Analytical and numerical approach to corporate operational risk modelling
Analityczne i numeryczne podejcie do modelowania ryzyka operacyjnego firmy

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Wrocław, March 2007
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The New Accord regulatory (Basel II)

- 1988 – the Basel Accord (Basel I) – credit risk,
- 1996 – market risk, (internal models, scorecards, VaR measure),
- 1999 – New Accord (Basel II), the definition of operational risk.

Definition 1  Operational risk is the risk of (direct or indirect) losses resulting from inadequate or failed internal processes and procedures, people and systems, or external events.

Classification of aggregated operational loss

- 8 business lines
- 7 loss types
Risk measurement methods for operational risk

- Basic Indicator Approach (BIA),
- Standardized Approach (SA),
- Advanced Measurement Approach (AMA).
  - Internal Measurement Approaches
  - Scorecard Approaches
  - Loss Distribution Approaches

Loss Distribution Approach

For each business line / loss type cell \((i,k)\) the compound variable

\[
S_{i,k}^{t} = \sum_{l=1}^{N_{i,k}^{t}} X_{l}^{i,k}, \quad (1)
\]

The total one year loss in year \(t\)

\[
S_{t} = \sum_{i=1}^{8} \sum_{k=1}^{7} S_{i,k}^{t} = \sum_{i=1}^{8} \sum_{k=1}^{7} \sum_{l=1}^{N_{i,k}^{t}} X_{l}^{i,k}. \quad (2)
\]
Basel II proposals

- risk measure – Value at Risk with $\alpha = 99.9\%$ or even $(99.95\% - 99.97\%)$,
- total capital charge $C_{op}$:
  \[
  C_{op} = \sum_{i,k} VaR_\alpha(S^{i,k}).
  \]
  (3)

with possible reduction due to correlation effects.

Difficulties in implementation LDA

- high confidence level $\implies$ estimation very difficult
- solutions via Extreme Value Theory (EVT), Peaks Over Threshold (POT),
- non-stationarity, dependence, inhomogeneity of data $\implies$ multivariate EVT
  and copulas necessary.
Motivation to model operational risk with ruin theory

- horizon of planning and making decisions,
- LDA dual to actuarial models in insurance – classical risk model:

\[ R_t = u + ct - \sum_{i=1}^{N_t} X_i, \]  

(4)

- LDA – looking for appropriate quantile of the total loss \( S_t \) to apply \( VaR \) method in order to find the capital charge \( C_{op} \)

Proposed approach – exploit the insurance risk model in a longer time horizon and use the probability of ruin as a risk measure instead of \( VaR \). Then: \( c \) – yearly intensity of capital charge \( C_{op} \) (or operational reserves) and \( u \) – arbitrary amount of capital that should never be exceeded under threat of bankruptcy. Instead of quantile of estimated variable, the problem of finding first the probability of ruin and then to invert it in order to find the required capital charge.
Market risk management in corporation – hedging.

Setting up the problem

Bank vs. corporation within market risk management

- horizon of planning and taking decisions,
- exposure to market risk: a set of portfolios vs. fluctuation of commodity prices and currencies,
- market activity: natural long position of corporation, hedging = minimizing market risk exposure, ≠ speculation
- risk measures: $VaR$ in banks; $RaR$, $EaR$ and $CFaR$ in non-financial corporations.
The aim:

- An analytical, correlation-based approach to calculate $RaR$ measure exploiting portfolio theory:

$$VaR(X_1 + X_2) = \sqrt{VaR(X_1)^2 + VaR(X_2)^2 + 2VaR(X_1)VaR(X_2)\rho_{X_1,X_2}}. \quad (5)$$

Assumptions:

