Pricing electricity risk by interest rate methods

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February 13, 2004

Abstract
We address a method for pricing electricity contracts based on valuation of ability to produce power, which is considered as the true underlying for electricity derivatives. This approach shows that an evaluation of free production capacity provides a framework where a change-of-numeraire transformation converts electricity forward market into the common settings of money market modeling. Using the toolkit of interest rate theory, we derive explicite option pricing formulas.

Key words: energy economics, futures markets, electricity prices, power derivatives.
1 Introduction

Beginning in the nineties a number of countries have deregulated their markets for electrical power. This involved the creation of competitive power markets, where electrical energy is traded as a commodity. One of the most popular financial products for electricity risk management is the power forward. The buyer of such an instrument is guaranteed the delivery of a pre–determined amount of electrical energy as a constant flow over a future period of time specified in the contract. The delivery is either physical, or settled financially. The importance of power forwards is comparable to those of forward contracts in other commodity markets since both the buyer and the writer insure themselves against possible harmful future price movements. Similarly, derivative instruments written on power forwards are also widely used to hedge the electricity price risk. However, the valuation of these contracts is still under discussion due to the lack of convincing economical pricing concepts. The point here is that the electrical energy is not economically storable. Thus, power forwards with non-overlapping delivery intervals seem to have different underlying commodities (electrical energy, delivered in different periods) without any opportunity to transfer one commodity into the other.

At the first glance, the pricing methodology in the style of interest rate theory seems appropriate, since a forward contract supplying 1 MWh of electrical power within a short delivery interval immediately after $\tau$ is analogous to a zero–bond maturing at the time $\tau$. However, for power forwards, we observe a peculiar price behavior: considering prices as a function of time to maturity, one notices that they are relatively stable for times long before the onset of the delivery period, but as the delivery approaches, prices begin to fluctuate. Clearly, this behavior is impossible for bonds, whose prices converge to one when approaching their maturity date. As a response to this, a line of research (see [5], [19], [3], [17]) has focused on modeling the evolution of the whole forward curve dynamics by

$$\frac{dp_t(\tau)}{p_t(\tau)} = \sum_{i=1}^{d} \sigma_i(\tau) dW_t^i, \quad d \in \mathbb{N}. \quad (1)$$

Here $p_t(\tau)$ denotes the price at time $t$ for the future delivery of 1 MWh of at time $\tau \geq t$, $(W_t^i)$ for $i = 1, \ldots, d$ are Brownian motions under risk–neutral probability measure and $(\sigma_i(\tau))$ denotes the volatility of term structure. The limit $\lim_{t \uparrow \tau} \sum_{i=1}^{d} \sigma_i(\tau)^2$ is positive reflecting the fluctuation of power forward prices near maturity. Specifying volatility structure, the dynamics (1) yields explicite option formulas (see [3]). However, purposing (1) directly, qualitative features of electricity risk remain unconsidered. Among them are questions of existence and interpretation of risk–neutral measures, risk hedging by real assets, explicite connection to interest rate models and reasons for increasing volatility at maturity. Moreover, the question of valuation and hedging for electricity options is frequently considered as a part
of hedging problems in incomplete markets and so electricity risk management is linked to pricing of weather derivatives and insurance–like instruments. Obviously, this point of view neglects the electricity production process: although electrical energy can not be stored, a hedging is still possible by production capacity investments. That is, a transparent and liquid market for contracts on availability of free electricity production capacities will help to price correctly, to reduce, and to avoid risk resulting from electricity production and trading.

In this work, we respond to these aspects considering pricing of electricity contracts within a production capacity market. It turns out that equilibrium asset prices are given by their future payoff, expected with respect to some equivalent measure. Using this framework, we apply a change–of–numeraire transformation (see [10]) to avail the toolkit of interest rate theory.

Let us mention some relations to other research in this field. In [8], the authors expose questions of electricity pricing and explain that the non–storability requires a modeling of production process. They suggest to use marginal fuel (gas, oil, etc.) prices to describe forward power prices, considering the fact that fuel is easily transformed into electrical power, provided an electricity production unit is rented for the delivery period. The work [20] and the recent paper [6] present a counterpart of this conception for hydro–electric power generation giving a treatment of operational flexibility valuation and dispatch management to hedge electricity contracts by optimal scheduling of hydro–electric power plants. Both approaches show that a detailed production–based modeling of electricity markets may shed light on how to price electricity contracts. We follow this insight in our paper. Another line of research (see [19], [2], [15]) focuses on modeling the stochastic process of spot price. Again, a production–based point of view exhibits reasons for high spot price volatility and highlights the role of real–time electricity auctions. Electricity auctions themselves are considered as a crucial point in deregulation of electricity industry and enjoy a considerable research interest (see [7], [1], and [11] – [14]).

The paper is organized as follows. First we introduce the equilibrium principle in a capacity market, in order to drive a contract valuation. In Section 3 an application to interest rate theory is discussed. An empirical study of our pricing methodology is presented in Section 4. The detailed proof of the existence of a symmetric equilibrium, on which the contract valuation is based, is given in Section 5.

2 An equilibrium principle for pricing electricity contracts

Here we present a valuation principle which comes from the realization that though electricity can not be stored, it can be produced and so the true underlying of electricity contracts
is the physical ability to produce power. Consequently, writers of contingent claims prefer those contracts which are perfectly replicated by an appropriate portfolio of real production units. For this reason, we observe that in electricity markets, financial assets typically mimic agreements on power production capacities. The market price for such an agreement is to be considered as a fair price for the financial asset which assures the agreements payoff.

