Ruin probabilities in infinite time

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Introduction

The standard mathematical model for insurance risk - the classical risk process $\{R_t\}_{t\geq 0}$ is given by

$$R_t = u + ct - \sum_{i=1}^{N_t} X_i.$$

The initial capital is denoted by u, the Poisson process N_t with intensity λ describes the number of claims in $(0, t]$ interval and claim severities are random, given by i.i.d. non-negative sequence $\{X_k\}_{k=1}^{\infty}$ with mean value μ and variance σ^2, independent of N_t. To cover its liability, the insurance company receives premium at a constant rate c per unit time, where $c = (1+\theta)\lambda\mu$ and $\theta > 0$ is called the relative safety loading.

The ruin is said to occur if the insurer's surplus reaches a specified lower bound, e.g. minus the initial capital. One measure of risk is the probability of an event such as this and thus serves as a useful tool in long range planning for the use of insurer's funds.

We define a claim surplus process $\{S_t\}_{t\geq 0}$ as

$$S_t = u - R_t = \sum_{i=1}^{N_t} X_i - ct.$$

The time to ruin is defined as

$$\tau(u) = \inf\{t \geq 0 : R_t < 0\} = \inf\{t \geq 0 : S_t > u\}.$$

Let $L = \sup_{0\leq t<\infty}\{S_t\}$. The ruin probability in infinite time, i.e. the probability that the capital of an insurance company ever drops below zero can be then written as

$$\psi(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{P}(L > u). \tag{1}$$

Considering claim amounts we distinguish between light- and heavy-tailed distributions.

Distribution $F_X(x)$ is said to be light-tailed, if there exist constants $a > 0$, $b > 0$ such that $\bar{F}_X(x) = 1 - F_X(x) \leq ae^{-bx}$ or, equivalently, if there exist $z > 0$, such that $M_X(z) < \infty$, where $M_X(z)$ is the moment generating function.

Distribution $F_X(x)$ is said to be heavy-tailed, if for all $a > 0$, $b > 0$ $\bar{F}_X(x) > ae^{-bx}$, or, equivalently, if $\forall z > 0$ $M_X(z) = \infty$.

We will study claim size distributions as in Table 1.

Table 1: Claim size distributions.

Light-tailed distributions			
Name	Parameters	Pdf	
Exponential	$\beta > 0$	$f_X(x) = \beta e^{-\beta x},$	$x \geq 0$
Gamma	$\alpha > 0,\ \beta > 0$	$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$	$x \geq 0$
Weibull	$c > 0,\ \tau \geq 1$	$f_X(x) = c\tau x^{\tau-1} e^{-cx^\tau},$	$x \geq 0$
Mixed exp's	$\beta_i > 0,\ \sum_{i=1}^n a_i = 1$	$f_X(x) = \sum_{i=1}^n (a_i \beta_i e^{-\beta_i x}),$	$x \geq 0$
Heavy-tailed distributions			
Name	Parameters	Pdf	
Weibull	$c > 0,\ \ \tau < 1$	$f_X(x) = c\tau x^{\tau-1} e^{-cx^\tau},$	$x \geq 0$
Lognormal	$\mu \in \mathbb{R}\ \ \sigma > 0$	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}},$	$x \geq 0$
Pareto	$\alpha > 0,\ \ \nu > 0$	$f_X(x) = \frac{\alpha}{\nu+x}\left(\frac{\nu}{\nu+x}\right)^\alpha,$	$x \geq 0$
Burr	$\alpha > 0,\ \nu > 0,\ \tau > 0$	$f_X(x) = \frac{\alpha\tau\nu^\alpha x^{\tau-1}}{(\nu+x^\tau)^{\alpha+1}},$	$x \geq 0$

The adjustment coefficient

The adjustment coefficient (called also the Lundberg exponent) plays a key role in calculating the ruin probability in the case of light-tailed claims. Let $\gamma = \sup_z M_X(z) < \infty$ and let R be a positive solution of the equation:

$$1 + (1+\theta)\mu R = M_X(R), \qquad R < \gamma. \tag{2}$$

If there exists a non-zero solution to the above equation, we call this R an adjustment coefficient. Clearly, $R = 0$ satisfies the equation (2), but there may exist a positive solution as well (this requires that X has a moment generating function). To see the plausibility of this result, note that $M_X(0) = 1$, $M_X^{'}(z) < 0$, $M_X^{"}(z) > 0$ and $M_X^{'}(0) = -\mu$. Hence, the curves $y = M_X(z)$ and $y = 1 + (1+\theta)\mu z$ may intersect, as shown in Figure 1.

