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A conditionally exponential decay approach to scaling in finance

Rafal Weron[a,*], Karina Weron[b], Aleksander Weron[c]

[a] *Institute of Mathematics, Wroclaw University of Technology, 50-370 Wroclaw, Poland*

[b] *Institute of Physics, Wroclaw University of Technology, 50-370 Wroclaw, Poland*

[c] *Hugo Steinhaus Center for Stochastic Methods, Wroclaw University of Technology, 50-370 Wroclaw, Poland*

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Abstract

We demonstrate how the basic ideas of the fractal and the heterogeneous market hypotheses lead to a rigorous mathematical model, which can be used to solve the problem of characterizing the distribution of price changes corresponding to the empirical scaling law of volatility for high-frequency data from the foreign exchange market. For this purpose, we adopt the conditionally exponential decay model, which describes asymptotic behaviour of general complex systems. We also discuss the overall rationale for why one might expect such scaling laws to hold for financial data. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Methods originating in statistical physics have been proved useful in analysing financial data [1–5]. This inspired us to test a model, originally developed to explain dielectric relaxation in dipolar materials [6–8], on financial markets. More specifically, we adopt the conditionally exponential decay (CED) model [9] and show how to use it to solve the problem of characterizing the distribution of returns (price changes) corresponding to the empirical scaling law of volatility for high-frequency data from the foreign exchange (FX) market [4,10].

* Corresponding author. Fax: +48-71-320-2654; e-mail: rweron@im.pwr.wroc.pl.

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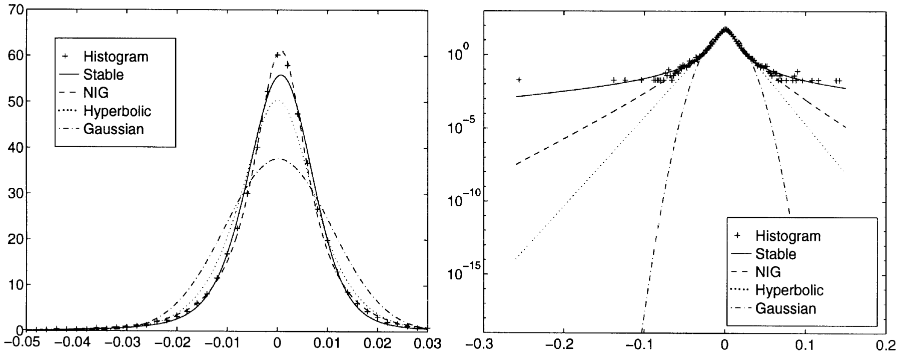

Fig. 1. Histogram of the empirical density (kernel estimator) of the Dow Jones Industrial Average (DJIA) daily returns from January 1901 till May 1996 and four estimating densities: stable, normal inverse Gaussian (NIG), hyperbolic and Gaussian (*left panel*). The same plot on a semilog scale (*right panel*).

In recent years, the set of available data from financial markets has grown rapidly. In the 1960s and 1970s, most of the empirical studies were based on yearly, quarterly, or monthly data. During the 1980s, the study of weekly and daily data led to the discovery of new properties such as autoregressive heteroskedasticity. The studies of intra-day data in the 1990s revealed a new wealth of complexity [11,12].

It is well known, starting from Mandelbrot [13,14] and Fama [15], that market returns are not normally distributed. However, this information has been downplayed or rationalized away over the years to maintain the crucial assumption of the traditional efficient market hypothesis (EMH). In conventional economies, markets are assumed to be efficient if all available information is reflected in current market prices [16]. In the search for satisfactory descriptive models of economic data, large numbers of distributions have been tried and even entire classes of distributional types have been constructed [17–21]. In any particular case it is always possible to find a distribution that fits the data well, provided one works within a suitably broad and flexible class of candidates, see Fig. 1, where some alternatives to the normal distribution – like the stable, the hyperbolic and the normal inverse Gaussian (NIG) distribution – are compared. Table 1 below contains the estimated parameters of these four laws [22].

However, it is not enough to fit given data through the choice of a "good distribution". It is much more important to explain return's data behaviour with a statistical model that predicts the data's main characteristics. This is the main problem we address in this paper.