- Company A: the producer of commodity $X_t$ denominated in foreign currency (USD),
- Main sources of risk: the commodity price $X_t$ and currency rate of exchange $Y_t$,
- Hedging: portfolio $H^X_t$ of commodity sell forward contracts on $X_t$ and portfolio $H^Y_t$ of currency sell forwards,
- problem of determining $RaR$: finding the distribution (quantiles and mean) of $\beta_t X_t Y_t + H^X_t + H^Y_t$, i.e. revenues from sale corrected by settlement results from hedging portfolios.
Definition 2

\[ RaR_t^\alpha = \mathbb{E}(\beta_t U_t + H_t^X + H_t^Y) - \mathbb{Q}_\alpha(\beta_t U_t + H_t^X + H_t^Y) \]  \hspace{1cm} (6)

**RaR algorithm**

For every \( t \) starting from 0 to the end of time horizon for our analysis:

1. First, compute the desired statistics for \( \beta_t X_t Y_t \).
2. Second, compute the statistics for PLN value of settlement result from commodity hedging portfolio \( -H_t^X \).
3. Then, obtain the summary statistics for \( \beta_t X_t Y_t + H_t^X \) + options premiums as a sum of results from point (1) and (2), corrected by obvious negative (close to -1) correlation between \( X_t Y_t \) and \( H_t^X \).
4. Next, compute the statistics for currency hedging portfolio \( H_t^Y \).
5. Finally obtain the total statistics of \( \beta_t X_t Y_t + H_t^X + H_t^Y \) by summing values from points (3) and (4) with necessary correction by correlation between \( X_t Y_t + H_t^X \) and \( H_t^Y \).
Modelling risk factors

Schwartz commodity model

Geometric Ornstein-Uhlenbeck process (Schwartz mean-reverting model):

\[
dX_t = \eta \left( \frac{\sigma^2}{2 \eta} + \log(k) - \log(X_t) \right) X_t dt + \sigma X_t dW_t. \tag{7}
\]

Necessary statistics of \( X_t \):

\[
E(X_t) = \exp \left( \mu_{t,X} + \frac{\sigma_{t,X}^2}{2} \right),
\]

\[
\text{Var}(X_t) = \exp(2\mu_{t,X} + \sigma_{t,X}^2)(\exp(\sigma_{t,X}^2) - 1),
\]

\[
Q_\alpha(X_t) = \exp \left( \mu_{t,X} + \sigma_{t,X} \Phi^{-1}(\alpha) \right).
\]

Calibration: method of moments and MLE.

Geometric Brownian Motion for currency price.
How to calculate RaR based on analytical correlations approach

Theorem 1 The RaR algorithm can be realized by an analytical correlations approach.

Proof:

(1) Let $U_t = X_t Y_t$ (lognormal variable) and assume independence $X_t$ of $Y_t$. The statistics for revenues are:

$$
\beta_t \mathbb{E}(U_t) = \beta_t \mathbb{E}(X_t) \mathbb{E}(Y_t) = \beta_t \exp \left( \mu_{t,X} + \mu_{t,Y} + \frac{\sigma^2_{t,X} + \sigma^2_{t,Y}}{2} \right), \quad (9)
$$

and

$$
\beta_t \mathcal{Q}_\alpha(U_t) = \beta_t \exp \left( \mu_{t,X} + \mu_{t,Y} + \sqrt{\sigma^2_{t,X} + \sigma^2_{t,Y}} \Phi^{-1}(\alpha) \right). \quad (10)
$$

(2) Commodity hedging portfolio $H_t^X \overset{\text{def}}{=} \gamma_t Y_t (K_t - X_t)$ is a difference of two dependent lognormal variables. Let $g(u,v)$ be a 2-dimensional lognormal density of $(U_t, V_t) \overset{\text{def}}{=} (X_t Y_t, K_t Y_t)$. Then, putting $q = \frac{\mathcal{Q}_\alpha(H_t^X)}{\gamma_t}$ yields
\[ P(\bar{H}_t^X < \bar{Q}_\alpha(\bar{H}_t^X)) = P(V_t < U_t + q) = \int_{\max\{0,-q\}}^{\infty} \int_0^{q+u} g(u,v) dvdu \]

\[ = \int_{\log(\max\{0,-q\}) - \mu_U}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \Phi\left(\frac{\log(q+\exp(\mu_U+\sigma_Uy)) - \mu_V - \rho_{UV}\sigma Vy}{\sigma_V\sqrt{1-\rho_{UV}^2}}\right) dxdy. \]  