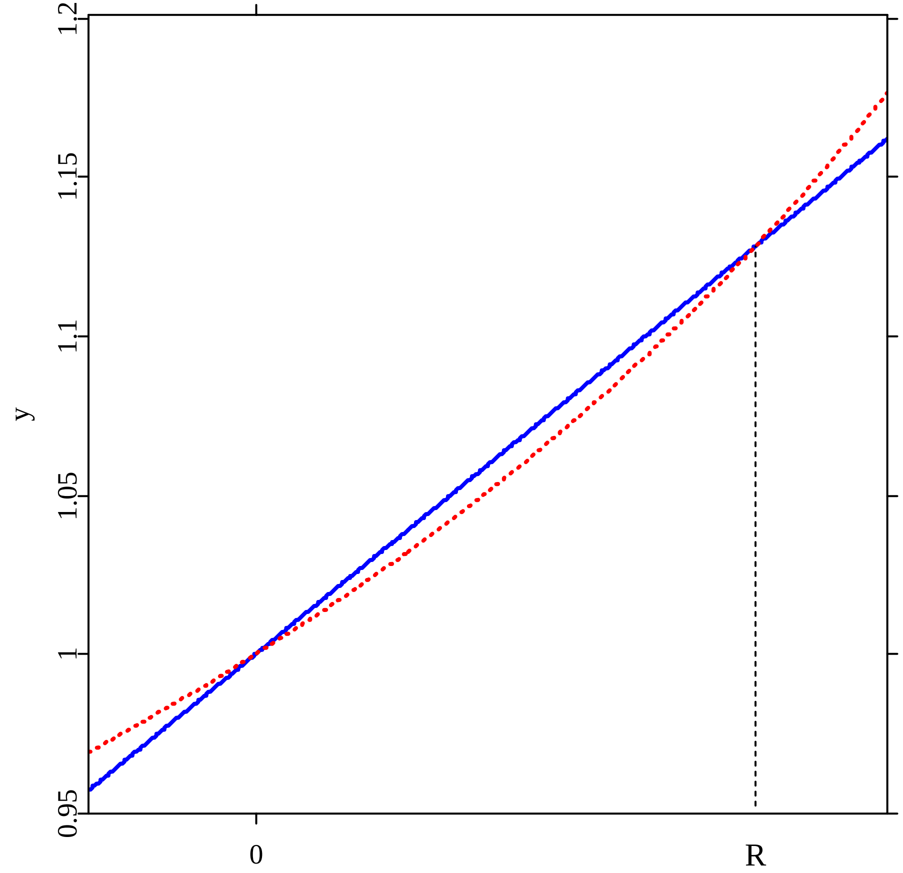

Figure 1: Illustration of the existence of the adjustment coefficient.

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Analytical solution to eq. (2) exists only for few claim distributions.

However, it is quite easy to obtain a numerical solution. The coefficient R satisfies the inequality:

$$R < \frac{2\theta\mu}{\mu^{(2)}}, \tag{3}$$

where $\mu^{(2)} = \mathbb{E}X_i^2$. Let $D(z) = 1 + (1+\theta)\mu z - M_X(z)$. Thus, the adjustment coefficient $R > 0$ satisfies the equation $D(R) = 0$. In order to get the solution one may use the Newton-Raphson formula

$$R_{j+1} = R_j - \frac{D(R_j)}{D'(R_j)}, \tag{4}$$

with $R_0 = 2\theta\mu/\mu^{(2)}$. Moreover, if it is possible to calculate the third raw moment $\mu^{(3)}$, we can obtain a sharper bound than (3):

$$R < \frac{12\mu\theta}{3\mu^{(2)} + \sqrt{9(\mu^{(2)})^2 + 24\mu\mu^{(3)}\theta}},$$

and use it as the initial condition in (4).

Exact ruin probabilities in infinite time

In order to present a ruin probability formula we use the representation of the ruin probability in terms of L and the decomposition of the maximum L as a sum of ladder heights $\{L_k\}_{k=1}^{\infty}$ - the sequence of independent and identically distributed variables with the density

$$f_{L_1}(x) = \bar{F}_X(x)/\mu. \tag{5}$$

One may also show that the number of ladder heights K is given by the geometric distribution with the parameter $q = \theta/(1+\theta)$. Thus, the random variable L may be expressed as

$$L = \sum_{i=1}^{K} L_i \tag{6}$$

and it has a compound geometric distribution. The above fact leads to the Pollaczeck–Khinchine formula for the ruin probability:

$$\psi(u) = 1 - \mathbb{P}(L \leq u) = 1 - \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta} \right)^n F_{L_1}^{*n}(u), \tag{7}$$

where $F_{L_1}^{*n}(u)$ denotes the nth convolution of the distribution function F_{L_1}.

We shall briefly present a collection of basic exact results on the ruin probability in infinite time. The ruin probability $\psi(u)$ is always considered as a function of the initial surplus u.

No initial capital

When $u = 0$ it is easy to obtain the exact formula

$$\psi(u) = \frac{1}{1+\theta},$$

Notice that the formula depends only on θ, regardless of the claim frequency rate λ and claim size distribution. The ruin probability is clearly inversely proportional to the safely loading.

Exponential claim amounts

One of the first results on ruin probability is the explicit formula for exponential claims with the parameter β, namely

$$\psi(u) = \frac{1}{1+\theta} \exp\left(-\frac{\theta\beta u}{1+\theta}\right). \tag{8}$$

We can observe the ruin probability decreases as the capital grows:

Table 2: The ruin probability for exponential claims with $\beta = 1/158893135.4$ and $\theta = 0.3$.

u	0	10^7	10^8	10^9	10^{10}
$\psi(u)$	0.76923077	0.75813955	0.66524508	0.18001426	0.00000038

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Gamma claim amounts

It was shown by Grandell and Segerdahl that for the gamma claim amount distribution with mean 1 and $\alpha \leq 1$ the exact value of ruin probability can be computed via the formula

$$\psi(u) = \frac{\theta(1-R/\alpha)\exp(-Ru)}{1+(1+\theta)R-(1+\theta)(1-R/\alpha)} + \frac{\alpha\theta\sin(\alpha\pi)}{\pi} \cdot I, \qquad (9)$$

where

$$I = \int_0^\infty \frac{x^\alpha \exp\left\{-(x+1)\alpha u\right\}}{\left[x^\alpha\left\{1+\alpha(1+\theta)(x+1)\right\}-\cos(\alpha\pi)\right]^2+\sin^2(\alpha\pi)}\,dx. \qquad (10)$$

