Supergraphs. Ideas, results and problems.

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Abstract

This paper introduces the concepts of a *supergraph* and *graphical complexity* of a permutation group, intended as a tool for investigating the structure of concrete permutation groups. Basic results are established and natural research problems suggested. This is an extended version of the paper *Supergraphs and graphical complexity of permutation groups*.

1. Introduction

We speak of concrete permutation groups in contrast with abstract groups they represent. Instead of abstract isomorphism we apply permutation isomorphism in our study: two permutation groups are considered as different unless they can be obtained from each other by renaming elements of the base sets. Obviously, the permutation groups viewed in this way give more insight into the the symmetry of an object than abstract groups, Figure 1 being here a striking example. Yet, the problems involving concrete permutation groups are usually much harder then their abstract group counterparts.

It is well-known that while every abstract group is isomorphic to the automorphism group of a graph, not every permutation group can be represented

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This research was partially done when the author was a Fulbright Visiting Scholar at Vanderbilt University.

2000 *Mathematics Subject Classification.* Primary: 05C25; Secondary: 05C65, 20B25, 20B35 automorphism group, concrete representability, complexity;

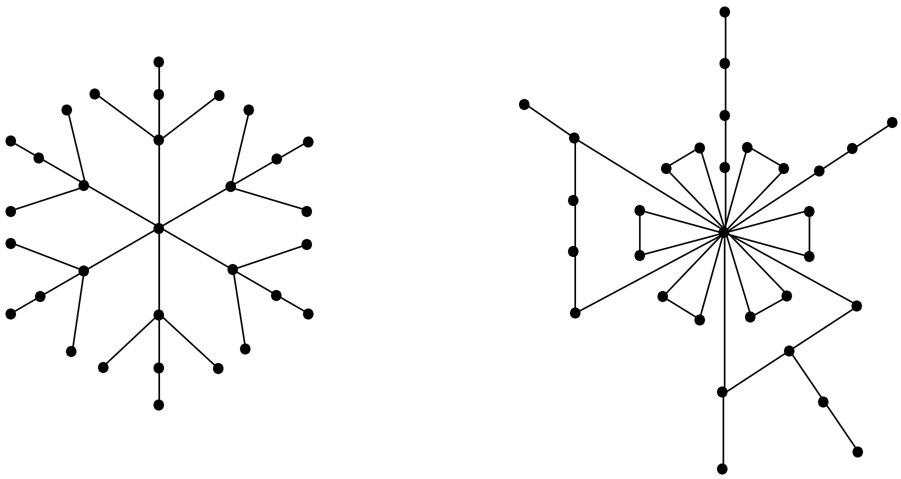

Figure 1: Graphs with automorphism groups isomorphic to $S_2\, wr\, S_6$.

as equal to the automorphism group of a concrete graph. This paper is the result of our attempt to find a simple graphical representation for concrete permutation groups, which could form a base for an exhaustive classification. Attempts to do such classifications via k-ary relations go back to Wielandt [45], but the concept generally turned out to be hard (see also [25, 36] and [3, 11, 13, 46] for further references). A trouble with k-ary relations is that they lack a structure that could be easily visualised. Supergraphs we introduce in this paper have superedges coming in a variety of types that have to be preserved by automorphism. This makes possible and often easy to see a solution of a problem and to construct proofs that can be subsequently reproduced in terms of relations. So, our hope is that this uniform graphical representation may prove not only to be more accessible, and interesting as a new point of view, but also helpful in attacking old problems.

Formal definitions are given in Section 2, where we also discuss diagrams, and prove the basic result. In Section 3, we introduce the concept of *graphical complexity* based on the rank of a supergraph, and establish the complexity of the simplest classes of permutation groups. Here we apply an approach used already in [27, 19, 20], where various constructions of permutation groups are studied with regard to the growth of complexity. The most interesting open problem and further directions of research are suggested in conclusion.

It is my pleasure to thank Peter Cameron, Wiesiek Dziobiak, Mark Ellingham, Peter Jipsen, Mikhail Klin, Wojtek Peisert, and Stephan Thomasseé who all, more or less directly, influenced the content of this paper.