(11)

Correlation coefficient $\rho_{UV}$ between $\log(U_t)$ and $\log(V_t)$

\[ \rho_{UV} = \frac{\log\left(1 + \rho_{XY,Y} \sqrt{(\exp(\sigma_{t,U}^2) - 1)(\exp(\sigma_{t,V}^2) - 1)}\right)}{\sigma_{t,U} \sigma_{t,V}}. \]  

(12)

Finding $\bar{Q}_\alpha(\bar{H}_t^X)$ requires inverting (11) equal to $\alpha$. The end of step (2) comes with $\mathbb{E}(\bar{H}_t^X) = \gamma_t \mathbb{E}(Y_t)(K_t - \mathbb{E}(X_t))$.

(3)

\[ \bar{Q}_\alpha(\beta_t U_t + \bar{H}_t^X) \approx \mathbb{E}(\beta_t U_t + \bar{H}_t^X) + p(\bar{H}_t^X) \]

\[ \pm \sqrt{\left(\bar{Q}_\alpha(\beta_t U_t) - \mathbb{E}(\beta_t U_t)\right)^2 + \left(\bar{Q}_\alpha(\bar{H}_t^X) - \mathbb{E}(\bar{H}_t^X)\right)^2 + 2(\bar{Q}_\alpha(\beta_t U_t) - \mathbb{E}(\beta_t U_t))(\bar{Q}_\alpha(\bar{H}_t^X) - \mathbb{E}(\bar{H}_t^X))\rho_{UH}}, \]
\( \rho_{UH} \) can be found (introducing \( \phi = \sqrt{\frac{\text{Var}(Y_t)}{\text{Var}(X_t Y_t)}} \)) by:

\[
\rho_{UH} = \frac{\phi K \rho_{XY,Y} - 1}{\sqrt{1 - \rho_{XY,Y}^2 + (K \phi - \rho_{XY,Y})^2}}. 
\]  

(13)

(5) Finally

\[
\begin{align*}
Q_{\alpha}(\beta_t U_t + H_t^X + H_t^Y) &\approx \mathbb{E}(\beta_t U_t + H_t^X + H_t^Y) + p(H_t^X) + p(H_t^Y) \\
\pm \sqrt{(Q_{\alpha}(\beta_t U_t + H_t^X) - \mathbb{E}(\beta_t U_t + H_t^X))^2 + (Q_{\alpha}(H_t^Y) - \mathbb{E}(H_t^Y))^2 + 2(Q_{\alpha}(\beta_t U_t + H_t^X) - \mathbb{E}(\beta_t U_t + H_t^X))(Q_{\alpha}(H_t^Y) - \mathbb{E}(H_t^Y))\rho_{UHH}}. 
\end{align*}
\]

with \( \rho_{UHH} \) (correlation between \( \beta_t U_t + H_t^X \) and \( H_t^Y \)) given by:

\[
\rho_{UHH} = -\rho_{UHY} = -\frac{(\beta - \gamma) \rho_{XY,Y} + \gamma \phi K}{\sqrt{(\beta - \gamma)^2(1 - \rho_{XY,Y}^2) + (K \gamma \phi + \rho_{XY,Y}(\beta - \gamma))^2}}. 
\]
Some basic aspects of actuarial risk theory.