The assumption on the mean is not restrictive since for claims X with arbitrary mean μ we have that $\psi_X(u) = \psi_{X/\mu}(u/\mu)$. As the gamma distribution is closed under scale changes we obtain that $\psi_{G(\alpha,\beta)}(u) = \psi_{G(\alpha,\alpha)}(\beta u/\alpha)$. Table 3 shows the ruin probability values for gamma claims with with $\alpha = 0.49345$, $\beta = 561633357.3$ and safety loading $\theta = 30\%$ with respect to the initial capital u.

Table 3: The ruin probability for gamma claims with $\alpha = 0.49345$, $\beta = 561633357.3$ and $\theta = 0.3$.

u	0	10^7	10^8	10^9	10^{10}
$\psi(u)$	0.76923070	0.75824563	0.66662131	0.18483976	0.00000050

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Mixture of two exponentials claim amounts

For the claim size distribution being a mixture of two exponentials with the parameters α, β and weights q, $1-q$, using the Laplace transform inversion, one may obtain an explicit formula, see Panjer and Willmot (1992):

$$\psi(u) = \frac{1}{(1+\theta)(r_2 - r_1)} \left\{ (\rho - r_1) \exp(-r_1 u) + (r2 - \rho) \exp(-r_2 u) \right\},$$

where

$$r_1 = \frac{\rho + \theta(\alpha+\beta) - \left[\{\rho + \theta(\alpha+\beta)\}^2 - 4\alpha\beta\theta(1+\theta) \right]^{1/2}}{2(1+\theta)},$$

$$r_2 = \frac{\rho + \theta(\alpha+\beta) + \left[\{\rho + \theta(\alpha+\beta)\}^2 - 4\alpha\beta\theta(1+\theta) \right]^{1/2}}{2(1+\theta)}.$$

and

$$p = \frac{q\alpha^{-1}}{q\alpha^{-1} + (1-q)\beta^{-1}}, \qquad \rho = \alpha(1-p) + \beta p.$$

Table 4 shows the ruin probability values for mixture of 2 exponentials claims with $\alpha = 1/190744933.98$, $\beta = 1/84535691.61$, $q = 0.78$ and $\theta = 30\%$ with respect to the initial capital u.

Table 4: The ruin probability for mixture of 2 exponentials claims with $\alpha = 1/190744933.98$, $\beta = 1/84535691.61$, $q = 0.78$ and $\theta = 0.3$

u	0	10^7	10^8	10^9	10^{10}
$\psi(u)$	0.76923077	0.75872977	0.67258748	0.21205921	0.00000214

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Approximations of the ruin probability in infinite time

When the claim size distribution is exponential (or closely related), simple analytic results for the ruin probability in infinite time may be possible. For more general claim amount distributions, e.g. heavy-tailed, the Laplace transform technique does not work and one may need some estimates. Here we will present 12 different well-known and not so well-known approximations. Next, numerical comparison of the different approximations is done.

Cramér–Lundberg approximation

One may obtain approximate formulae for $\psi(u)$ for large u. Cramér–Lundberg's asymptotic ruin formula is given by

$$\psi_{CL}(u) = Ce^{-Ru}, \tag{11}$$

where $C = \theta\mu / \left\{ M_X'(R) - \mu(1+\theta) \right\}$. The classical Cramér–Lundberg approximation yields quite accurate results, however we must remember that it requires the adjustment coefficient to exist, therefore merely the light-tailed distributions can be taken into consideration. For exponentially distributed claims the formula (11) yields an exact result.

In Table 5 the Cramér–Lundberg approximation for mixture of 2 exponentials claims with $\alpha = 1/190744933.98$, $\beta = 1/84535691.61$, $q = 0.78$ and the relative safety loading $\theta = 30\%$ with respect to the initial capital u is given.

Table 5: The Cramér–Lundberg approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_{CL}(u)$	0.76139296	0.75172213	0.67002910	0.21205910	0.00000214

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Exponential approximation

This approximation was proposed and derived by De Vylder (1996).
It requires the first three moments to be finite.

$$\psi_E(u) = \exp\left\{-1 - \frac{2\mu\theta u - \mu^{(2)}}{\sqrt{(\mu^{(2)})^2 + (4/3)\theta\mu\mu^{(3)}}}\right\}. \qquad (12)$$

Table 6 shows the results of the exponential approximation for mixture of 2 exponentials claims with respect to u.

Table 6: The exponential approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_E(u)$	0.80689909	0.79634099	0.70732285	0.21617418	0.00000154

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Lundberg approximation

The following formula, called the Lundberg approximation, comes from Grandell (2000). It requires the first three moments to be finite.

$$\psi_L(u) = \left\{ 1 + \left(\theta u - \frac{\mu^{(2)}}{2\mu} \right) \frac{4\theta\mu^2\mu^{(3)}}{3(\mu^{(2)})^3} \right\} \exp\left(\frac{-2\mu\theta u}{\mu^{(2)}} \right). \quad (13)$$

In Table 7 the Lundberg approximation for mixture of 2 exponentials claims with respect to u is given.