Motivation

Permutation groups arise "in nature" as groups of permutations of various kinds of mathematical objects which preserve the underlying structure. They are usually called *symmetry groups* or *automorphism groups*. Given a kind of objects, it is natural to ask what groups may occur (or may be represented) as the symmetry or automorphism groups of these of objects. This *representability question* has actually two versions: an *abstract* one, asking about groups considered up to abstract isomorphism, and a *concrete* one, asking about concrete permutation groups, i.e. given as the sets of permutations.

In the literature, usually, only the abstract version is considered. Results on concrete representations of permutation groups are isolated and rare. The first obvious reason is that permutation groups have been studied so far mainly in connection with applications to abstract group theory. The other reason is that concrete versions of the representability question are usually much harder, often considered as hopeless.

For graphs, the problem is known as *König's problem*. In its abstract version it has a simple (and not difficult to prove) solution (due to Frucht): every group is abstractly isomorphic to the automorphism group of a suitable graph. As far as concrete version of König's problems is considered only isolated results are known (for general reference see [46, p. 88] and [3]; for some particular results see [46, p. 98]).

Now, for many problems in computer science and general algebra involving permutation groups, it is important which concrete permutation groups may occur under conditions of the problem, how these groups act on underlying sets, and it is often the case that the problem is not compatible with abstract isomorphism, at all.

To give a simple example, the symmetry groups of operations play an important role in counting the total number of n-ary operations in clones (composition closed sets of operations). The number of different operations that can be obtained from a given operation $f = f(x_1, x_2, \ldots, x_n)$ by permutations of variables is the index $[S_n : G(f)]$ of the symmetry group $G(f)$ of f in the full symmetric group S_n. Yet, this observation is only the first step in a more general solution. At some further step it is important if there exists a commutative binary operation in the clone or not, since the way of counting in both the cases is very different. Now, the existence of such operation depends, in particular, on the concrete structure of $G(f)$: if there is a block in $G(f)$ of size $n/2$ which is not an orbit, then a commutative operation can be obtained, just by identifying variables corresponding to the block and its complement (see [26] for details; see also [15, 27] for an example arising in connection with symmetries of formal languages and boolean functions, and [28] for examples arising in the study of equations in semigroups). These yield enough motivation to investigate combinatorial structure of concrete (and in particular, intransitive) permutation groups.

There is also, perhaps even more important, a general motivation. Consider graphs in Figure 1, each drawn as symmetric as possible. Everybody agrees that the first one has (somehow) higher degree of symmetry than the other, in spite of that the automorphism groups of these graphs are both isomorphic. So, keeping in mind, that "the investigation of symmetries of a given mathematical structure has always yield the most powerful results" (E. Artin, Geometric Algebra, Interscience, New York, 1957), it is natural and reasonable to undertake the study of concrete permutation groups on their own. Certainly the concrete representation problems are harder than abstract counterparts. Therefore we should not expect any simple complete solution, but instead, we should break the problem into more detailed natural questions, and proceed step by step.

In [27], a new approach was proposed, that as I realized afterwards, may be applied to all concrete permutation group representation problems. Namely, one may ask whether the class of permutation groups in question is closed on various natural constructions of permutation groups. Of course, constructions that make sense in this connection are those that from given permutation groups allow to obtain other concrete permutation groups. It turned out however that not many such constructions were considered in the literature so far. Even the reduction theorems, showing how intransitive groups can be obtained from transitive constituents, and how imprimitive groups are constructed from primitive ones, contain only few details related to abstract groups (see. [21, p. 63, p. 88], [31, p. 66], [44, p. 13], [16, p. 65]). In particular, there were no special interest in the combinatorial structure of intransitive permutation groups, since from abstract point of view, these groups are just direct or subdirect products of transitive groups.

In [27], the reduction theorems are formulated with all the details, and the most natural constructions of permutation groups are distinguished. These include, in particular, *parallel power* and *parallel wreath product*, which yield no new abstract groups, but yield essentially new intransitive and imprimitive actions. The investigation of closeness of various classes of permutation groups on these and other operations yielded already interesting and even quite unexpected results. For example, the very first study of whether the class of automorphism groups of graphs is closed on direct product led to distinguishing the so called unique groups and graphs, and in consequence, to full description of self-complementary symmetric graphs [34, 35].