The efficient market hypothesis (EMH), including arbitrage pricing theory (APT) and the capital asset pricing model (CAPM), was very successful in making the mathematical environment easier, but unfortunately is not justified by the real data. Instead, there is a need to seek for a market hypothesis that fits the observed data better and takes into account why markets exist to begin with. In the place of EMH, the fractal and the heterogeneous market hypotheses (FMH resp. HMH) have been proposed recently [12,23].

Table 1

Distribution	Estimated parameters	
α-stable	$\hat{\alpha} = 1.6147$	$\hat{\sigma} = 0.0051$
	$\hat{\beta} = -0.1581$	$\hat{\mu} = 5.8 \times 10^{-5}$
NIG	$\hat{\chi} = 4.8918 \times 10^{-5}$	$\hat{\psi} = 4.438 \times 10^{3}$
	$\hat{\beta} = -5.6673$	$\hat{\mu} = 7.7 \times 10^{-4}$
Hyperbolic	$\hat{\chi} = 1.3053 \times 10^{-6}$	$\hat{\psi} = 2.1466 \times 10^{4}$
	$\hat{\beta} = -6.885$	$\hat{\mu} = 8.3868 \times 10^{-4}$
Gaussian	$\hat{\sigma} = 0.0106$	$\hat{\mu} = 1.8 \times 10^{-4}$

Based on current developments of chaos theory and using the fractal objects whose disparate parts are self-similar, these hypotheses provide a new framework for a more precise modelling of turbulence, discontinuity, and non-periodicity that truly characterize today's financial markets.

2. Fractal and Heterogeneous Market Hypotheses

The success of the fractal approach gave rise to the hypothesis that the market itself is fractal, with heterogeneous trading behaviour. The fractal market hypothesis (FMH) emphasizes the impact of information and investment horizons on the behaviour of investors. In traditional finance theory, the investor is generic. Basically, an investor is anyone who wants to buy, sell, or hold a security because of the available information, which is also treated as a generic item. The investor is considered a rational price-taker, i.e. someone who always wants to maximize return and knows how to value current information. This generic approach, where information and investors are general cases, implies that all types of information impact all investors equally. That is where it fails.

The following assumptions were proposed for the FMH [23]: the market is made up of many individuals with a large number of different investment horizons; information has a different impact on different investment horizons; the stability of the market is largely a matter of liquidity (balancing of supply and demand). Liquidity is present when the market is composed of many investors with many different investment horizons; prices reflect a combination of short-term technical trading and long-term fundamental valuation; if a security has no tie to the economic cycle, then there will be no long-term trend. Trading, liquidity, and short-term information will dominate.

The purpose of the FMH is to give a model of investor behaviour and market price movements that fits our observations. When markets are considered stable, the EMH and CAPM seem to work fine. However, during panics and stampedes, these models break down, like singularities in physics. This is not unexpected, because the EMH, APT, and the CAPM are equilibrium models. They cannot handle the transition to turbulence. Unlike the EMH, the FMH says that information is valued according to the investment horizon of the investor. The key fact is that under the FMH the market is stable when it has no characteristic time scale or investment horizon. Instability occurs when the market loses its fractal structure and assumes a fairly uniform investment horizon.

A statistical study of financial data from the fractal point of view is based on the analysis of time intervals Δt of different sizes. A reasonable question to ask is: what is the relation between volatility and the size of time intervals? The answer to this question is the scaling law reported in Müller et al. [11], which relates the mean of the absolute logarithmic price change – known also as the volatility – in foreign exchange markets

$$v(t_i) \equiv \frac{1}{n}\sum_{k=1}^{n} |x(t_{i-k}) - x(t_{i-k} - \Delta t)| = c(\Delta t)^D , \tag{1}$$

where c is an empirical constant and D is the empirical drift exponent. In spite of its elementary nature, a scaling law study is immediately able to reject the Gaussian hypothesis and reveal an important property of financial time series. For the Gaussian case the above formula is true with a drift exponent of 0.5, while the empirical values of drift exponents D are significantly different from that value. These and other recently found properties of empirical time series have led the researchers at Olsen & Associates to the heterogeneous market hypothesis (HMH) as opposed to the assumption of a homogeneous market where all participants interpret news and react to news in the same way. The HMH is characterized by the following interpretations of the empirical findings: different actors in the heterogeneous market have different time horizons and dealing frequencies; the market is heterogeneous with a fractal structure of the participants' time horizons as it consists of short-, medium- and long-term components; different actors are likely to settle for different prices and decide to execute their transactions in different market situations (in other words, they create volatility); the market is also heterogeneous in the geographic location of the participants.