Table 7: The Lundberg approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_L(u)$	0.68952377	0.68317733	0.62709804	0.22624195	0.00000031

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Beekman–Bowers approximation

The Beekman–Bowers approximation uses the following representation of the ruin probability.

$$\psi(u) = \mathbb{P}(L > u) = \mathbb{P}(L > 0)\mathbb{P}(L > u|L > 0). \tag{14}$$

The idea of the approximation is to replace the conditional probability $1 - \mathbb{P}(L > u|L > 0)$ with a gamma distribution function $G(u)$ by fitting first two moments. This leads to:

$$\psi_{BB}(u) = \frac{1}{1+\theta}(1 - G(u)), \tag{15}$$

where the parameters α, β of G are given by

$$\alpha = \left\{1 + \left(\frac{4\mu\mu^{(3)}}{3(\mu^{(2)})^2} - 1\right)\theta\right\}/(1+\theta), \beta = 2\mu\theta/\left\{\mu^{(2)} + \left(\frac{4\mu\mu^{(3)}}{3\mu^{(2)}} - \mu^{(2)}\right)\theta\right\}.$$

The Beekman–Bowers approximation gives rather accurate results. In the exponential case it becomes the exact formula. It can be used only for distributions with finite first three moments.

Table 8: The Beekman–Bowers approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_{BB}(u)$	0.76923077	0.75876182	0.67379297	0.21161637	0.00000224

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Table 8 shows the results of the Beekman–Bowers approximation for mixture of 2 exponentials claims with $\alpha = 1/190744933.98$, $\beta = 1/84535691.61$, $q = 0.78$ and $\theta = 30\%$ with respect to u. The results justify the thesis the approximation yields accurate results.

Renyi approximation

The Renyi approximation may be derived from (15) when we replace the gamma distribution function G with an exponential one, matching only the first moment. Hence, it can be regarded as a simplified version of the Beekman–Bowers approximation.

$$\psi_R(u) = \frac{1}{1+\theta} \exp\left\{ -\frac{2\mu\theta u}{\mu^{(2)}(1+\theta)} \right\}. \tag{16}$$

Table 9: The Renyi approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_R(u)$	0.76923077	0.75937197	0.67613874	0.21176217	0.00000192

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De Vylder approximation

The idea of this approximation is to replace the claim surplus process S_t with the claim surplus process $\bar{S}_t$ with exponentially distributed claims such that the three moments of the process coincide, namely $ES_t^k = E\bar{S}_t^k$ for $k = 1, 2, 3$. The process $\bar{S}_t$ is determined by the three parameters $(\bar{\lambda}, \bar{\theta}, \bar{\beta})$. Thus the parameters must satisfy

$$\bar{\lambda} = \frac{9\lambda {\mu^{(2)}}^3}{2{\mu^{(3)}}^2}, \qquad \bar{\theta} = \frac{2\mu\mu^{(3)}}{3{\mu^{(2)}}^2}\theta, \qquad \text{and} \qquad \bar{\beta} = \frac{3\mu^{(2)}}{\mu^{(3)}}.$$

Then De Vylder's approximation is given by

$$\psi_{DV}(u) = \frac{1}{1+\bar{\theta}} \exp\left(-\frac{\bar{\theta}\bar{\beta}u}{1+\bar{\theta}} \right). \tag{17}$$

Table 10 shows the results of the De Vylder approximation for mixture of 2 exponentials claims with $\alpha = 1/190744933.98$, $\beta = 1/84535691.61$, $q = 0.78$ and the relative safety loading $\theta = 30\%$ with respect to the initial capital u. The approximation gives surprisingly good results.

Table 10: The De Vylder approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_{DV}(u)$	0.76308137	0.75337907	0.67142556	0.21224673	0.00000211

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4-moment gamma De Vylder approximation

The 4-moment gamma De Vylder approximation, proposed by Burnecki, Miśta and Weron (2003) is based on De Vylder's idea to replace the claim surplus process S_t with another one $\bar{S}_t$ for which the expression for $\psi(u)$ is explicit. This time we calculate the parameters of the new process with gamma distributed claims and apply the exact formula (9) for the ruin probability. To this end we match the four moments of S_t and $\bar{S}_t$. First, let us note that the claim surplus process $\bar{S}_t$ with gamma claims is determined by the four parameters $(\bar{\lambda}, \bar{\theta}, \bar{\mu}, \bar{\mu}^{(2)})$.

Since

$$
\begin{aligned}
\mathbb{E}S_t &= -\theta\lambda\mu t, \\
\mathbb{E}S_t^2 &= \lambda\mu^{(2)}t + (\theta\lambda\mu t)^2, \\
\mathbb{E}S_t^3 &= \lambda\mu^{(3)}t - 3(\lambda\mu^{(2)}t)(\theta\lambda\mu t) - (\theta\lambda\mu t)^2, \\
\mathbb{E}S_t^4 &= \lambda\mu^{(4)}t - 4(\lambda\mu^{(3)}t)(\theta\lambda\mu t) + 3(\lambda\mu^{(2)}t)^2 + 6(\lambda\mu^{(2)}t)(\theta\lambda\mu t)^2 + (\theta\lambda\mu t)^4
\end{aligned}
$$