What we apparently needed to continue this research was a nice classification of permutation groups based on the symmetries of natural structures, that would make possible to distinguish classes of permutation groups of similar complexity degree. The candidates we considered in this connection included edge-colored graphs, hypergraphs, colored hypergraphs, and relational structures. A combined classification based on these structures had this drawback that it was not uniform, and in addition, hypergraphs and relational structures had no nice graphical representation.

In this paper we introduce *supergraphs* and show that they fit very nice to our purpose. Every permutation group is the automorphism group of a supergraph, and this leads to a natural classification of permutation groups. Supergraphs have simple graphical and symbolic representations that can be conveniently used in the proofs.

2. Supergraphs

A supergraph consists of vertices and superedges; each superedge, like an edge in a simple graph, joins two elements, and the only difference is that these elements may be also other superedges.

Formally, a *supergraph* is a pair (V, S) of sets V of *vertices* and S of *superedges* such that $S \subseteq E^i(V)$, for some $i > 0$, where $E^0(V) = V$ and $E^i(V) = E(E^{i-1}(V))$ for $i > 1$, with $E(X)$ denoting the set of unordered pairs of X. To fit the simple idea described above we require also that S is *hereditary* in the following sense: if $\{s, t\} \in S$, then $s, t \in S$, as well.

We consider only finite simple supergraphs, i.e., with both V and S finite, and with no loops (each superedge is a two-element set or a vertex). A superedge $\{s, t\} \in E^i(V) \setminus E^{i-1}(V)$ is said to be of *order* i. Thus vertices are superedges of order 0. The *order of a supergraph* is the maximum of the orders of its superedges.

Supergraphs have natural and simple representations, both graphical and symbolic. In diagrams vertices are drawn as circles, and other superedges as lines joining suitable elements. A superedge joined with another superedge or a vertex has a filled circle on it, such that, all lines joining the superedge with other elements come out from this very circle. In such a way the picture consists only of lines and two kinds of circles (in a three-dimensional picture, without line crossings, filled circles could be removed). To give a precise definition we start from symbolic representation.

In this representation, to denote superedges, we use *words* consisting of symbols of vertices and brackets, in the natural way. Thus, (xy) denotes the superedge joining vertices x and y, and (st) denotes the superedge joining elements s and t. Then, for example, $(c(ab))$ is a superedge joining vertex c with edge (ab), and $((ab)(c(ab)))$ is a superedge joining superedge (ab) with the superedge $(c(ab))$. The *order* of a superedge coincides with what is sometimes called the *rank* of the corresponding word.

Note that this notation is not quite unique: the elements in brackets can be written in either order. Formally, the strings denoting superedges may be considered as *idempotent commutative groupoid* words (IC-*words*, in short), put in their normal form (since by definition we do not allow loops, we do not have substrings of the form ss). A superedge s is *contained* in a superedge t if the word corresponding to s is a subword of t (up to commutativity).

By definition, if $s \in S$, for a supergraph S then each superedge contained in s also belongs to S. Therefore, usually it is enough to deal only with *maximal superedges*, those not contained in other superedges. In particular, only maximal superedges are listed in symbolic representation. From this point of view, a supergraph is simply a set of noncomparable IC-words, and from now on, we adopt convention, that in notation (V, S), the set S contains only maximal superedges.

Now, in order to draw a diagram of a supergraph (V, S), we first make a list of all superedges (which can be easily derived from S), and then we draw in turns: vertices, superedges of order 1, superedges of order 2, etc. To distinguish between usual line crossings and the starting points of superedges, for the latter we use filled circles. For example, for $V = \{1, 2, 3, 4\}$ and

$$S = \{((12)(14)), ((12)(23)), ((12)(34)), ((23)(34)), ((34)(14)), (13), (24)\},$$

$K = (V, S)$ is a supergraph with six superedges of order 1 (which are (12),

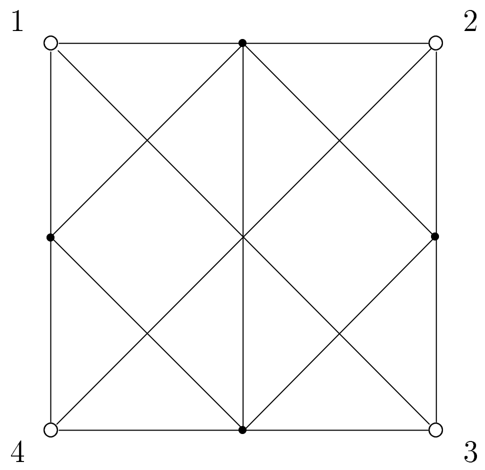

Figure 2: A supergraph K on 4 vertices.