The market participants of the HMH also differ in the other aspects beyond the time horizons and the geographic locations: they can have different degrees of risk aversion, institutional constraints, and transactions costs [12,10].

Guillaume et al. [10] present stylized facts concerning the spot intra-daily FX market. They uncover a new wealth of structures that demonstrates the complexity of this market. Further, they group empirical regularities of the FX market data under three major topics: the distribution of price changes, the process of price formation, and the heterogeneous structure of the market. Independently Gallucio et al. [4] addressed the same problem and found that FX rates have a far more complex nature than random walk, since hidden correlations are present in the data. In both papers the following problem is posed:

Problem. *How to characterize the mathematical structure of the distributions of price changes corresponding to the empirical scaling law for volatility*?

In what follows we show how to use the CED model to solve this problem.

3. The CED model

The CED approach clarifies the ideas of the FMH or the HMH and provides a rigorous mathematical framework for further analysis of financial complex processes. Moreover, the obtained distributional form of returns has important implications for theoretical and empirical analyses in economics and finance, since portfolio and option pricing theories are typically based on distributional assumptions [22].

The market is made up of participants, from tick traders to long-term investors. Each has a different investment horizon that can be ordered in time. When all investors with different horizons are trading simultaneously the market is stable. The stability of the market relies, however, not only on a random diversification of the investment horizons of the participants but also on the fact that the different horizons value the importance of the information flow differently. Hence, both the information flow and the investment horizons should have their own contribution to the observed global market features. In general, the locally random markets have a global statistical structure that is non-random.

We assume that the model is a discrete time economy with a finite number of trading dates from time 0 to time T and its uncertainty has a global impact on the market index returns on the interval $[0,T]$. As a proxy of the capital market index we can use a stock index, for example DJIA or S&P500, and as a proxy of the FX market index for a particular currency – the prevailing exchange rate, i.e. the local currency rate against one of the global currencies (e.g. USD, DEM). In the family of all world investors let us identify those N who are acting on a given market described by a chosen index. Call them $I_{1N}, I_{2N},\dots,I_{NN}$. Let R_{iN} be the positive (or the absolute value of negative) part of the ith investor's return. Each ith investor is related to a cluster of agents acting simultaneously on common complement markets. The influence of this cluster of agents is of the type of short-range (intra-cluster) interactions and is reflected by a random risk-aversion factor A_i. The long-range type of interactions are imposed on the ith investor by the inter-cluster relationship manifested by the random risk factors B^i_j for all $j \neq i$. They reflect how fast the information flows to the ith investor, see [24].

Condition CED1. For the ith investor the following property holds:

$$\begin{aligned}\phi_{iN}(r|a,b) &= \mathbf{P}(R_{iN} \geqslant r | A_i = a, b_N^{-1}\max(B^i_1,\dots,B^i_{i-1},B^i_{i+1},\dots,B^i_N) = b)\\ &= \exp(-[a\min(r,b)]^c)\,, \end{aligned} \tag{2}$$

where r, a and b are non-negative constants, b_N is a suitable, positive normalizing constant and $c \geqslant 1$.

The dependence in the CED model measured by the conditional return excess decays similarly as in some conditionally heteroscedastic models, i.e. in an exponential way, see (2), but reflects both short- as well as long-range effects. This idea concerns systems in which the behaviour of each individual entity strongly depends on its short- and long-range random interactions.

Condition CED2. Investors have different investment horizons ("short-range interaction") affected by different information sets ("long-range interaction"). The investment horizon of the investor is reflected by the random variable A_i, while $\{B^i_j,\ j = 1,2,\dots,N,\ j \neq i\}$ reflect the information flow to this investor.