- classical risk process
- ruin probability in finite and infinite time horizon
- light- and heavy-tailed distributions
- adjustment coefficient

Definition 3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying Poisson process $\{N_t\}_{t \geq 0}$ with intensity $\lambda$, and sequence $\{X_k\}_{k=1}^{\infty}$ of positive, i.i.d. random variables, with mean $\mu$ and variance $\sigma^2$. Furthermore, we assume that $\{X_k\}$ and $\{N_t\}$ are independent. The classical risk process $\{R_t\}_{t \geq 0}$ is given by

$$R_t = u + ct - \sum_{i=1}^{N_t} X_i, \quad c > 0, \; u \geq 0. \quad (14)$$
To introduce the term *ruin probability*, first define the time to ruin as

\[
\tau(u) = \inf\{t \geq 0 : R_t < 0\}.
\] (15)

**Definition 4** The ruin probability in finite time \( T \) is given by

\[
\psi(u, T) = \mathbb{P}(\tau(u) \leq T)
\]

and ruin probability in infinite time is defined as

\[
\psi(u) = \mathbb{P}(\tau(u) < \infty).
\] (16)

**Definition 5** Let \( \gamma = \sup_z M_X(z) < \infty \) and let \( R \) be a positive solution of the equation

\[
1 + (1 + \theta)\mu R = M_X(R), \quad R < \gamma.
\] (17)

If there exists a non-zero solution to the above equation, we call such \( R \) an adjustment coefficient (or Lundberg exponent).
Ruin probability in finite time horizon.

Exact ruin probabilities in finite time

Exponential loss amounts (β = 1, c=1)

\[
\psi(u, T) = \lambda \exp \{-(1 - \lambda)u\} - \frac{1}{\pi} \int_0^\pi \frac{f_1(x)f_2(x)}{f_3(x)} dx, \tag{18}
\]

where

\[
f_1(x) = \lambda \exp \left\{2\sqrt{\lambda}T \cos x - (1 + \lambda)T + u \left(\sqrt{\lambda} \cos x - 1\right)\right\},
\]

\[
f_2(x) = \cos \left(u\sqrt{\lambda} \sin x\right) - \cos \left(u\sqrt{\lambda} \sin x + 2x\right), \text{ and } f_3(x) = 1 + \lambda - 2\sqrt{\lambda} \cos x.
\]
Approximations of the ruin probability in finite time

Monte Carlo method
Segerdahl normal approximation
Diffusion approximation
Corrected diffusion approximation
Finite time De Vylder approximation

The idea of the De Vylder approximation – replace the claim surplus process with the one exponential claims fitting first three moments:

\[
\bar{\beta} = \frac{3\mu^{(2)}}{\mu^{(3)}}, \quad \bar{\lambda} = \frac{9\lambda\mu^{(2)}}{2\mu^{(3)^2}}, \quad \text{and} \quad \bar{\theta} = \frac{2\mu\mu^{(3)}}{3\mu^{(2)^2}} \theta.
\]

Next, employ the exact, exponential case formula.
Numerical comparison of the finite time approximations

5 approximations – mixture of 2 exponentials case, $\theta = 30\%$.

Figure 1: Monte Carlo (left panel), the relative error (right panel). Segerdahl (short-dashed blue line), diffusion (dotted red line), corrected diffusion (solid black line) and finite time De Vylder (long-dashed green line). $T$ fixed and $u$ varying.
Figure 2: The exact ruin probability obtained via Monte Carlo simulations (left panel), the relative error of the approximations (right panel). The Segerdahl (short-dashed blue line), diffusion (dotted red line), corrected diffusion (solid black line) and finite time De Vylder (long-dashed green line) approximations. The mixture of two exponentials case with $u$ fixed and $T$ varying.
Infinite horizon.

Exact ruin probabilities

No initial capital. \((u = 0)\)

Exponential claims. (explicit, analytical)

Gamma claims. (numerical integration from 0 to \(\infty\))

Mixture of \(n\) exponentials claims. (analytical result for \(n = 2\))