(14), (13), (23), (24), (34)), and five superedges of order 2 (listed in S). It is pictured in Figure 2.

While given a set of noncomparable IC-words we may easily draw a corresponding diagram (at least, in principle), the converse is not so obvious. It is even not quite obvious that the diagram determines the supergraph uniquely. To see this, we need to observe that the maximal superedges are distinguished by the fact that these are the only lines having no filled circle on themselves. Once we remove them from the diagram, and then remove filled circles of "degree 2" (those being no longer starting points for other superedges) we get a diagram of order less by one. Thus, using simple induction we may prove that the diagram determine uniquely the supergraph.

The *automorphism group* $\mathrm{Aut}(S)$ of a supergraph $S = (V, S)$ may be defined and viewed into two ways: 1) as the set of all permutations on V preserving the diagram (for every superedge $s \in S$ its image under the action induced by a permutation of V is again in S), and 2) as the symmetry group of the set of words representing maximal superedges (i.e. the set of all permutations preserving the set of these words). So, investigating automorphism groups of supergraphs we may work both with graphical and symbolic representation. This makes supergraphs easier objects to study then k-ary relations, and may be even a way to solve analogous problems on

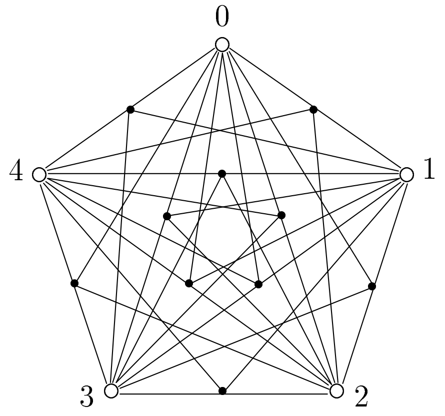

Figure 3: Supergraph with double transitive automorphism group $AGL_1(5)$.

the symmetry groups of relations.

For example, in the study of the automorphism groups of graphs, a very useful invariant is the degree of a vertex. For supergraphs we may also consider analogously the *degree of a superedge* defined as the number of outcoming other superedges, and even the degrees with respect to types of superedges (these where used successfully in [20]). In Figure 2 every vertex has degree 3, but there are three types of edges joining a pair of vertices: those of degree 3, 2, and 0, respectively. It follows that every automorphism preserves or exchange the pairs $(1,2)$ with $(3,4)$, $(1,3)$ with $(2,4)$, and $(1,4)$ with $(2,3)$. It is not difficult to see that each of this exchangings is actually possible, and that the full automorphism group is just the Klein 4-group with its regular action.

In Figure 3, a supergraph is given whose automorphism group is the one-dimensional linear affine group $AGL_1(5)$ over the five-element field. It may be instructive to check that this is indeed the case. First observe that this supergraph has 20 superedges of order 2 (the starting points are filled circles), and (the remaining) 10 superedges of order 1. As it is well known (cf. [16]), $AGL_1(5)$ can be presented as the permutation group generated by two cycles: $(0,1,2,3,4)$ and $(1,2,4,3)$ (the first is translation by 1, and the second is multiplication by 2). From Figure 3 one sees immediately that the first cycle preserves the supergraph structure. In order to see that the

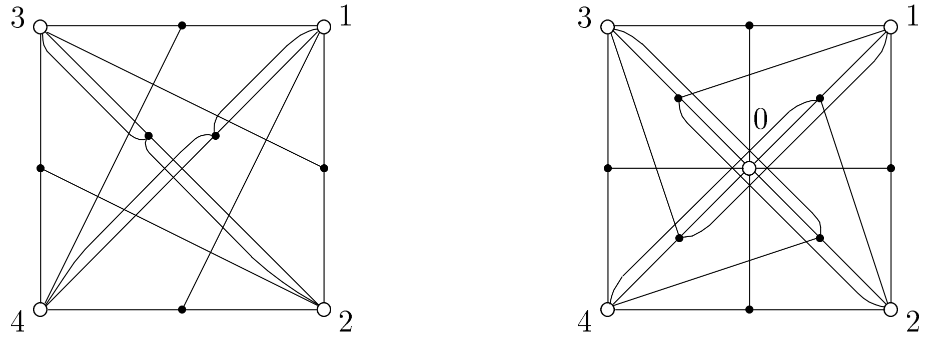

Figure 4: Decomposition of the supergraph of Figure 3.