We describe electricity market at discrete equidistant times \( t = 0, \ldots, T, T \in \mathbb{N} \) on the filtered probability space \( (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t=0}^T) \), where \( \mathcal{F}_t \) is the information available to the market participants at the time \( t = 0, \ldots, T \). Let us start deterministically \( P(A) \in \{0, 1\} \) for all \( A \in \mathcal{F}_0 \) and restrict ourselves to consider adapted processes. Assume that \( \mathcal{E} = \mathcal{E}^{phys} \cup \mathcal{E}^{fin} \) is a finite set of tradeable assets, where \( \mathcal{E}^{phys} \) denotes physical assets (production capacity agreements) and \( \mathcal{E}^{fin} \) stands for financial assets. Let \( (R_t)_{t=1}^T \) be an \( \mathbb{R}^\mathcal{E} \)-valued process describing revenues, where \( R_t(e) \) is the revenue from holding the asset \( e \in \mathcal{E} \) within \([t-1, t]\). Suppose that \( I \in \mathbb{N} \) agents may share the assets. An agent \( i = 1, \ldots, I \) is determined by \((x_i, U_i)\), where \( x_i \in [0, \infty] \) denotes its initial endowment and \( U_i \) is its utility function.

\[
(2) \quad U_i \in \{ U \in C^1[0, \infty[ : U' \text{ is positive, strictly decreasing with } \lim_{z \to \infty} U'_i(z) = 0 \}. 
\]

At times \( \{0, 1, 2, \ldots, T\} \) the agents \( i = 1, \ldots, I \) trade the assets, which are arbitrarily divisible and short positions are allowed. At the end of each period, agents obtain their part of revenues and re–allocate their wealth. We agree to write \( (\hat{F}_t = F_t/N_t)_{t=0}^T \) for asset prices \( (F_t)_{t=0}^T \) expressed in the units of savings security whose price process we denote by \((N_t)_{t=0}^T\). Under additional assumptions, we calculate equilibrium asset prices for given

\[
(3) \quad (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P), \quad (N_t)_{t=0}^T, (R_t)_{t=1}^T, (U_i, x_i)_{i=1}^I.
\]

Let us explain the notion of equilibrium we use. Write \( S_t = (S_t(e))_{e \in \mathcal{E}} \) to denote the price vector of all physical and financial assets \( e \in \mathcal{E} \) at time \( t \). A trading strategy \( ((\theta_t, \vartheta_t))_{t=0}^T \) determines the number \( \theta_t \) of savings security units and the part \( \vartheta_t(e) \) of each asset \( e \in \mathcal{E} \) held by the agent within \([t, t+1]\). The strategy \( ((\theta_t, \vartheta_t))_{t=0}^T \) is called self–financed, if

\[
(4) \quad X_{t+1} = X_t + \theta_t(N_{t+1} - N_t) + \vartheta_t \circ (S_{t+1} - S_t + R_{t+1}) \quad \text{for all } t = 0, \ldots, T - 1,
\]

where \( (X_t = \theta_t N_t + \vartheta_t \circ S_t)_{t=0}^T \) denotes the wealth of this strategy. Let us point out that a self–financed \( ((\theta_t, \vartheta_t))_{t=0}^T \) is uniquely determined by its initial wealth \( x = \theta_0 N_0 + \vartheta_0 \circ S_0 \) and by its asset positions \( \vartheta = (\vartheta_t)_{t=0}^T \). In fact, savings security positions \( (\theta_t)_{t=0}^T \) are reconstructed from \((x, \vartheta)\) recursively by

\[
(5) \quad \theta_t = (X_t^{x, \vartheta, S} - \vartheta_t \circ S_t)/N_t,
\]

\[
(6) \quad X_{t+1}^{x, \vartheta, S} = X_t^{x, \vartheta, S} + \theta_t(N_{t+1} - N_t) + \vartheta_t \circ (S_{t+1} - S_t + R_{t+1})
\]
starting at \( t = 0 \) with \( X_t^{x,\vartheta,S} = x \). Thus, each element from

\[ (x, \vartheta) : x \in [0, \infty[, \vartheta = (\vartheta_t)_{t=0}^T \text{ is } (\mathcal{F}_t)_{t=0}^T \text{–adapted} \] 

(7)

corresponds by (5) and (6) to a unique self–financed strategy implying that the set (7) gives a parameterization of all self–financed strategies and obviously \((X_t^{x,\vartheta,S})_{t=0}^T\) is the wealth of the strategy determined by \( (x, \vartheta) \). For given initial endowment \( x \in [0, \infty[ \), asset prices \( S = (S_t)_{t=0}^T \), and utility function \( U \), introduce admissible positions by

\[ A(x, S, U) := \{ \vartheta = (\vartheta_t)_{t=0}^{T-1} : \hat{X}_t^{x,\vartheta,S} \geq 0, \ t = 0, \ldots, T, \ E(U(\hat{X}_T^{x,\vartheta,S})) < \infty \} \]

We suppose that each agent behaves rationally: given prices \( S \), the agent chooses strategy \( \vartheta^* \in A(x, S, U) \) which maximizes \( A(x, S, U) \rightarrow \mathbb{R}, \ \vartheta \mapsto E(U(\hat{X}_T^{x,\vartheta,S})) \).

**Definition 1.** An equilibrium \((S^*, \vartheta^1, \ldots, \vartheta^I, \ldots)\) of electricity market with agents \((x_i, U_i)_{i=1}^I\) consists of price process \( S^* \) and agent’s positions \((\vartheta^i_{*})_{i=1}^I\) such that market clears as

\[ \sum_{i=1}^I \vartheta^i_{*}(e) = 1 \text{ for all } e \in \mathcal{E}^{phys}, \quad \sum_{i=1}^I \vartheta^i_{*}(e) = 0 \text{ for all } e \in \mathcal{E}^{fin}, \ t = 0, \ldots, T, \]

and \( \vartheta^i_{*} \) maximizes \( \vartheta \mapsto E(U_i(\hat{X}_T^{x_i,\vartheta,S^*})) \) on \( A(x_i, S^*, U_i) \) for each \( i = 1, \ldots, I \).

To ensure the existence of the equilibrium, we upgrade our analysis by additional assumptions.