and for the gamma distribution $\bar{\mu}^{(3)} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}}(2\bar{\mu}^{(2)} - \bar{\mu}^2)$, $\bar{\mu}^{(4)} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)} - \bar{\mu}^2)(3\bar{\mu}^{(2)} - 2\bar{\mu}^2)$, the parameters $(\bar{\lambda}, \bar{\theta}, \bar{\mu}, \bar{\mu}^{(2)})$ must satisfy the equations

$$
\theta\lambda\mu = \bar{\theta}\bar{\lambda}\bar{\mu}, \qquad \lambda\mu^{(2)} = \bar{\lambda}\bar{\mu}^{(2)},
$$

$$
\lambda\mu^{(3)} = \bar{\lambda}(2\bar{\mu}^{(2)} - \bar{\mu}^2)\bar{\mu}^{(2)}/\bar{\mu}^2, \qquad \lambda\mu^{(4)} = \bar{\lambda}(2\bar{\mu}^{(2)} - \bar{\mu}^2)(3\bar{\mu}^{(2)} - 2\bar{\mu}^2)\bar{\mu}^{(2)}/\bar{\mu}^2.
$$

Hence

$$\bar{\lambda} = \frac{\lambda(\mu^{(3)})^2(\mu^{(2)})^3}{\left\{\mu^{(2)}\mu^{(4)} - 2(\mu^{(3)})^2\right\}\left\{2\mu^{(2)}\mu^{(4)} - 3(\mu^{(3)})^2\right\}}, \bar{\theta} = \frac{\theta\mu\left\{2(\mu^{(3)})^2 - \mu^{(2)}\mu^{(4)}\right\}}{(\mu^{(2)})^2\mu^{(3)}},$$

$$\bar{\mu} = \frac{3(\mu^{(3)})^2 - 2\mu^{(2)}\mu^{(4)}}{\mu^{(2)}\mu^{(3)}},\ \bar{\mu}^{(2)} = \frac{\left\{\mu^{(2)}\mu^{(4)} - 2(\mu^{(3)})^2\right\}\left\{2\mu^{(2)}\mu^{(4)} - 3(\mu^{(3)})^2\right\}}{(\mu^{(2)}\mu^{(3)})^2}.$$

We also need to assume that $\mu^{(2)}\mu^{(4)} < \frac{3}{2}(\mu^3)^2$ to ensure that $\bar{\mu}^{(2)} > \bar{\mu}^2$ and$\bar{\mu}, \bar{\mu}^{(2)} > 0$. In case this assumption can not be fulfilled, we simply set $\bar{\mu} = \mu$ and do not calculate the fourth moment. This case leads to

$$\bar{\lambda} = \frac{2\lambda(\mu^{(2)})^2}{\mu(\mu^{(3)} + \mu^{(2)}\mu)},\ \ \bar{\theta} = \frac{\theta\mu(\mu^{(3)} + \mu^{(2)}\mu)}{2(\mu^{(2)})^2},\ \ \bar{\mu} = \mu,\ \ \bar{\mu}^{(2)} = \frac{\mu(\mu^{(3)} + \mu^{(2)}\mu)}{2\mu^{(2)}}. \tag{18}$$

All in all, we get the approximation

$$\psi_{4MGDV}(u) = \frac{\bar{\theta}(1-\frac{R}{\bar{\alpha}})e^{-\frac{\bar{\beta}R}{\bar{\alpha}}u}}{1+(1+\bar{\theta})R-(1+\bar{\theta})(1-\frac{R}{\bar{\alpha}})} + \frac{\bar{\alpha}\bar{\theta}sin(\bar{\alpha}\pi)}{\pi} \cdot I, \quad (19)$$

where

$$I = \int_0^\infty \frac{x^{\bar{\alpha}}e^{-(x+1)\bar{\beta}u}\,dx}{\left[x^{\bar{\alpha}}\left\{1+\bar{\alpha}(1+\bar{\theta})(x+1)\right\}-cos(\bar{\alpha}\pi)\right]^2+sin^2(\bar{\alpha}\pi)},$$

and $(\bar{\alpha},\bar{\beta})$ are given by $\bar{\alpha} = \bar{\mu}^2/\left(\bar{\mu}^{(2)}-\bar{\mu}^2\right)$, $\bar{\beta} = \bar{\mu}/\left(\bar{\mu}^{(2)}-\bar{\mu}^2\right)$.

In the exponential and gamma case this method gives the exact result. For other claim distributions the first four (or three) moments have to exist. Burnecki, Miśta and Weron showed numerically that the method gives a slight correction to the De Vylder approximation, which is often regarded as the best among 'simple' approximations.

In Table 11 the 4-moment gamma De Vylder approximation for mixture of 2 exponentials claims and $\theta = 30\%$ with respect to u is given. The most striking impression of Table 11 is certainly the extremely good accuracy of the simple 4-moment gamma De Vylder approximation. The relative error with respect to the exact values presented in Table 4 is always below 0.3%.

Table 11: The 4-moment gamma De Vylder approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi(u)$	0.76746161	0.75702255	0.67221498	0.21209805	0.00000213

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Heavy traffic approximation

The term 'heavy traffic' comes from queuing theory. In risk theory it means that on the average the premiums exceed only slightly the expected claims. It implies that safety loading is positive and small.

$$\psi_{HT}(u) = \exp\left(-\frac{2\theta\mu u}{\mu^{(2)}}\right). \tag{20}$$

Table 12: The heavy traffic approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_{HT}(u)$	1.00000000	0.98337076	0.84561548	0.18695163	0.00000005

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Light traffic approximation

The queuing theory term 'light traffic' has an obvious interpretation also in risk theory, namely, on the average, the premiums are much larger than the expected claims. It implies that θ is large.

$$\psi_{LT}(u) = \lambda \int_u^\infty \bar{F}_X(x)dx. \tag{21}$$

The method gives accurate results merely for huge values of θ.