second cycle does, as well, we decompose the supergraph into two parts and draw them separately in a way that makes the statement, again, obvious: see Figure 4. The only thing to check is that the supergraphs in Figures 3 and 4 have exactly the same superedges. Now, from Figure 3 we also see that transposition $(2,4)$ does not preserve the structure of the supergraph (the image of the superedge $((40)1)$ under this transposition is $((20)1)$, and the latter does not belong to the supergraph). Therefore, since $AGL_1(5)$ is a maximal subgroup of S_5, the automorphism group of the supergraph given in Figures 3 and 4 is precisely $AGL_1(5)$.

Of course, the practical use of diagrams, as in case of graphs (or even more), is restricted only to small supergraphs. What however counts here is that the very idea of existence of graphical representation may be very useful in arguments (see [20] for further examples).

Before we state our first result we fix some notational conventions. By a group we mean always a permutation group, unless otherwise stated. We say often briefly that a graph (digraph, colored graph,or other mathematical object) *represents* a permutation group $G = (X, G)$ meaning that the automorphism (symmetry) group of this object is equal (up to permutation isomorphism) to G (i.e., the elements of the base sets may be identified in such a way that the utomorphisms coincide with the permutations in G). The automorphism (or symmetry) group of an object S is denoted generally $\mathrm{Aut}(S)$. If the base set is clear from the context we use short notation $S = (X, S)$, for a upergraph on a set X of vertices, and similarly, $G = (X, G)$

for a permutation group on the same set X. We also write informally $x, s \in S$ both for vertices and superedges. We use the notation $w = w(x_1, x_2, \ldots, x_m)$ to denote a word of in m letters $x_1, x_2, \ldots, x_m$. Following common practice, writing down concrete words, the external pair of brackets is usually omitted. Below, $\log x$ stands for $\log_2 x$, and we put $\log x = 0$ for $x = 0$

Theorem 1 *For every permutation group $G = (X, G)$ on an n-element set X, there exists a supergraph $S = (X, S)$ such that $\mathrm{Aut}(S) = G$, and the order of S is less than $2 + \log(n-2)$.*

Proof. It is well-known (and easy to see) that each finite permutation group $G = (X, G)$ is the automorphism group of a relational structure (X, R), where R is an $(n-1)$-ary relation on X [16, p. 43]. Indeed, it is enough to fix an $(n-1)$-tuple $(x_1 x_2 \ldots x_{n-1})$ of distinct elements of X and take R to be the least set containing $(x_1 x_2 \ldots x_{n-1})$ and closed on permutations in G (under its natural action on k-tuples).

So, all we need is to show that for every relational structure (X, R), with a single relation R, there exists a supergraph (X, S) such that $\mathrm{Aut}(S) = \mathrm{Aut}(R)$. This is, in fact, a problem of encoding relations into supergraphs. The construction below will be useful also later.

Let us consider words of the form $x_1(x_2(\ldots x_{m-1}(x_m x_{m+1})\ldots))$, which we denote briefly $x_1 x_2 \ldots x_{m-1}(x_m x_{m+1})$ (we leave the last pair of brackets, because $x_m x_{m+1}$ can be written in the reverse order). Then, there is an obvious one-to-one correspondence between m-tuples $(x_1, x_2, \ldots, x_m)$ of elements X and words of the form $x_1 x_2 \ldots x_{m-1}(x_m x_1)$. It is also obvious that the supergraph S consisting of all superedges corresponding in such a way to m-tuples in a relational structure (X, R) has the same automorphism group as (X, R).

This natural correspondence yields the bound $n-1$ for the order of the supergraph. To get better bound we must find a more compact encoding. What we need is to assign to every m an asymmetric IC-word $w = w(x_1, x_2, \ldots, x_m)$ (i.e., one with the property that each permutation of variables yields a different word; for example, $x(yz) = x(zy)$ is not asymmetric). Above, we have used uniformly the word $w = x_1 x_2 \ldots x_{m-1}(x_m x_1)$ for all m. Now we look for words of lower order.