**Assumption 1:** The one–period revenue is integrable and bounded from below:

\[ E(|\hat{R}_t(e)|) < \infty, \quad \text{essinf } \hat{R}_t(e) > -\infty \quad \text{for all } e \in \mathcal{E}^{phys}, \ t = 1, \ldots, T. \]

(8)

**Assumption 2:** All contracts lose their values at the final date:

\[ S_T(e) = 0 \quad \text{for all } e \in \mathcal{E}. \]

(9)

**Assumption 3:** All agents \((x_i, U_i)_{i=1}^I\) are equal: There exists a utility function \( U \) and an initial endowment \( x \in [0, \infty[ \) such that

\[ U_i = U, \quad x_i = x \quad \text{for all } i = 1, \ldots, I. \]

(10)

The first assumption is reasonable since the capacity holder runs the unit if the electricity price covers the variable costs of production, otherwise, the unit is idle and causes merely fixed costs. Hence, the one-period loss is bounded by the fixed costs aggregated within one period. The second assumption is justified if we suppose that agents trade contracts which
are valid for the period $[0, T]$ and so the market prices vanish as the agreements expire. The third assumption helps to avoid an exact description of agents endowments and their utility functions which are not observed in reality. As we assumed that all agents are equal, it is naturally to suppose that they hold the same positions $\vartheta^*$:

\begin{equation}
\vartheta^*_t(e) = 1/I \quad \text{for all } e \in \mathcal{E}^{phys}, \quad \vartheta^*_t(e) = 0, \quad \text{for all } e \in \mathcal{E}^{fin} \quad \text{for all } t = 0, \ldots, T.
\end{equation}

Such an equilibrium $(S^*, \vartheta^*, \ldots, \vartheta^*)$ is called symmetric. In the last section we show that a symmetric equilibrium exists, provided all agents are sufficiently wealthy. Moreover, there is a measure $Q$ equivalent to $P$ such that equilibrium asset prices are given by their future revenues, expected with respect to $Q$:

\begin{equation}
\hat{S}^*_t(e) = E_Q \left( \sum_{u=t+1}^{T} \hat{R}_u(e) \mid \mathcal{F}_t \right), \quad t = 0, \ldots, T, \quad e \in \mathcal{E}.
\end{equation}

### 3 Interest rate formulation

In the previous section, we have outlined that mild assumptions ensure the existence of equilibrium and provide contract valuation by (12). This formula suggests that equilibrium pricing is performed by the equivalent–measure–methodology common for financial modeling. If the market data (3) with (8) – (10) are given, then we are able to explicitly price electricity contracts. However, in reality, most quantities in (3) are not known, instead, one usually observes exchange prices for various financial products. Hence, to overcome the unknown quantities in (12), we have to describe the asset dynamics directly under $Q$ such that the observed exchange prices are explained as best as possible. Let us see how to proceed in this way for the case of power forward market.

Suppose there is an electricity market with (3) and (8) – (10) where all agents are sufficiently wealthy. Choose domestic currency as follows:

\begin{equation}
\text{currency unit at } t \text{ is 1 MWh, delivered within a short interval immediately after } t.
\end{equation}

Suppose that the savings security $(N_t)_{t=0}^{T}$ is a bank account in EURO paying a constant interest rate $r > 0$, which means that

\begin{equation}
e^{-rt}N_t \text{ is the reciprocal EURO-price at time } t \text{ for electricity delivered at } t.
\end{equation}

In the symmetric equilibrium, there exists a measure $Q$ such that the market price $p_t(\tau)$ at the time $t$ for power forward maturing at $\tau$ is given by

\begin{equation}
p_t(\tau) = N_t E_Q \left( \frac{1}{N_{\tau}} \mid \mathcal{F}_t \right) \quad \tau = 0, \ldots, T, \quad t = 0, \ldots, \tau.
\end{equation}
since due to the numeraire (13), all power forward prices finish at one: \( p_t(t) = 1 \) for all \( t = 0, \ldots, T \). So we describe their dynamics by interest rate methodology to. To do so, we apply Heath–Jarrow–Morton (HJM) formulation with spot martingale measure, which uses the wealth of the self financing strategy investing entirely in just maturing bonds as the standard numeraire security and supposes that asset prices, expressed in units of this numeraire follow martingales with respect to the spot martingale measure (see [4], [18]). Thus, to accomplish the analogy of electricity market to money market in the above form, we have to choose the wealth of the self financing strategy investing entirely in just maturing power forwards as the new numeraire. For this reason, we introduce the sliding MWh \( (B_t)_{t=0}^T \) defined by

\[
B_t = \Pi_{u=1}^t p_{u-1}(u)^{-1}, \quad t = 0, \ldots, T
\]

which mimics the wealth of this strategy. Choosing \( (B_t)_{t=0}^T \) as numeraire, we have to change from \( Q \) to the spot martingale measure \( \tilde{Q} \) by

\[
d\tilde{Q} := \frac{N_0 B_T}{N_T B_0} dQ
\]

in order to ensure the martingalizing property:

\[
\text{for each process } (F_t)_{t=0}^T \text{ such that } (F_t/N_t)_{t=0}^T \text{ is a } Q \text{-martingale, } (F_t/B_t)_{t=0}^T \text{ is a } \tilde{Q} \text{-martingale.}
\]

As a consequence to this, the discounted electricity forward prices

\[
(\hat{p}_t(\tau) := p_t(\tau)/B_t)_{t=0}^T \text{ are } \tilde{Q} \text{-martingales for all } \tau = 0, \ldots, T.
\]

Moreover, the discounted savings security

\[
(\hat{N}_t := N_t/B_t)_{t=0}^T \text{ is a } \tilde{Q} \text{-martingale.}
\]

Now we turn to valuation of the European call option with strike price of \( K \) EURO and maturity date \( \tau_1 \) written on the price of a power forward maturing at \( \tau_2 \) where \( 0 \leq \tau_1 \leq \tau_2 \leq T \) and \( \tau_1, \tau_2 \in \{0, \ldots, T\} \). The crucial point is that option strike price is given in EURO, which is a foreign currency in the electricity market. Thus, at time \( t \), the European call price in EURO is

\[
C_t = B_t E_{\tilde{Q}}((p_{\tau_1}(\tau_2) - Ke^{-r\tau_1}N_{\tau_1})^+ B_{\tau_1}^{-1} | \mathcal{F}_t)/(e^{-rt}N_t)
\]

\[
= B_t E_{\tilde{Q}}((\hat{p}_{\tau_1}(\tau_2) - Ke^{-r\tau_1}\hat{N}_{\tau_1})^+ | \mathcal{F}_t)/(e^{-rt}N_t)
\]

Here \( Ke^{-r\tau_1}N_{\tau_1} \) is the strike price in MWh, \( B_{\tau_1}^{-1}, B_t \) occur due to discounting and undiscounting, and the division by \( e^{-rt}N_t \) transforms from MWh back to EURO.
For the remainder of this section, we substantiate the above formula by purposing $\tilde{Q}$-dynamics for $(N_t)_t$, $(p_t(\tau))_t$ and $(B_t)_t$ within the Heath–Jarrow–Morton framework. Therefore, we need an adequate model restatement in continuous time.