Table 13: The light traffic approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_{LT}(u)$	0.76923077	0.72475312	0.43087903	0.00361367	0.00000000

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Heavy-light traffic approximation

The crude idea of this approximation is to combine heavy and light approximation:

$$\psi_{HLT}(u) = \frac{\theta}{1+\theta}\psi_{LT}\left(\frac{\theta u}{1+\theta}\right) + \frac{1}{(1+\theta)^2}\psi_{HT}(u), \tag{22}$$

The particular features of this one is that it is exact for the exponential case and asymptotically correct both in light and heavy traffic.

Table 14: The heavy-light traffic approximation.

u	0	10^7	10^8	10^9	10^{10}
$\psi_{HLT}(u)$	0.76923077	0.75696126	0.65517763	0.15895288	0.00000091

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Heavy-tailed claims approximation

First, let us introduce the class of subexponential distributions $\mathcal{S}$, namely

$$\mathcal{S} = \left\{ F : \lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\bar{F}(x)} = 2 \right\}. \tag{23}$$

Here $F^{*2}(x)$ is the convulsion square. In terms of random variables (23) means $P(X_1 + X_2 > x) \sim 2P(X_1 > x)$, $x \to \infty$, where X_1, X_2 are independent random variables with distribution F.

The class contains lognormal and Weibull (for $\tau < 1$) distributions. Moreover, all distributions with a regularly varying tail (e.g. loggamma, Pareto and Burr distributions) are subexponential.

For subexponential distributions we can formulate the following approximation of the ruin probability. If $F \in \mathcal{S}$, then the asymptotic formula for large u is given by

$$\psi_{HT}(u) = \frac{1}{\theta\mu}\left(\mu - \int_0^u \bar{F}(x)dx\right), \tag{24}$$

The approximation is considered to be inaccurate. The problem is a very slow rate of convergence as $u \to \infty$. Even though the approximation is asymptotically correct in the tail, one may have to go out to values of $\psi(u)$ which are unrealistically small before the fit is reasonable. However, we will show that it is not always the case.

Computer approximation via the Pollaczeck–Khinchine formula

Pollaczeck–Khinchine formula (7) can be used to derive explicit solutions for only a few claim amount distributions. For the rest, in order to calculate the ruin probability, the Monte Carlo method can be applied to (1) and (6). The main problem is to simulate random variables from the density $f_{L_1}(x)$. Only four of the considered distributions lead to a close form of the density, namely

- for exponential claims, $f_{L_1}(x)$ is the density of the same exponential distribution,
- for mixture of exponentials claims, $f_{L_1}(x)$ is the density of the mixture of exponential distribution with the weights $\left(\frac{a_1}{\beta_1}/\left\{\sum_{i=1}^{n}\left(\frac{a_i}{\beta_i}\right)\right\}, \cdots, \frac{a_n}{\beta_n}/\left\{\sum_{i=1}^{n}\left(\frac{a_i}{\beta_i}\right)\right\}\right)$,

- for Pareto claims, $f_{L_1}(x)$ is the density of the Pareto distribution with the parameters $\alpha - 1$ and ν,
- for Burr claims, $f_{L_1}(x)$ is the density of the transformed beta distribution.

For other studied here distributions in order to generate random variables L_k we use formula (5) and controlled, numerical integration. We note that the methodology based on the Pollaczeck–Khinchine formula works for all considered claim distributions and can be chosen as the reference method for calculating the ruin probabilty.

Summary of the approximations

Table 15: Indication when the approximations can be applied

	Exp.	Gamma	Weib.	Mix.Exp.	Lognor.	Loggam.	Pareto	Burr
Cramér	+	+	–	+	–	–	–	–
Exponential	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
Lundberg	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
B-B	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
Renyi	+	+	+	+	+	$\beta > 2$	$\alpha > 2$	$\alpha\tau > 2$
De Vylder	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
4M GDeV	+	+	+	+	+	$\beta > 3$	$\alpha > 3$	$\alpha\tau > 3$
H-Traffic	+	+	+	+	+	$\beta > 2$	$\alpha > 2$	$\alpha\tau > 2$
L-Traffic	+	+	+	+	+	+	+	+
H-L-Traffic	+	+	+	+	+	$\beta > 2$	$\alpha > 2$	$\alpha\tau > 2$
H-tailed	–	–	$0<\tau<1$	–	+	+	+	+
Pol-Khin	+	+	+	+	+	+	+	+

Numerical comparison of the infinite time approximations

We will now illustrate all 12 approximations presented before. To this end we consider three claim amount distributions which were best fitted to the catastrophe data, namely the mixture of two exponentials, lognormal and Pareto distributions. The parameters of the distributions are: $\alpha = 1/190744933.98$, $\beta = 1/84535691.61$, $q = 0.78$ (mixture), $\mu = 18.44$, $\sigma = 1.13$ (lognormal), and $\alpha = 2.39$, $\lambda = 3.03 \cdot 10^8$ (Pareto). The ruin probability will be depicted as a function of the initial capital u ranging from USD 0 to 10 billions. The relative safety loading is set to 30%.