The probability that the return R_{iN} will not be less than r is conditioned by the value a taken by the random variable A_i and by the value b taken by the maximum of the set of random variables $\{B^i_j,\ j = 1,2,\dots,N,\ j \neq i\}$. Therefore (2) can be rewritten as follows:

$$\phi_{iN}(r|a,b) = \begin{cases} 1 & \text{for } r = 0\,, \\ \exp(-(ar)^c) & \text{for } r < b\,, \\ \exp(-(ab)^c) & \text{for } r \geqslant b\,, \end{cases}$$

i.e. the conditional return excess $\phi_{iN}(r|a,b)$ decays exponentially with a decay rate a and exponent c as r tends to the value b. Then it takes a constant value $\ll 1$. The basic statistical assumption is that

Condition CED3. The random variables $A_1, A_2, \dots$ and $B^i_1, B^i_2, \dots$ form independent and convergent (with respect to addition and maximum, respectively [9]) sequences of non-negative, independent, identically distributed (iid) random variables. The variables $R_{1N}, \dots, R_{NN}$ are also non-negative, iid for each N.

Observe that the dependence on external conditions is expressed by the relationship (2) of each R_{iN} with A_i and $\max(B^i_1, \dots, B^i_{i-1}, B^i_{i+1}, \dots, B^i_N)$. Condition CED3 can be partially justified by the following argument. Institutional trading is a major factor in the determination of security prices. If professional investment managers have similar beliefs, then the iid distributions assumption may hold as a first approximation. Professional managers are likely to have similar beliefs because they have access to similar information sources. This uniformity of information over time would tend to generate similar beliefs [24].

The basic result which allows us to see the structure of distributions describing market returns is the following: if the global behaviour of the market is given by

$$\phi(r) = \mathbf{P}\left(\lim_{N\to\infty} r_N \min(R_{1N}, \dots, R_{NN}) \geqslant r \right), \tag{3}$$

where r_N is a suitable, positive normalizing constant, then we have that under conditions CED1-CED3 the function $\phi(r)$ fulfils the global return equation:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r}(r) = -\alpha\lambda(\lambda r)^{\alpha-1} \left(1 - \exp\left(-\frac{(\lambda r)^{-\alpha}}{k} \right) \right) \phi(r)\,, \tag{4}$$

where the parameters $\lambda > 0$, $k > 0$ and $\alpha > 0$ are determined by the limiting procedure in (3).

A more general type of the above equation has been recently studied in the context of complex stochastic systems in [7,9]. It turns out that the solution of (4) has

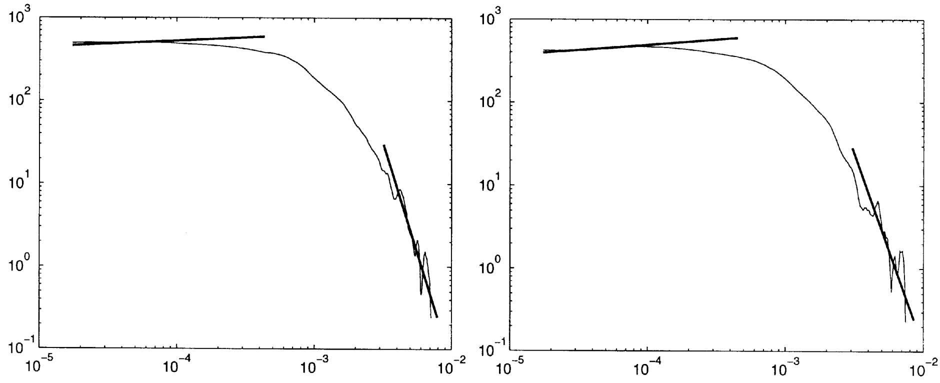

Fig. 2. Double logarithmic plots of the empirical densities (kernel estimators) of the USD/DEM exchange rate 30 min returns for positive (*left panel*) and absolute value of negative returns (*right panel*). Bold lines represent the two power-laws, see (6). For positive returns the maximum likelihood estimation yielded the following values: $\alpha^+ = 1.0781$, $\lambda^+ = 1156$ and $k^+ = 0.2508$.

the following integral form:

$$\phi(r) = \exp\left[-\frac{1}{k}\int_0^{k(\lambda r)^\alpha}(1-e^{-1/s})\,\mathrm{d}s\right].$$

The function $\phi(r)$ monotonically decreases from $\phi(0)=1$ to $\phi(\infty)=0$. Moreover, the probability density of the global return $f(r) = -\mathrm{d}\phi(r)/\mathrm{d}r$ is given by the formula

$$f(r)=\alpha\lambda(\lambda r)^{\alpha-1}\left[1-\exp\left(-\frac{(\lambda r)^{-\alpha}}{k}\right)\right]\exp\left[-\frac{1}{k}\int_0^{k(\lambda r)^\alpha}\left(1-e^{-1/s}\right)\,\mathrm{d}s\right], \quad (5)$$

and exhibits the two power-laws property, see Fig. 2

$$f(r) \propto \begin{cases} (\lambda r)^{\alpha-1} & \text{for } \lambda r \ll 1, \\ (\lambda r)^{-\alpha/k-1} & \text{for } \lambda r \gg 1\,. \end{cases} \quad (6)$$