Let $(W_t)_{t \in [0,T]}$ be a $d$-dimensional Brownian motion given on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \tilde{Q})$ where the filtration is right-continuous, $\tilde{Q}$-completed version of the Brownian filtration. Suppose (see assumptions HJM.1, HJM.2 from [18]) that the power forward prices are given by

$$p_t(\tau) = \exp(-\int_0^\tau f_t(u)du) \quad \text{for all } t = [0, \tau], \tau \in [0, T]$$

where the forward rates $(f_t(\tau))_{t \in [0,T]}$ for each $\tau \in [0, T]$ fulfill

$$f_t(\tau) = f_0(\tau) + \int_0^t \alpha_u(\tau)du + \int_0^t \sigma_u(\tau)dW_u$$

with some Borel measurable function $f_0(\cdot) : [0, T] \to \mathbb{R}$ and coefficients

$$\alpha : \Delta \times \Omega \to \mathbb{R}, \quad \sigma : \Delta \times \Omega \to \mathbb{R}^d$$

defined on $\Delta = \{(u, t) : 0 \leq u \leq t \leq T\}$ such that $(\alpha_u(t))_{u \in [0,t]}, (\sigma_u(t))_{u \in [0,t]}$ are for each $t \in [0, T]$ adapted processes satisfying

$$\int_0^t \alpha_u(t)du < \infty, \quad \int_0^t \sigma_u(t)^2du < \infty \quad \text{for all } t \in [0, T].$$

In analogy to (15), we introduce

$$B_t := \exp(\int_0^t f_u(u)du) \quad t \in [0, T]$$

and according to (18), require that

$$\tilde{p}_t(\tau) := p_t(\tau)/B_t \in [0,\tau] \quad \text{are } \tilde{Q}-martingales for all } \tau \in [0, T].$$

Consequently, $\alpha$ is uniquely determined by $\sigma_r$:

$$\alpha_t(\tau) = \sigma_t(\tau) \int_\tau^\tau \sigma_t(u)du \quad \text{for all } t \in [0, \tau], \tau \in [0, T],$$

since using (21), (22), and the Proposition 2.2.1 from [4] we have

$$d\tilde{p}_t(\tau) = \tilde{p}_t(\tau)(a_t(\tau) + \frac{1}{2}\|s_t(\tau)\|^2)dt + \tilde{p}_t(\tau)s_t(\tau)dW_t$$

with coefficients

$$a_t(\tau) = \int_\tau^\tau \alpha_t(u)du, \quad s_t(\tau) = \int_\tau^\tau \sigma_t(u)du \quad \text{for all } t \in [0, \tau], \tau \in [0, T].$$
Hence, (24) implies that $a_t(\tau) + \frac{1}{2}\|s_t(\tau)\|^2 = 0$ for all $t \in [0, \tau]$, $t \in [0, T]$ and differentiating this equation with respect to $t$ we obtain (25), which with (26) finally provides a substantiated version of (18):

\[(28) \quad d\hat{p}_t(\tau) = \hat{p}_t(\tau)s_t(\tau)dW_t \quad \text{for all} \quad \tau \in [0, T].\]

Now, the continuous–time counterpart of positive $\tilde{Q}$–martingale (18) will be an exponential $\tilde{Q}$–martingale, due to the Brownian framework:

\[(29) \quad d\tilde{N}_t = \tilde{N}_tv_t dW_t.\]

Now we sketch the use of HJM modeling following [4]. Specify (this is a modeling part) the volatilities

\[(30) \quad (v_t)_{t \in [0,T]}, \quad (\sigma_t(\tau))_{t \in [0,\tau]} \quad \text{for} \quad \tau \in [0, T]\]

and observe today’s forward rates

\[f_0(\tau) = -\frac{\partial}{\partial \tau} \ln p_0(\tau) \quad \tau \in [0, T].\]

Then $f_t(\tau)$ for $(t, \tau) \in \Delta$ is determined by (22) and power forward prices by (21). We may now price arbitrary electricity derivatives. Let us illustrate how it works by valuing European call written on a power forward. To obtain a closed–form solution, we focus on deterministic volatility structures.

**Proposition 1.** Suppose that

\[(31) \quad s_t(\tau) - v_t \text{ is deterministic for all } t \in [0, \tau] \text{ and } \tau \in [0, T],\]

then EURO–price $C_t^*$ at the time $t \in [0, T]$ for European call with strike price $K > 0$ EURO, time to maturity $\tau_1 \in [t, T]$ written on power forward with time to maturity $\tau_2 \in [\tau_1, T]$ is given by

\[(32) \quad C_t^* = E_t(\tau_2)N(d_1) - e^{-\tau(\tau_1-t)}KN(d_2)\]

where $E_t(\tau_2)$ is the EURO–price at the time $t$ for the underlying forward and