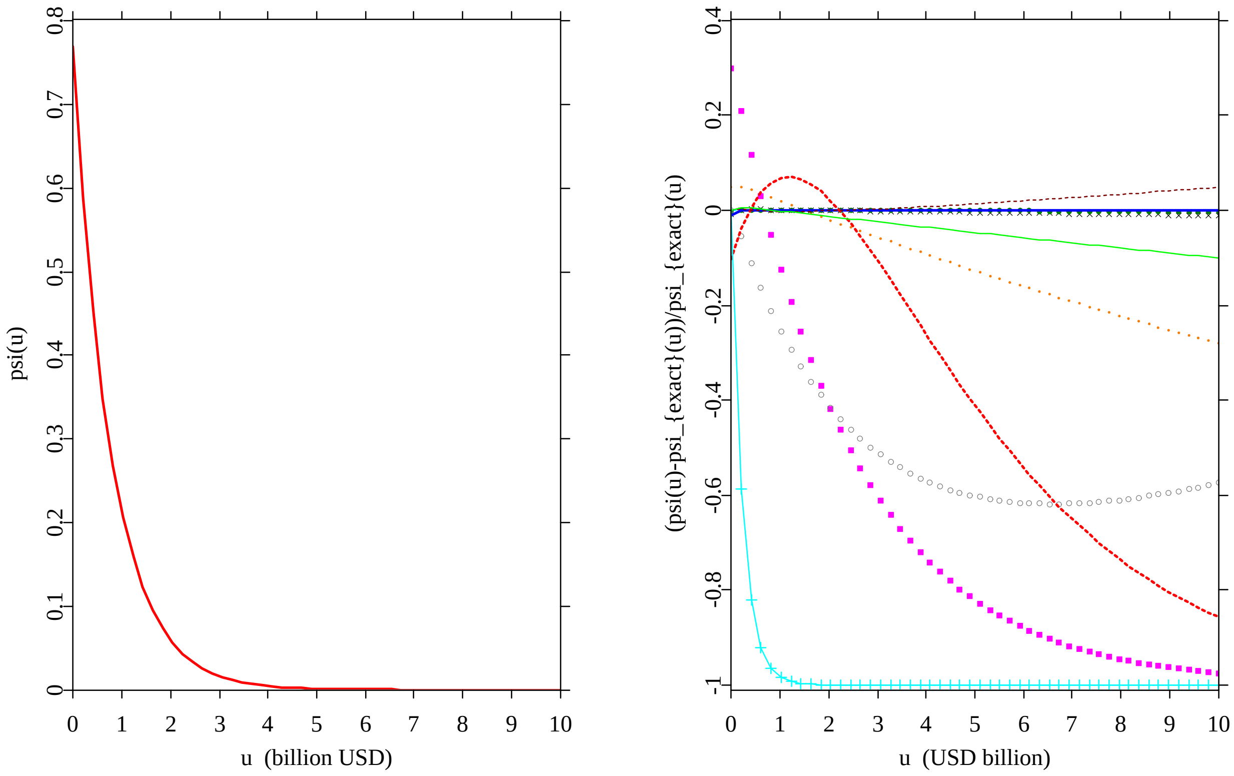

Figure 2: The exact value of the ruin probability (left panel), the relative error of the approximations (right panel). The Cramér-Lundberg (blue line), exponential (orange circles), Lundberg (red dashed line), Beekman–Bowers (dashed brown line), Renyi (green line), De Vylder (black "x"), 4-moment gamma De Vylder (dark green circles), heavy traffic (magenta rectangles), light traffic (cyan line with crosses) and heavy-light traffic (grey circles) approximations. The mixture of 2 exps case.