Hence, the global return distribution is characterized by the following three parameters: α, λ and k, where α is the shape and λ is the scale parameter. In general, parameter α slows down, in comparison with an individual investor, the return rate $\alpha\lambda(\lambda r)^{\alpha-1}$ of the global market return distribution. Let us stress here the role of the parameter k. It decides how fast the information flow is spread out in the market. If $k \to 0$ then the long-range interaction is neglected and (4) takes the form of the Weibull probability density function.

How can we fit the CED model to financial data? One way to do it is to use the most commonly employed estimation procedure among practitioners – the maximum likelihood (ML) method. The ML method is a prescription for producing an estimator $\hat{\theta}$ of the vector of parameters – in our case $\theta=(\alpha,\lambda,k)$, called the maximum likelihood estimator (MLE). The prescription is to consider the likelihood function

$$L_\theta(R_1,\ldots,R_n) = \prod_{k=1}^{n} f(R_k)\,,$$

where $R=(R_1,\ldots,R_n)$ is the sample (e.g. 30 min returns of the USD/DEM exchange rate) and f is the probability density, as a function of θ, and to let the MLE estimator $\hat{\theta}$ be any estimator such that it is the value of the argument θ which maximizes $L_\theta(R)$. Observe that $L_\theta(R)$ obtains a maximum exactly for the same values of θ as the so-called log-likelihood function

$$\log L_\theta(R_1,\ldots,R_n) = \sum_{k=1}^{n} \log f(R_k)\,.$$

This observation usually leads to much easier maximization algorithms.

From formula (5) we can calculate the log-likelihood function of the CED model

$$\log L_\theta(R_1,\ldots,R_n) = \sum_{k=1}^{n} \left\{ \log \alpha\lambda^\alpha + (\alpha-1)\log(R_k) + \log\left[1-\exp\left(-\frac{(\lambda R_k)^{-\alpha}}{k}\right)\right]\right\} - \sum_{k=1}^{n}\left\{\frac{1}{k}\int_0^{k(\lambda R_k)^\alpha} \left(1-e^{-1/s}\right) ds\right\}.$$

Then using the Nelder–Mead simplex minimization procedure (MATLAB implementation) applied to the function

$$-\log L_\theta(R_1,\ldots,R_n)\,,$$

we obtain estimates of α, λ, and k.

4. High-frequency data

In the last section of this paper we discuss the scaling law for volatility using high-frequency data from the foreign exchange market. The considered data set was released by Olsen & Associates for the HFDF-II conference. It included exchange rates of major currencies (of which the USD/DEM and the DEM/FRF exchange rates are studied below) from 1 January 1996 to 31 December 1996. The data came in files where GMT time and FX rates were reported sequentially in 30 min intervals, thus the number of data was 17 520 for each exchange rate. As in [4], data are international quotations of foreign currencies available from Reuters, Knight-Ridder and Telerate, and do not correspond to real prices in the global FX market.

The FX spot market is a 24 h global market, mostly inactive during weekends and national holidays. The first observation of the week arrives at 22:30 Greenwich Mean Time (GMT) on Sunday with the opening of the Asian markets and the last observation comes from the West Coast of the USA at about 22:30 GMT on Friday. Due to these features of the FX market, we have excluded inactive periods from the calculations. In contrast to traditional high-frequency analysis [11,12], where inactive periods are included, this method is much better suited for comparison of scaling laws for different instruments, some of which trade more and some less often. For example in 1996, the market for the most actively traded exchange rate – USD/DEM – was active for 80% of the time, whereas the market for the DEM/FRF exchange rate – only for 71% of the time.