\[
\begin{align*}
   d_1 &= \frac{1}{\Sigma}(\ln(E_t(\tau_2)_K) + \tau(\tau_1-t) + \frac{1}{2}\Sigma^2) \\
   d_2 &= d_1 - \Sigma \\
   \Sigma^2 &= \Sigma^2(t, \tau_1, \tau_2) = \int_t^{\tau_1} \|s_u(\tau_2) - v_u\|^2 du.
\end{align*}
\]
Proof. Using the new measure
\[ dQ' = \frac{\tilde{N}_t}{t}d\tilde{Q}, \]
we rewrite the formula (20) as
\[
C_t^* = B_t E_Q(\tilde{N}_t(\frac{\hat{p}_{\tau_1}(\tau_2)}{N_{\tau_1}} - e^{-r\tau_1}K)^+ | \mathcal{F}_t)/(e^{-rt}N_t)
\]
\[
= B_t E_Q'((\frac{\hat{p}_{\tau_1}(\tau_2)}{N_{\tau_1}} - e^{-r\tau_1}K)^+ | \mathcal{F}_t)
\]
\[
(34) = E_Q'((e^{rt}\frac{\hat{p}_{\tau_1}(\tau_2)}{N_{\tau_1}} - e^{-r(\tau_1-t)}K)^+ | \mathcal{F}_t).
\]

Introduce
\[
E_u(\tau_2) = e^{ru}\frac{\hat{p}_u(\tau_2)}{N_u} = e^{ru}\frac{p_u(\tau_2)}{N_u} = p_u(\tau_2)(e^{-ru}N_u)^{-1} \quad \text{for all } u \in [0, \tau_1]
\]
which is in view of (14) interpreted as the EURO price at time \( u \) for electricity delivered at time \( \tau_2 \). Define
\[
\mathcal{E}_u(\tau_2) = e^{-ru}E_u(\tau_2) = \frac{p_u(\tau_2)}{N_u} \quad u \in [0, \tau_1]
\]
which possesses the stochastic differential
\[
d\mathcal{E}_u(\tau_2) = \mathcal{E}_u(\tau_2)(\gamma_u du + \beta_u dW_u)
\]
where Ito formula yields coefficients
\[
(36) \quad \gamma_u = \|v_u\|^2 - v_us_u(\tau_2), \quad \beta_u = s_t(\tau_2) - v_u \quad \text{for all } u \in [0, \tau_1].
\]
Let us write the solution to (35) as
\[
\mathcal{E}_u(\tau_2) = \mathcal{E}_0(\tau_2)e^{L_u \tau_1 - \frac{1}{2}[L]_u} \quad \text{for all } u \in [0, \tau_1]
\]
\[
(37) = \mathcal{E}_t(\tau_2) \exp \left( L_u - L_t - \frac{1}{2}([L]_u - [L]_t) \right) \quad \text{for all } u \in [t, \tau_1]
\]
where
\[
L_u = \int_0^u \gamma_q dq + \int_0^u \beta_q dW_q, \quad [L]_u = \int_0^u \|\beta_q\|^2 dq \quad \text{for all } u \in [0, \tau_1].
\]
With this quantities, (34) reads as
\[
C_t^* = E_Q'((E_t(\tau_2) \exp(L_{\tau_1} - L_t - \frac{1}{2}([L]_{\tau_1} - [L]_t) - e^{-r(\tau_1-t)K})^+ | \mathcal{F}_t).
\]
According to Girsanov theorem, \((L_u)_{u \in [0, \tau_1]}\) follows a continuous martingale under \(Q'\). Using the deterministic time change
\[
l(u) = \inf\{q \in [0, \tau_1] : [L]_q > u\} \quad \text{for all } u \in [0, [L]_{\tau_1}]
\]
we verify with time–change properties for continuous semimartingales (see [16], Theorem 4.6, chapter 3) that
\[
W'_{u} := L_{l(u)}, \quad F'_{u} := F_{l(u)} \quad \text{for all } u \in [0, \tau_1]
\]
defines a \(Q'\)–Brownian motion \((W'_{u}, F'_{u})_{u \in [0, \tau_1]}\), satisfying
\[
L_u = W'_{[L]_u} \quad \text{almost surely for all } u \in [0, \tau_1].
\]
Since the quadratic variation \([L]\) is deterministic,
\[
G := L_{\tau_1} - L_t = W'_{[L]_{\tau_1}} - W'_{[L]_t}
\]
follows due to (33), (36), and (38) under \(Q'\) a centered Gaussian distribution with variance
\[
E_{Q'}(G^2) = [L]_{\tau_1} - [L]_t = \Sigma^2(t, \tau_1, \tau_2).
\]
Thus, we obtain from (39) with (40) and (41) that
\[
C^*_t = E_{Q'}((E_t(\tau_2) \exp(G - \frac{1}{2} \Sigma^2) - e^{-r(\tau_1-t)}K)^+ \mid F_t).
\]
Being an increment, \(G\) is \(Q'\)–independent from \(F'_{[L]_t} = F_{l([L]_t)}\) and also \(Q'\)–independent from \(F_t \subseteq F_{l([L]_t)}\) where the inclusion holds due to \(t \leq l([L]_t)\). The expression (32) follows now from (42) by a straight–forward derivation. \(\square\)

Let us point out that \(C^*_t\) in (39) is alternatively calculated using Black–Scholes formula
\[
C^*_t = BS(E_t(\tau_2), K, \tau_1, t, r, \sqrt{\Sigma^2(t, \tau_1, \tau_2)/(\tau_1 - t)})
\]
with \(\Sigma^2(t, \tau_1, \tau_2)\) from (33) and
\[
BS(s, k, \tau, t, r, \sigma) := sN(d_+) - e^{-r(\tau-t)}kN(d_-)
\]
with
\[
d_+ = \frac{1}{\sigma \sqrt{\tau - t}} (\ln\left(\frac{s}{k}\right) + (r + \frac{1}{2} \sigma^2)(\tau - t)), \quad d_- = d_+ - \sigma \sqrt{\tau - t}.
\]
The relation (44) is useful for calibrating the model parameters: given implied Black–Scholes volatilities for observed option prices, we obtain \(\sqrt{\Sigma^2(t, \tau_1, \tau_2)/(\tau_1 - t)}\) for different parameters \(t, \tau_1, \tau_2\) to adjust volatility structure (30). In the following section, we outline the use of this technique exemplarily discussing option data from the power exchange NordPool.
4 Example

An essential part of our pricing approach to disentangle the term structure behavior of forward prices in MWh from effects caused by spot price fluctuation. That is, we have first to apply standard tools of interest rate theory to model power forward prices in MWh, hereafter, we establish a connection to EURO prices. Whereas the first steps proceeds along the field-tested techniques (see [17]), the assignment to money units in the second step seems to involve some unusual considerations. Let us focus on them.