A survey of approximations

Cramér–Lundberg approximation

Exponential approximation

Lundberg approximation

Beekman–Bowers approximation

Renyi approximation

De Vylder approximation

Heavy traffic approximation

Light traffic approximation

Heavy-light traffic approximation

Heavy-tailed claims approximation
4-moment gamma De Vylder approximation
Parameters determining the new process with gamma claims

\[ \tilde{\lambda} = \frac{\lambda \mu^{(3)}^2 (\mu^{(2)})^3}{(\mu^{(2)} \mu^{(4)} - 2(\mu^{(3)})^2)(2 \mu^{(2)} \mu^{(4)} - 3(\mu^{(3)})^2)}, \quad \tilde{\theta} = \frac{\theta \mu(2(\mu^{(3)})^2 - \mu^{(2)} \mu^{(4)})}{(\mu^{(2)})^2 \mu^{(3)}}, \]
\[ \tilde{\mu} = \frac{3(\mu^{(3)})^2 - 2(\mu^{(2)} \mu^{(4)})}{\mu^{(2)} \mu^{(3)}}, \quad \tilde{\mu}^{(2)} = \frac{(\mu^{(2)} \mu^{(4)} - 2(\mu^{(3)})^2)(2 \mu^{(2)} \mu^{(4)} - 3(\mu^{(3)})^2)}{(\mu^{(2)} \mu^{(3)})^2}. \]

4MG approximation

\[ \psi_{4MG}(u) = \frac{\tilde{\theta}(1 - \frac{R}{\tilde{\alpha}})e^{-\tilde{\beta} \frac{R}{\tilde{\alpha}} u}}{1 + (1 + \tilde{\theta})R - (1 + \tilde{\theta})(1 - \frac{R}{\tilde{\alpha}})} + \frac{\tilde{\alpha} \tilde{\theta} \sin(\tilde{\alpha} \pi)}{\pi} \cdot I, \quad (19) \]

where

\[ I = \int_{0}^{\infty} \frac{x^{\tilde{\alpha}} e^{-(x+1)\tilde{\beta} u} \, dx}{[x^{\tilde{\alpha}} (1 + \tilde{\alpha}(1 + \tilde{\theta})(x + 1)) - \cos(\tilde{\alpha} \pi)]^2 + \sin^2(\tilde{\alpha} \pi)}, \]

and \( \tilde{\alpha} = \frac{\tilde{\mu}^2}{\tilde{\mu}^{(2)} - \tilde{\mu}^2}, \quad \tilde{\beta} = \frac{\tilde{\mu}}{\tilde{\mu}^{(2)} - \tilde{\mu}^2}. \)
Computer approximation via Pollaczek–Khinchin formula

\[
\psi(u) = \mathbb{P}(M > u) = \frac{\theta}{1 + \theta} \sum_{n=0}^{\infty} \left( \frac{1}{1 + \theta} \right)^n \overline{B}_0^n(u),
\]

(20)

$\overline{B}_0$ – tail of the distribution corresponding to the density $b_0(x) = \frac{F_X(x)}{\mu}$.

Since $\psi(u) = EZ$, where $Z = 1(M > u)$, it may be generated as follows.

**SIMULATION ALGORITHM**

1. Generate a random variable $K$ from the geometric distribution with $p = \frac{1}{1+\theta}$,
2. Generate random variables $X_1, X_2, \cdots, X_K$ from the density $b_0(x)$,
3. Calculate $M = X_1 + X_2 + \cdots + X_K$,
4. If $M > u$, let $Z = 1$, otherwise let $Z = 0$, 
Proposition 1  The density $b_0(x)$ has a closed form only for four of the considered distributions:

- **exponential** $\implies b_0(x)$ exponential,
- **mixture of exponentials** $\implies b_0(x)$ mixture of exponentials with the weights $\left( \frac{a_1}{\beta_1}, \ldots, \frac{a_n}{\beta_n} \right)$,
- **Pareto** $\implies b_0(x)$ Pareto with $(\alpha - 1, \nu)$,
- **Burr** $\implies b_0(x)$ transformed beta.
Numerical comparison of the methods

The relative error of 12 methods w.r.t. exact values (mixture of exponentials) and Pollaczek–Khinchin approximation as a reference method (lognormal case).