STFruin16.xpl

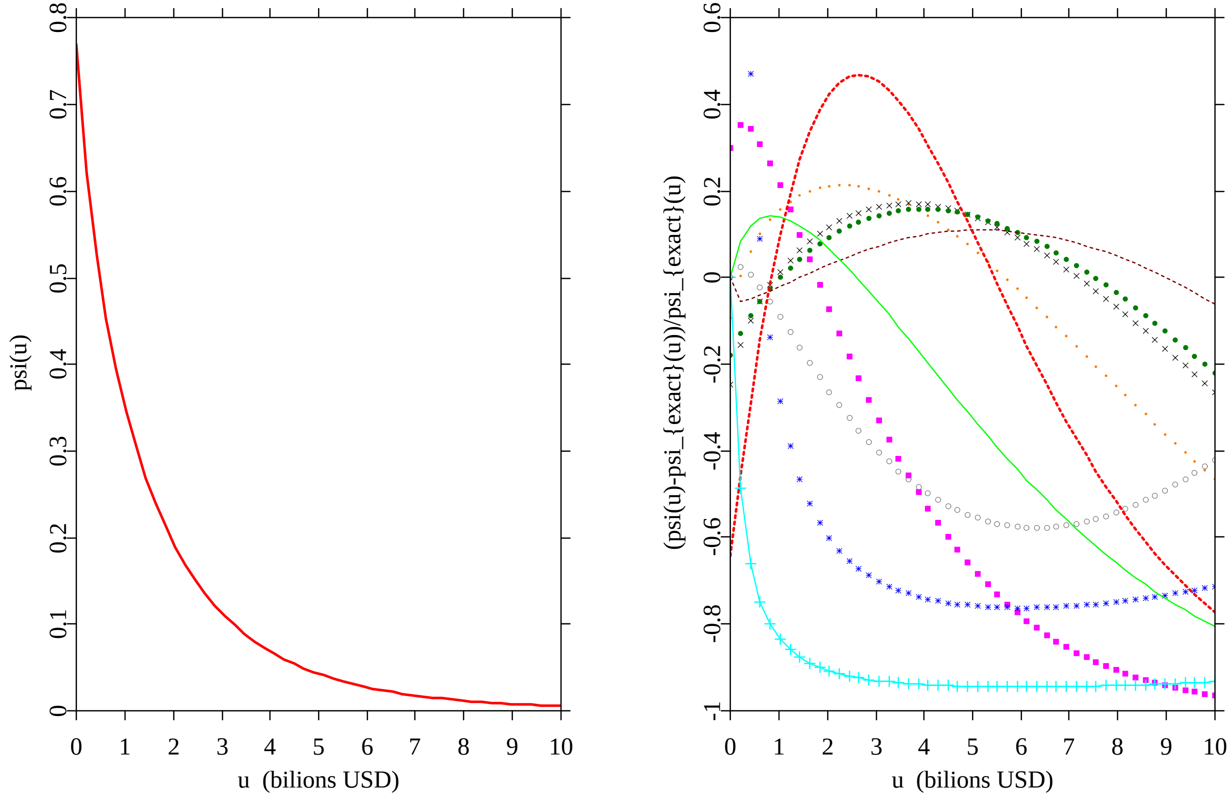

Figure 3: The exact value of the ruin probability (left panel), the relative error of the approximations (right panel). The exponential (orange circles), Lundberg (red dashed line), Beekman–Bowers (dashed brown line), Renyi (green line), De Vylder (black "x"), 4-moment gamma De Vylder (dark green circles), heavy traffic (magenta rectangles), light traffic (cyan line with crosses), heavy-light traffic (grey circles) and heavy tailed claims (blue stars) approximations. The lognormal case.

STFruin17.xpl

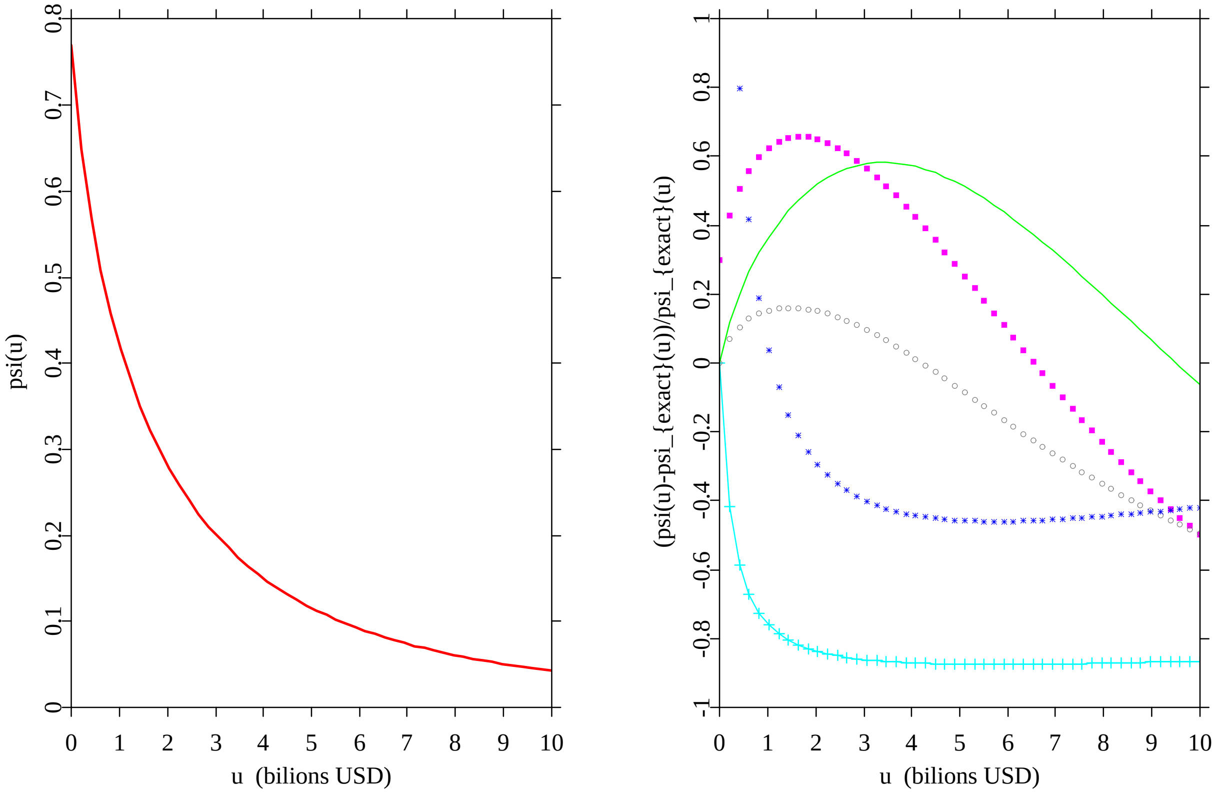

Figure 4: The exact value of the ruin probability (left panel), the relative error of the approximations (right panel). Renyi (green line), heavy traffic (magenta rectangles), light traffic (cyan line with crosses), heavy-light traffic (grey circles) and heavy tailed claims (blue stars) approximations. The Pareto case.

STFruin18.xpl