Suppose that we have chosen the simplest one–factor model for the forward rate dynamics specifying in (22) the constant and deterministic forward rate volatility $\sigma_u(\tau) = \sigma \in ]0, \infty[$ for all $u \in [0, \tau], \tau \in [0, T]$. According to (27), the forward price volatility is determined by $s_t(\tau) = \sigma(\tau - t)$ and the dynamics (28) is given by

$$d\hat{p}_t(\tau) = \hat{p}_t(\tau)\sigma(\tau - t)dW^1_t,$$

where $(W^1_t)_{t \in [0, T]}$ is a Brownian motion under the spot martingale measure $\tilde{Q}$. To describe the dependence between $\hat{p}_t(\tau)$ and $N_u$ by the correlation parameter $\rho \in [-1, 1]$, we chose for (29) the dynamics

$$d\hat{N}_t = \hat{N}_t v(\rho dW^1_t + \sqrt{1 - \rho^2} dW^2_t),$$

where $v \in ]0, \infty[$ is deterministic and constant and $(W^2_t)_{t \in [0, T]}$ is a Brownian motion independent from $(W^1_t)_{t \in [0, T]}$. Let us examine the effects of $\rho$ and $v$ on option prices to explain how these model parameters are identified from observed market data.

Since the volatilities are deterministic, EURO–prices for European calls are calculated from (32) with $\Sigma^2$ as in (33)

$$\Sigma^2(t, \tau_1, \tau_2) = \int_t^{\tau_1} (|\sigma(\tau_2 - u) - \rho v|^2 + |v\sqrt{1 - \rho^2}|^2) du$$

$$= \sigma^2 \frac{(\tau_2 - t)^3 - (\tau_2 - \tau_1)^3}{3} - 2\sigma \rho \frac{(\tau_2 - t)^2 - (\tau_2 - \tau_1)^2}{2} + v^2(\tau_1 - t).$$

For the case the delivery time $\tau_2$ coincides with options expiry date $\tau_1 = \tau_2 =: \tau$, we obtain the limit

$$\lim_{t \uparrow \tau} \frac{\Sigma^2(t, \tau, \tau)}{\tau - t} = \lim_{t \uparrow \tau} \left( \sigma^2 \frac{(\tau - t)^2}{3} - \sigma \rho (\tau - t) + v^2 \right) = v^2$$

showing that for the market model with parameters $\sigma, v, \rho$ as above, the implied Black–Scholes volatility for European calls with $\tau_1 = \tau_2 = \tau$ tends to $v$ as the time approaches options maturity date. The remaining parameter $\rho$ is identified from the rate this convergence

$$\frac{\partial}{\partial t} \bigg|_{t=\tau} \frac{\Sigma^2(t, \tau, \tau)}{\tau - t} = \rho \sigma.$$
Thus, positive (negative) $\rho$ correspond to the increasing (decreasing) implied volatilities at maturity. The following numerical example shows how to utilize these considerations to be consistent with real market prices.

Consider the price data for the European call option OPT-EC250FWSO-03 quoted at the NordPool’s Eltermin market (see [21]). The underlying is a power forward FWD-FWSO-03 supplying 3672 MWh of electrical energy within the period from 01.05.2003 to 30.09.2003 at the constant intensity of 1 MW. For commensurability reasons, the prices of the underlying and option strikes are listed in Norwegian kronen (NOK) for 1 MWh of energy delivered as a constant flow over the corresponding time interval. The option OPT-EC250FWSO-03 with the strike price $K = 250$ NOK has expired on 17.04.2003, 13 days before the start of underling’s delivery. The Figure 1 illustrates the Black–Scholes implied volatility during the trading time of the option from 20.11.2002 to 15.04.2003 calculated with short rate $r = 0.03$. At the times far to maturity, we observe a step-like graph indicating that option prices may be settled by the Black–Scholes formula with some pre-specified volatility. Since near maturity the volatility steps disappear, we assume that supply and demand have determined option price at these times. Moreover, we observe a decrease of the implied volatility near maturity. Supposing that the market is described by the one-factor model with parameters $\sigma, v, \rho$ as above, we
conclude from the Figure 1 that $\rho$ is negative and $v$ is around 0.3. However, this estimates neglect the time difference of 13 days between the start of delivery and expiry date. Further, we do not consider that the power forward supplies energy not instantaneously, but during a period of four months. In this sense, the above procedure is rather an illustration of our techniques than a quantitative analysis.

5 Appendix

Let us show that if the agent’s endowment is sufficiently large, then a symmetric equilibrium exist. We use the following notations:

$$\hat{R} := \frac{1}{I} \sum_{e \in E^{phys}} \sum_{t=1}^{T} \hat{R}_t(e), \quad r := \operatorname{essinf} \hat{R} > -\infty$$

**Proposition 2.** Consider an electricity market where the revenues $(R_t)_{t=1}^{T}$ fulfill (8) and agents $(x_i, U_i)_{i=1}^{I}$ satisfy (10) with endowment $x \in ]0, \infty[$ and utility function $U$.

(i) The mapping

$$s(\cdot) : ]0, \infty[ \rightarrow \mathbb{R}, \quad \varepsilon \mapsto \frac{E(U'(\varepsilon + \hat{R} - r)\hat{R})}{E(U'(\varepsilon + \hat{R} - r))}$$

defines a continuous function $s(\cdot)$ with $s(\varepsilon) \geq r$ for all $\varepsilon > 0$.