Figure 3: More effective methods (left): the Cramér–Lundberg (solid blue line), exponential (short-dashed brown line), Beekman–Bowers (dotted red line), De Vylder (medium-dashed black line) and 4-moment gamma De Vylder (long-dashed green line). Less effective (right): Lundberg (short-dashed red line), Renyi (dotted blue line), heavy traffic (solid magenta line), light traffic (long-dashed green line) and heavy-light traffic (medium-dashed brown line). The mixture of two exponentials case.
Figure 4: More effective methods (left panel): the exponential (dotted blue line), Beekman–Bowers (short-dashed brown line), heavy-light traffic (solid red line), De Vylder (medium-dashed black line) and 4-moment gamma De Vylder (long-dashed green line). Less effective methods (right panel): Lundberg (short-dashed red line), heavy traffic (solid magenta line), light traffic (long-dashed green line), Renyi (medium-dashed brown line) and subexponential (dotted blue line). The log-normal case.
Diffusion model – mixture of exponentials losses.

Diffusion risk process

\[ R_t = R_0 + \beta t + \sigma B_t - \sum_{n=1}^{N_t} X_n, \quad (21) \]

The infinitesimal generator

\[ A f(x) = \beta f'(x) + \frac{1}{2} \sigma^2 f''(x) + \lambda \int_0^\infty F_X(dy)(f(x-y) - f(x)). \quad (22) \]

**Lemma 1** Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded function with \( f \in C^2 B^2(\mathbb{R}_+) \). Then, by Itô’s formula and the martingale representation, the following formula holds

\[ f(R_{\tau \wedge t}) = f(R_0) + \int_0^{\tau \wedge t} Af(R_s) \, ds + M_t, \quad (23) \]

for \( M \) being \( F_t \)-martingale starting from zero \( (M_0 = 0) \).
Laplace transform of claims being rational function

The ruin caused by a jump and ruin by continuity

\[ A_j = \{ R_\tau < 0, \; \tau < \infty \}, \; \text{and} \; A_c = \{ R_\tau = 0, \; \tau < \infty \}. \]

The ruin probability of the risk process starting from \( R_0 \)

\[ P_{R_0}(\tau < \infty) = P_{R_0}(A_j) + P_{R_0}(A_c). \tag{24} \]

An important assumption

\[ L_X(\nu) = E \exp(-\nu X) = \frac{P_X(\nu)}{Q_X(\nu)} \tag{25} \]

\( P_X, Q_X \) – polynomials with no common complex roots and the leading coefficient for \( Q_X \) equal to 1.

**Proposition 2** For \( R_0 > 0 \), \( P_{R_0}(\tau < \infty) = 1 \) if and only if \( \beta \leq \lambda EX \).
Two versions of the Cramèr-Lundberg equation:

\[ Q_X(\gamma) = -P_X(\gamma) \frac{\lambda}{\beta \gamma + \frac{1}{2} \sigma^2 \gamma^2 - \lambda} \]  

(26)

modified version

\[ Q_X(\gamma)(\beta \gamma + \frac{1}{2} \sigma^2 \gamma^2 - \lambda) = -\lambda P_X(\gamma). \]  

(27)

**Proposition 3**  For diffusion risk process \( R_t \) with \( L_X(\nu) = \frac{P_X(\nu)}{Q_X(\nu)} \),

\( \text{degree}(Q) = m. \)

(i) If \( P_{\text{ruin}} < 1 \), then the Cramèr-Lundberg equation (27) has precisely \( m + 1 \) solutions \((\gamma_l)_{1 \leq l \leq m+1}\) with \( \text{Re}(\gamma_l) < 0 \),

(ii) \( \gamma_l : \text{Re}(\gamma_l) < 0 \) is a solution to (26) if and only if \( \gamma_l \) is a solution to the modified Cramèr-Lundberg equation (27) with \( q(\gamma_l) = \beta \gamma_l + \frac{1}{2} \sigma^2 \gamma_l^2 - \lambda \neq 0 \),

