Solving quadratic equations in complex domain

Consider a quadratic equation

\[ ax^2 + bx + c = 0 \]  \hspace{1cm} (1)

where \( a, b, c \in \mathbb{R} \).

Let \( \Delta = b^2 - 4ac \) be the discriminant of (1).

If \( \Delta > 0 \), then the equation (1) has two distinct real roots given by

\[ x_1 = \frac{-b - \sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a} \]  \hspace{1cm} (2)

If \( \Delta = 0 \), then the equation has a double real root given by

\[ x_{1,2} = \frac{-b}{2a} \]

Finally, if \( \Delta < 0 \), then the equation has no real roots.

On the other hand, if follows from the Fundamental Theorem of Algebra that each complex polynomial of degree \( n \) has \( n \) complex roots (possibly all non-real), counting their multiplicities. For example:

\( x^4 - 1 \) (degree 4) has four roots: 1, \(-1\), \(i\) and \(-i\), as it can be decomposed as

\[ x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x - i)(x + i) \]

\( x^4 - 1 \) has four simple roots: 1, \(-1\), \(i\) and \(-i\), and one root 5 of multiplicity 3.

It follows that the equation (1) always has complex roots. Let us take a closer look at the whole situation.

Consider the equation (1) again but now suppose that \( a, b, c \) are complex. Suppose that the discriminant \( \Delta = b^2 - 4ac \) is non-zero, and that \( \delta \) is such that \( \delta^2 = \Delta \)\(^1\). The equation (1) then has two distinct complex roots given by

\[ x_1 = \frac{-b - \delta}{2a}, \quad x_2 = \frac{-b + \delta}{2a} \]  \hspace{1cm} (3)

\(^1\)It is an easy exercise to prove that for each non-zero complex number \( w \) the equation \( z^2 = w \) has exactly two solutions \( z_1, z_2 \) and \( z_2 = -z_1 \).
Example. The discriminant of $x^2 - 4x + 5$ is $\Delta = -9$. Writing $-9 = (3i)^2$, we get the following roots of $x^2 - 4x + 5$:

$$x_1 = \frac{4 - 3i}{2} = 2 - \frac{3}{2}i, \quad x_2 = \frac{4 + 3i}{2} = 2 + \frac{3}{2}i$$

Example. The discriminant of $x^2 + (1 + i)x + i$ is:

$$\Delta = (2 + 2i)^2 - 4i = 8i - 4i = 4i$$

Writing $4i$ in the trigonometric form, $4i = 4(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$, we see that one of the two complex roots of order 2 of $\Delta$ is:

$$w_1 = 2 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \sqrt{2} + i\sqrt{2}$$

and the other is $w_2 = -\sqrt{2} - i\sqrt{2}$.

Therefore the roots of $x^2 + (1 + i)x + i$ are:

$$x_1 = \frac{-(1 + i) - (\sqrt{2} + i\sqrt{2})}{2} = \frac{-1 - \sqrt{2}}{2} + \frac{-1 - \sqrt{2}}{2}i$$

$$x_2 = \frac{-(1 + i) + (\sqrt{2} + i\sqrt{2})}{2} = \frac{-1 + \sqrt{2}}{2} + \frac{-1 + \sqrt{2}}{2}i$$

Note. If the coefficients $a, b, c$ in (1) are real, the discriminant $\Delta$ is real. If $\Delta < 0$, then we can write $\Delta = (\sqrt{-\Delta} i)^2$. From (3) we then get:

$$x_1 = \frac{-b - \sqrt{-\Delta} i}{2a}, \quad x_2 = \frac{-b + \sqrt{-\Delta} i}{2a} \quad (4)$$

Example. Solve the equation:

$$x^4 + x^2 + 1 = 0 \quad (5)$$

Putting $x^2 = t$ brings (5) into the quadratic equation:

$$t^2 + t + 1 = 0 \quad (6)$$

whose discriminant is $\Delta = -3$. In view of (4), the roots of (6) are:

$$t_1 = \frac{-1 - \sqrt{3} i}{2}, \quad t_2 = \frac{-1 + \sqrt{3} i}{2} \quad (7)$$
In that way we have the following two equations to solve with $x$:

$$x^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad x^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad (8)$$

Since $-\frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos \frac{4}{3}\pi + i\sin \frac{4}{3}\pi$, the first equation in (8) has the solutions:

$$x_1 = \cos \frac{4}{6}\pi + i\sin \frac{4}{6}\pi = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad x_2 = -x_1 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Similarly, since $-\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos \frac{2}{3}\pi + i\sin \frac{2}{3}\pi$, the second equation in (8) has the solutions:

$$x_3 = \cos \frac{2}{6}\pi + i\sin \frac{2}{6}\pi = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad x_2 = -x_1 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

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