(ii) If the initial endowment $x \in ]0, \infty[$ is sufficiently large in the sense that

$$x > \inf_{\varepsilon > 0} (\varepsilon + s(\varepsilon)) - r,$$

then there exists a solution $s^* \in \mathbb{R}$ to the equation

$$E\left(\frac{U'(x - s^* + \hat{R})}{E(U'(x - s^* + \hat{R}))} \hat{R}\right) = s, \quad s \in ]-\infty, x + r[.$$

(iii) If (46) is fulfilled and $s^*$ solves (47), then define a new probability measure $Q$ by

$$dQ := \frac{U'(x - s^* + \hat{R})}{E(U'(x - s^* + \hat{R}))}dP.$$

The price process $S^*$, given by

$$\hat{S}^*_t(e) = E_Q\left(\sum_{j=t+1}^{T} \hat{R}_j(e) \mid \mathcal{F}_t\right), \quad t = 0, \ldots, T, \quad e \in E$$

satisfies (9) and $(S^*, \vartheta^*, \ldots, \vartheta^*)$ with $\vartheta^*$ from (11) is a symmetric equilibrium.
Proof. (i) For all $\varepsilon > 0$, the random variable $U'(\varepsilon + \hat{R} - r)\hat{R}$ is integrable since
\[ |U'(\varepsilon + \hat{R} - r)\hat{R}| \leq U'(\varepsilon)|\hat{R}|. \]
Moreover, for each sequence $(\varepsilon_j)_{j \in \mathbb{N}} \subset ]0, \infty[$ converging to $\varepsilon > 0$, we have
\[ \lim_{j \to \infty} U'(\varepsilon_j + \hat{R} - r)\hat{R} = U'(\varepsilon + \hat{R} - r)\hat{R} \]
almost sure dominated by the integrable $U'(\inf_{j \in \mathbb{N}} \varepsilon_j + \hat{R} - r)|\hat{R}|$, which shows the continuity of the numerator $\varepsilon \mapsto E(U'(\varepsilon + \hat{R} - r)\hat{R})$ in (45). The same arguments apply to show the continuity of the denominator $\varepsilon \mapsto E(U'(\varepsilon + \hat{R} - r))$ which is positive since $U'$ is strictly positive. This gives the continuity of $s(\cdot)$. To show that $s(\cdot)$ is bounded from below by $r$, we consider $U'(\varepsilon + \hat{R} - r)/E(U'(\varepsilon + \hat{R} - r))$ as density of a new measure and interprete $s(\varepsilon)$ as the expectation of $\hat{R}$ with respect to this new measure. Then $s(\varepsilon) \geq r$ is an immediate consequence of $\hat{R} \geq r$.

(ii) If $x$ satisfies (46), then there exist $\varepsilon_0$ with
\[ x + r - \varepsilon_0 - s(\varepsilon_0) > 0. \]
Choosing $\varepsilon_1 > 0$ with $\varepsilon_1 > \varepsilon_0$ and $\varepsilon_1 > x$, we see that
\[ x + r - \varepsilon_1 - s(\varepsilon_1) < 0, \]
since $r \leq s(\varepsilon_1)$. Now, the intermediate value theorem yields $\varepsilon^* \in ]\varepsilon_0, \varepsilon_1[ \subset ]0, \infty[$ such that
\[ x - s(\varepsilon^*) = \varepsilon^* - r. \]
By definition of the function $s(\cdot)$, we have $E(U'(\varepsilon^* + \hat{R} - r)(\hat{R} - s(\varepsilon^*))) = 0$ and (52) gives
\[ E(U'(x - s(\varepsilon^*) + \hat{R})(\hat{R} - s(\varepsilon^*))) = 0. \]
Set $s^* = s(\varepsilon^*)$ to rewrite the identity (53) equivalently as
\[ E\left( \frac{U'(x - s^* + \hat{R})}{E(U'(x - s^* + \hat{R}))} \right) = s^*. \]

(iii) From (49) which gives (9) where $S$ is replaced by $S^*$ we conclude that
\[ \hat{X}_T^{x, \delta^*, S^*} = x - \frac{1}{T} \sum_{e \in E^{\text{phys}}} \hat{S}_0(e) + \frac{1}{T} \sum_{e \in E^{\text{phys}}} \sum_{t=1}^{T} \hat{R}_t = x - s^* + \hat{R} \]
where the last equality is obtained from (49) and (54) by verifying
\[ \frac{1}{T} \sum_{e \in E} \hat{S}_0(e) = E\left( \frac{U'(x - s^* + \hat{R})}{E(U'(x - s^* + \hat{R}))} \hat{R}(\mathcal{F}_0) \right) = s^*. \]
We have the strict positivity and with (54) the integrability:

\[ \tilde{X}_T^{x,\vartheta^*,S^*} = \varepsilon^* + \tilde{R} - r \geq \varepsilon^* > 0, \quad E_Q(\tilde{X}_T^{x,\vartheta^*,S^*}) = x < \infty. \]

According to the definition (49),

\[ (L_{t+1} := \tilde{S}_{t+1}^* - \tilde{S}_t^* + \tilde{R}_{t+1})_{t=0}^{T-1} \]

are \( Q \)-martingale increments, which gives the \( Q \)-martingale property of \( (\tilde{X}_t^{x,S^*,\vartheta^*})_{t=0}^{T} \) due to the recursion

\[ \tilde{X}_{t+1}^{x,\vartheta^*,S} = \hat{X}_t^{x,\vartheta^*,S} + \vartheta_t(\tilde{S}_{t+1} - \tilde{S}_t + \tilde{R}_{t+1}), \quad \tilde{X}_0^{x,\vartheta^*,S} = x \quad \text{for all } \vartheta \in A(x,S,U). \]

Combining this with (56) we deduce

\[ \tilde{X}_t^{x,\vartheta^*,S} = E_Q(\tilde{X}_T^{x,\vartheta^*,S^*} | \mathcal{F}_t) > 0 \quad \text{for all } t = 0, \ldots, T. \]

and so the admissibility \( \vartheta^* \in A(x,S^*,U) \) of the constant strategy \( \vartheta^* \) follows with

\[ E(U(\tilde{X}_T^{x,\vartheta^*,S^*})^-) < \infty \]

since \( U(\tilde{X}_T^{x,\vartheta^*,S^*}) \geq U(\varepsilon^*) \) is bounded from below.