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(iii) If $(\tilde{\gamma}_k)_{1 \leq k \leq m}$ are any distinct $m$ of the $(m + 1)$ solutions to (26) with $\Re e(\tilde{\gamma}_k) < 0$ and $q(\tilde{\gamma}_k) = \beta \tilde{\gamma}_k + \frac{1}{2} \sigma^2 \tilde{\gamma}_k^2 - \lambda \neq 0$, it holds for all $R_0 > 0$ that

$$
\sum_{k=1}^{m} r_k \frac{\lambda}{q(\tilde{\gamma}_k)} P_{R_0}(A_c) - \left( \sum_{k=1}^{m} r_k \right) P_{R_0}(A_j) = \sum_{k=1}^{m} r_k \frac{\lambda \exp(\tilde{\gamma}_k R_0)}{q(\tilde{\gamma}_k)},
$$

(28)

with $r_k$ given by

$$
r_k = - \frac{P_X(\tilde{\gamma}_k)}{\tilde{\gamma}_k \prod_{k' \neq k} (\tilde{\gamma}_k - \tilde{\gamma}_{k'})},
$$

(29)

(iv) If $P_{ruin} < 1$ and all the solutions $(\gamma_l)_{1 \leq l \leq m+1}$ to (26) with $\Re e(\gamma_l) < 0$ are distinct and $q(\gamma_l) \neq 0$, using (28) twice with, say, $(\tilde{\gamma}_k)_{1 \leq k \leq m} = (\gamma_1, \ldots, \gamma_{m-1}, \gamma_{m+s})$, $s = \{0, 1\}$, we obtain a system of 2 linear equations with unknowns $P_{R_0}(A_j)$ and $P_{R_0}(A_c)$, that can be solved uniquely provided matrix of coefficients is non-singular.
Proof:
(ii) and (iv) obvious. Proof of (iii) follows by showing

\[ \mathcal{A}f(x) = 0, \quad x \geq 0. \]  \hspace{1cm} (30)

for the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) of the form

\[
f(x) = \begin{cases} 
\sum_{k=1}^{m} c_k \exp(\gamma_k x) & x \geq 0 \\
K & x < 0,
\end{cases}
\]  \hspace{1cm} (31)

where \( c_k = \frac{\lambda r_k}{q(\gamma_k)} \), \( K = -\sum_{k=1}^{m} r_k \) and \( r_k \) as in (29) and next applying Lemma (1) and Proposition (2).

The proof of (i) goes through Rouché theorem from complex function theory. \( \Box \)
Mixture of exponentials claims

Mixture of \( m \) exponential distributions belongs to family of distributions with rational Laplace transform, as

\[
L_X(\nu) = \sum_{i=1}^{m} a_i \frac{\delta_i}{\delta_i + \nu} = \frac{\sum_{i=1}^{m} a_i \delta_i \prod_{j \neq i} (\delta_j + \nu)}{\prod_{j=1}^{m} (\delta_j + \nu)} = \frac{P_X(\nu)}{Q_X(\nu)}
\]

is well defined for \( \nu > -\min_{i=1,\ldots,m}\{\delta_i\} \) and \( P_X, Q_X \) are polynomials respectively of degree \( m \) and \( \leq m - 1 \). The modified Cramèr-Lundberg equation (27) takes a form:

\[
\prod_{j=1}^{m} (\delta_j + \gamma)(\beta \gamma + \frac{1}{2} \sigma^2 \gamma^2 - \lambda) = -\lambda \sum_{i=1}^{m} a_i \delta_i \prod_{j \neq i} (\delta_j + \gamma).
\]
Theorem 2 \( R_t \) – diffusion risk process with losses following mixture of \( m \) exponential distributions. To ensure \( P_{\text{ruin}} < 1 \) assume drift coefficient \( \beta < \lambda \sum_{i=1}^{m} \frac{a_i}{\delta_i} \). Then