Adequate analysis of the high-frequency (intra-daily) data relies on an explicit definition of the variables under study. These include the price, the change of price, and the volatility. The price at time t_i is defined as

$$x(t_i) \equiv x(t_i, \Delta t) \equiv \tfrac{1}{2}(\log p_{\mathrm{bid}}(t_i) + \log p_{\mathrm{ask}}(t_i)) ,$$

where $\{t_i\}$ is the sequence of the regular spaced in time data, Δt is the time interval ($\Delta t = 30\,\mathrm{min}$, $\Delta t = 1\,\mathrm{h}$, etc.) and $p_{\mathrm{bid}}(t_i)$ ($p_{\mathrm{ask}}(t_i)$) is the arithmetic average of the bid (ask) quotes just prior to and just after time t_i. The definition takes the average of the bid and ask price rather than either the bid or the ask series as a better approximation of the transaction price. The change of price $r(t_i)$ at time t_i is defined as

$$r(t_i) \equiv r(t_i, \Delta t) \equiv (x(t_i) - x(t_i - \Delta t)) .$$

In fact, this is the change of the logarithmic price and is often referred to as *return*. Note that it depends on the parameter Δt, which represents the investment horizon. The change of price, rather than the price itself, is the variable of interest for traders who try to maximize short-term investment returns. The volatility $v(t_i)$ at time t_i is defined as

$$v(t_i) \equiv v(t_i, \Delta t, T) \equiv \frac{1}{n} \sum_{k=1}^{n} |r(t_{i-k}, \Delta t)| , \qquad (7)$$

where T is the sample period over which the volatility is computed (e.g. one day, one year) and n is a positive integer with $T = n\,\Delta t$. A usual example is the computation of the daily volatility as the average daily volatility over one year: $T = 1$ year, $n = 250$ (working) days and $\Delta t = 1$ day.

As we have already mentioned, one of the most striking facts is the fractal structure of FX rates. Observe that formula (7) gives a good estimator of the mean value of the probability distribution of returns. Thus estimating the parameters of a CED distribution describing asset returns and approximating the mean value of the CED distribution by $1/\lambda$ we can compare the empirical and theoretical scaling laws for volatility. This is illustrated in Fig. 3, where the scaling laws for volatility of the USD/DEM and the DEM/FRF exchange rates and of the estimated CED models are compared. The fit is very good, especially for the USD/DEM exchange rate.

Our findings ($D_{emp}, D_{CED} < 0.5$) are quite different from some of the earlier results [10,12], where the exponent D of about 0.58 for the major FX rates was measured. The differences were probably caused by the different time intervals taken into account – in their case – 1987–1993, or by the sampling method. As we have said earlier, in our analysis we excluded the inactive periods from the data. This seems to be consistent with the results of Gallucio et al. [4], where an exponent of about 0.45 was measured for the USD/DEM exchange rate and the 1 October 1992–30 September 1993 period.

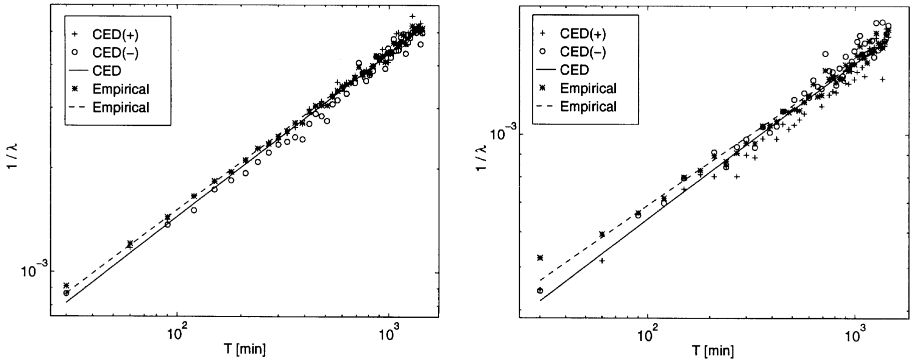

Fig. 3. Scaling law for the USD/DEM exchange rate with exponent $D_{emp} = 0.4614$ and the approximating CED density with $D_{CED} = 0.4777$ (*left panel*). Scaling law for the DEM/FRF exchange rate with exponent $D_{emp} = 0.3249$ and the approximating CED density with $D_{CED} = 0.3477$ (*right panel*).

These studies demonstrate that, in principle, it is possible to estimate the empirical drift exponent D, see formula (1), of the scaling law reported in [11,12] from the CED approach. Thus we have shown that the CED model can be used to solve the problem of Guillaume et al. [10].

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