Let us show that for all \( \vartheta \in A(x,S^*,U) \) the wealth \( (\tilde{X}_t^{x,\vartheta,S^*})_{t=0}^{T} \) is a \( Q \)-martingale. Given \( \vartheta \in A(x,S^*,U) \), define a bounded \( \vartheta^M \) by

\[ \vartheta^M_t(\omega) := \begin{cases} \vartheta_t(\omega) & \text{if } |\vartheta_t(\omega)| \leq M \\ M \cdot \text{sign} \vartheta_t(\omega) & \text{if } |\vartheta_t(\omega)| > M \end{cases} \quad \text{for all } t = 0, \ldots, T-1 \]

Then for all \( t = 0, \ldots, T-1 \) we have the monotone convergence for both the positive and the negative part as

\[ \lim_{M \to \infty} (\vartheta^M_t L_{t+1})^+ = (\vartheta_t L_{t+1})^+, \quad \lim_{M \to \infty} (\vartheta^M_t L_{t+1})^- = (\vartheta_t L_{t+1})^- \]

Moreover, since \( \vartheta^M_t \) is bounded and \( L_{t+1} \) is a \( Q \)-martingale increment, we obtain

\[ 0 = E_Q(\vartheta^M_t L_{t+1}) = E_Q((\vartheta^M_t L_{t+1})^+) - E_Q((\vartheta^M_t L_{t+1})^-) \quad \text{for all } M > 0. \]

Further, \( \tilde{X}_t^{x,\vartheta,S^*} + \vartheta_t L_{t+1} = \hat{X}_{t+1}^{x,\vartheta,S^*} \geq 0 \) implies that \( (\vartheta_t L_{t+1})^- \leq \tilde{X}_t^{x,\vartheta,S^*} \), hence

\[ (\vartheta^M_t L_{t+1})^- \leq (\vartheta_t L_{t+1})^- \leq \tilde{X}_t^{x,\vartheta,S^*} \quad \text{for all } M > 0. \]

If \( X_t^{x,\vartheta,S^*} \in L^1(\Omega,\mathcal{F},Q) \), then it follows from (59) and (60) that

\[ E_Q((\vartheta^M_t L_{t+1})^+) = E_Q((\vartheta^M_t L_{t+1})^-) \leq E_Q(\tilde{X}_t^{x,\vartheta,S^*}) \quad \text{for all } M > 0. \]
and the monotone convergence in (58) ensures that \((\vartheta_t L_{t+1})^+, (\vartheta_t L_{t+1})^- \in L^1(\Omega, \mathcal{F}, Q)\) and that the random variables \((\vartheta_t^M L_{t+1})^+, (\vartheta_t^M L_{t+1})^-)_{M>0}\) tend for \(M \to \infty\) to \((\vartheta_t L_{t+1})^+\) and \((\vartheta_t L_{t+1})^-\) respectively in \(L^1(\Omega, \mathcal{F}, Q)\)-sense which shows that

\[
0 = \lim_{M \to \infty} E_Q(\vartheta_t^M L_{t+1}|\mathcal{F}_t) = E_Q(\vartheta_t L_{t+1}|\mathcal{F}_t).
\]

Thus, for \(\vartheta \in \mathcal{A}(x, S^*, U)\) we conclude by induction for all \(t = 0, \ldots, T - 1\) that \(\hat{X}_{t+1}^{x, \vartheta, S^*}\) is integrable with respect to \(Q\) and satisfies \(E_Q(\hat{X}_{t+1}^{x, \vartheta, S^*}|\mathcal{F}_t) = \hat{X}_t^{x, \vartheta, S^*}\) giving the \(Q\)-martingale property for \((\hat{X}_t^{x, \vartheta, S^*})_{t=0}^T\).

Note that for each \(U\) from (2), the inverse function \(J := U' \cdot 1\) maps \(]0, \sup_{z>0} U'(z)[\) onto \(]0, \infty[\) and satisfies the inequality

\[
U(J(b)) \geq U(a) + b(J(b) - a) \quad \text{for all } a \in ]0, \infty[, \quad b \in ]0, \sup_{z>0} U'(z)[.
\]

To see that the asset prices \(S^*\) belong to a symmetric equilibrium, we apply (61) with \(a := \hat{X}_{T}^{x, \vartheta, S^*}\) for arbitrary \(\vartheta \in \mathcal{A}(x, S^*, U)\) and \(b := U'(x - s^* + \hat{R})\). By (55), we are led to

\[
U(\hat{X}_{T}^{x, \vartheta, S^*}) = U(J(U'(x - s^* + \hat{R}))) \geq U(\hat{X}_{T}^{x, \vartheta, S^*}) + U'(x - s^* + \hat{R})(\hat{X}_{T}^{x, \vartheta, S^*} - \hat{X}_{T}^{x, \vartheta, S^*}).
\]

We now calculate the expectation on both sides of the previous inequality taking (48) into account:

\[
E(U(\hat{X}_{T}^{x, \vartheta, S^*})) \geq E(U(\hat{X}_{T}^{x, \vartheta, S^*}) + E_Q(\hat{X}_{T}^{x, \vartheta, S^*} - \hat{X}_{T}^{x, \vartheta, S^*})E(U'(x - s^* + \hat{R})).
\]

The \(Q\)-martingale property yields \(E_Q(\hat{X}_{T}^{x, \vartheta, S^*} - \hat{X}_{T}^{x, \vartheta, S^*}) = 0\), which implies

\[
E(U(\hat{X}_{T}^{x, \vartheta, S^*})) \geq E(U(\hat{X}_{T}^{x, \vartheta, S^*})) \quad \text{for all } \vartheta \in \mathcal{A}(x, S^*, U)
\]

showing that that for asset prices \(S^*\) the best strategy is in fact to hold the constant part \(1/I\) of each physical asset and no financial asset positions.

\[
\square
\]

References