(i) the Cramér-Lundberg equation (32) has precisely \( m+1 \) solutions \( (\gamma_k)_{1 \leq k \leq m+1} \) with \( \Re(\gamma_k) < 0 \),

(ii) If all the solutions \( (\gamma_k)_{1 \leq k \leq m+1} \) to (32) with \( \Re(\gamma_k) < 0 \) are distinct and \( q(\gamma_k) = \beta \gamma_k + \frac{1}{2} \sigma^2 \gamma_k^2 - \lambda \neq 0 \), then for all \( R_0 > 0 \)

\[
P_{\text{ruin}} = \frac{\sum_{m}^{(1)} \sum_{m+1}^{(3)} - \sum_{m+1}^{(1)} \sum_{m}^{(3)} - \sum_{m}^{(3)} \sum_{m+1}^{(2)} + \sum_{m+1}^{(3)} \sum_{m}^{(2)}}{\sum_{m}^{(1)} \sum_{m+1}^{(2)} - \sum_{m+1}^{(1)} \sum_{m}^{(2)}} , \tag{33}
\]

with \( \Sigma_{l}^{(1)} = \sum_{k \neq l} r_k \), \( \Sigma_{l}^{(2)} = \sum_{k \neq l} \frac{\lambda r_k}{q(\gamma_k)} \), \( \Sigma_{l}^{(3)} = \sum_{k \neq l} \frac{\lambda r_k \exp(\gamma_k R_0)}{q(\gamma_k)} \)

and

\[
r_k = -\frac{\sum_{i=1}^{m} a_i \delta_i \prod_{j \neq i} (\delta_j + \gamma_k)}{\gamma_k \prod_{k' \neq k} (\gamma_k - \gamma_{k'})} .
\]
Proof:
The proof follows by appropriate exploiting Propositions (3) and (2) and solving suitable system of linear equations. □

Importance of the analytical result – mixtures of distributions are naturally exploited distributions in operational risk modelling and diffusion component makes the model closer to real life.
Building operational reserves.

Ruin probability criterion

- The idea: setting the level of capital charge $c$ for operational risk dual to finding insurance premium.
- Notation: $S_t$ – the aggregate loss over $(0, t]$ (compound Poisson variable in classical model (14)), $W = S_{t+1} - S_t$ – the one-year aggregate loss.
- Long-run horizon operational reserves calculation problem: obtain such a level of operational capital charge $c(W)$ to be sufficient to cover each year the aggregate loss $W$, with critical level of capital $u$ (profit presumed in budget) and accepted level of ruin probability $\psi$.
- Solution: inverting various approximate formulae for the probability of ruin $\psi$ to obtain capital charge $c(W)$. 
The definition of the adjustment coefficient for the portfolio:
\[ \mathbb{E}(e^{RW}) = e^{Rc(W)}, \]

The reserve amount formula
\[ c(W) = R^{-1} \ln \{ \mathbb{E}(e^{RW}) \} = R^{-1} C_W (R), \]

with \( C_W \) denoting the c.g.f. of \( W \).

Inverting various approximations of \( \psi \) in order to find \( c(W) \) generally leads to a system of two or three equations.

**Beekman–Bowers approximation**
**Diffusion approximation**
**De Vylder approximation**
**Subexponential approximation**
**Cramér–Lundberg approximation** – an example

Operational reserve calculation based on the Cramér-Lundberg approximation can be reduced to the system of equations

\[
\begin{align*}
    c(W) &= R^{-1} C_W(R) \\
    R &= \frac{1}{u} \left\{ -\ln \psi + \ln \frac{\mu Y \theta}{M_x'(R) - \mu X (1 + \theta)} \right\} \\
    (1 + \theta) &= \frac{c(W)}{E(W)}.
\end{align*}
\]

with unknowns \( c(W), \theta \) and \( R \).

**Extensions**

- After calculating the whole business operational capital charge, the problem of decomposition it into individual business risk lines / loss types,
- Extension of the decision problem by allowing for additional insurance of operational risk \( \rightarrow \) truncated variables.
References


