 e^{-H} \left[a(1-s)^{H-1/2} - bs^{H-1/2}\right]
\]
and
\[
\tilde{z}(s, t) = \theta_2 b \left[(s - e^{-\alpha})^{H-1/2} - s^{H-1/2}\right].
\]

For fixed \( s \in (0, 1) \), we get from the mean value theorem \( e^{\alpha \tilde{z}(s, t)} \to -\theta_2 b(H - 1/2)s^{H-1/2-1} \) as \( t \to \infty \), and by using the mean value theorem
Note that $Q_\infty(s)$ is integrable on $(0, 1)$. Put $Q(s, t) := |\tilde{w}(s, t) + \tilde{z}(s, t)|^2 - |\tilde{w}(s, t)|^2 - |\tilde{z}(s, t)|^2$. In what follows, we will show that

$$\left| \int_0^1 \left[ e^{t+H(z-1)} Q(s, t)1_{(e^{-t}, 1)}(s) - Q_\infty(s) \right] ds \right| \to 0,$$

Fix $\epsilon > 0$ appropriately small and set

$$\int_0^1 \left[ e^{t+H(z-1)} Q(s, t)1_{(e^{-t}, 1)}(s) - Q_\infty(s) \right] ds$$

$$= \int_0^{-\epsilon} \ldots ds + \int_{-\epsilon}^{-\epsilon+\epsilon} \ldots ds + \int_{-\epsilon+\epsilon}^1 \ldots ds$$

$$=: J_1(t) + J_2(t) + J_3(t).$$

We immediately obtain

$$|J_1(t)| \to 0,$$

since $Q_\infty(s)$ is integrable on $(0, 1)$. Next, we have

$$|J_2(t)| \leq \int_{-\epsilon}^{-\epsilon+\epsilon} e^{t+H(z-1)} |Q(s, t)| ds + \int_{-\epsilon}^{-\epsilon+\epsilon} |Q_\infty(s)| ds \equiv J_{21}(t) + J_{22}(t).$$

Using (30), we obtain

$$J_{21}(t) = \int_{-\epsilon}^{-\epsilon+\epsilon} e^{t+H(z-1)} |Q(s, t)| ds$$

$$= \int_{-\epsilon}^{-\epsilon+\epsilon} e^{t+H(z-1)} |Q(s, t)|1_{(0, 1/|1+(1/2)/\epsilon|)}(s) ds$$

$$+ \int_{-\epsilon}^{-\epsilon+\epsilon} e^{t+H(z-1)} |Q(s, t)|1_{(1/|1+(1/2)/\epsilon|, 1)}(s) ds.$$
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\[
\begin{align*}
&\leq \int_{e^{-t}}^{e^{-t+\epsilon}} e^{t + \epsilon H (\alpha - 1)} (\alpha + 1) \| \tilde{z}(s, t) \| \tilde{w}(s, t) |^{\alpha - 1} \\
&\quad + \int_{e^{-t}}^{e^{-t+\epsilon}} e^{t + \epsilon H (\alpha - 1)} \| \tilde{z}(s, t) \| \tilde{w}(s, t) |^{\alpha - 1} \, ds \\
&\leq d_1 \int_{e^{-t}}^{e^{-t+\epsilon}} (s - e^{-t})^{H - 1/2 - 1} \, ds \leq d_2 \epsilon^{H - 1/2}.
\end{align*}
\]

Here $d_1$ and $d_2$ are the appropriate constants independent of $\epsilon$, $s$, and $t$.

For $J_{22}(t)$, we get

\[
J_{22}(t) \leq d_3 \int_{e^{-t+\epsilon}}^{e^{-t+\epsilon+\epsilon}} s^{H - 1/2 - 1} \, ds = \frac{d_3}{H - 1/2} \left[ (e^{-t} + \epsilon)^{H - 1/2} - (e^{-t})^{H - 1/2} \right],
\]

where $d_3$ is the appropriate constant independent of $\epsilon$, $s$, and $t$. Finally, we obtain

\[
\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} J_{2i}(t) = 0,
\]

$i = 1, 2$, which implies that $\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} |J_{2i}(t)| = 0$.

Now, we proceed with $J_3(t)$. For fixed $s \in (0, 1)$ we have

\[
e^{t + \epsilon H (\alpha - 1)} Q(s, t) 1_{(e^{-t+\epsilon}, 1)}(s) \overset{\epsilon \to 0}{\longrightarrow} Q_{\infty}(s) 1_{(e^{-t}, 1)}(s).
\]

Now, using (30) and DCT we show in a standard way that $|J_3(t)| \longrightarrow 0$ as $t \to \infty$. Finally, combining the results for every $J_i(t)$, $i = 1, 2, 3$, we obtain

\[
\left| \int_0^1 [e^{t + \epsilon H (\alpha - 1)} Q(s, t) 1_{(e^{-t+\epsilon}, 1)}(s) - Q_{\infty}(s)] \, ds \right| \overset{\epsilon \to 0}{\longrightarrow} 0
\]

and consequently

\[
I_3(t) \sim e^{-t(1-H)} \int_0^1 Q_{\infty}(s) \, ds
\]

as $t \to \infty$.

For the last term we show in a standard way that

\[
I_4(t) \sim e^{-t(1-H)} \int_t^\infty R_{\infty}(s) \, ds.
\]

as $t \to \infty$. 

Finally, combining the results for $I_j(t), j = 1, \ldots, 4$, we obtain

$$I(\theta_1; \theta_2; t) \sim e^{-(t-H)} \left\{ |a|^2 \int_0^\infty P_\infty(s) \, ds + \int_0^3 Q_\infty(s) \, ds + \int_1^\infty R_\infty(s) \, ds \right\}$$

as $t \to \infty$, which ends part (iii) of the theorem.

**Corollary 2.1.** The fractional O–U $\alpha$-stable process $Z_2(t)$ defined in (11) does not have long memory in the sense of (8).

**Proof.** From Theorems (2.1), (2.2), and (2.3) we see that $r(\theta_1; \theta_2; t)$ decays exponentially as $t \to \infty$, therefore

$$\sum_{i=1}^\infty |r(\theta_1; \theta_2; t)| < \infty.$$

**Corollary 2.2.** The fractional O–U $\alpha$-stable processes $Z_1(t)$ given in (10) and $Z_2(t)$ defined in (11) are two different stationary processes.

**Proof.** Since the rate of convergence of the function $r(\theta_1; \theta_2; t)$ is different for both processes, their finite-dimensional distributions can not be identical.

The above results show that, similarly to the Gaussian case, the Lamperti transformation of the LFSM (1) results in the exponential decay of $r(\theta_1; \theta_2; t)$ for $Z_2(t)$, while the long memory property of the noise in the Langevin equation (9) transfers to its solution $Z_1(t)$. The presented approach to long memory, basing on the asymptotic properties of the measure of dependence $r(\theta_1; \theta_2; t)$, should be viewed as one of the existing approaches. The final formulation of the problem of LRD in the $\alpha$-stable case, as well as the behaviour of other dependence measures, is a subject of our further research.

**REFERENCES**