We prove that for all $k \geq 3$ and $n \geq 4$ satisfying $n \leq 2^{k-1} + 1$ there exists an asymmetric IC-word $w(x_1, x_2, \ldots, x_n)$ in n letters of order k.

The proof is by induction on k. For $k = 3$, the words $((xy)(zt))(x(zu))$ in 5 letters, and $((xy)(zy))(x(zu))$ in 4 letters, are as required.

For induction step, define $u(x_1, x_2) = (x_1x_2)$, and

$$u(x_1, \ldots, x_{2k}) = u(x_1, \ldots, x_k)u(x_{k+1}, \ldots, x_{2k}).$$

Suppose that the claim above is true for a fixed k, and let $w(x_1, x_2, \ldots, x_m)$ be an asymmetric IC-word in $m = 2^{k-1} + 1$ letters and of order k. Then, consider the word

$$u(x_1, y_1, \ldots, x_{m-1}y_{m-1})w(x_1, x_2, \ldots, x_{m-1}z).$$

This word is, by assumption, of order $k+1$, in $2(m-1)+1 = 2^k+1$ letters, and it can be easily seen to be asymmetric. Hence, it proves the required statement for $n = 2^k + 1$. For $2^{k-1} + 1 < n < 2^k + 1$ it is enough to put $y_1 = \ldots = y_r$ with $r = 2^k + 2 - n$. A key observation is that, in any case, the word u on the left hand side is not asymmetric itself, in contrast with the word $w(x_1, x_2, \ldots, x_{m-1}z)$ on the right hand side, and therefore the resulting word is always asymmetric. This completes the induction proof.

Now, having the claim proved, it is routine to see that given $n \geq 4$, the minimal order of an asymmetric word in n letters does not exceed $\lceil 1+\log(n-1)\rceil$. Taking into account, that according to the argument at the beginning of the proof, it is enough to apply asymmetric words in $n-1$ letters, the order of a supergraph S we construct (using these words as corresponding to m-tuples) does not exceed $\lceil 1 + \log(n-2)\rceil$.

It remains to consider cases $n = 2$ and 3. For $n = 2$ there are two groups: S_2 represented by a graph, and I_2 (trivial) represented by a directed graph. For $n = 3$, there are four groups (up to permutation isomorphism) that can be denoted S_3, S_2, C_3, I_3. Again they are represented either by a graph or a directed graph. Since, directed edges can be represented in supergraphs as edges $x(xy)$, in any case, the order of a supergraph is less or equal to 2, and the result follows. $\square$

Note that the proof above contains a method that given a k-ary relation representing a permutation group makes possible to produce a supergraph representing this group.

3. Graphical complexity

In general, by a graphical complexity of a permutation group P we mean the degree of complexity of a graphical structure whose symmetry group (automorphism group) is P. In this paper, the result above suggest to define the *graphical complexity* of G as follows. By $\mathrm{gc}(G)$ we denote the least k such that supergraph $S = (X, S)$ of order k represents G. Theorem 1 shows that $\mathrm{gc}(G)$ is well-defined, and $\mathrm{gc}(G) < 2 + \log(|X| - 2)$. Since there is a finite number of supergraphs of a given order on X, we see that $\mathrm{gc}(G)$ is, in principle, *computable*.

The trivial graphs without edges represent the full symmetric groups S_n, and therefore $\mathrm{gc}(S_n) = 1$. The groups with $\mathrm{gc}(G) \leq 1$ are exactly the automorphism groups of graphs. It is also not difficult to see, using the encoding from the proof above, that for automorphism groups of directed graphs we have $\mathrm{gc}(G) \leq 2$. Taking into account the representation of $AGL_1(5)$ in Figure 3, one also easily infer that $\mathrm{gc}(AGL_1(5)) = 2$. A natural question arising in this connection, whether there are supergraphs with higher complexity, is answered by the next result.

Theorem 2 *For every $k \geq 1$ there exists a permutation group G such that $\mathrm{gc}(G_k) = k$.*

Proof. The most widely used example of groups that are difficult to represent as a symmetry groups of given objects are the alternating groups A_n. From the previous theorem we have $\mathrm{gc}(A_n) < 2 + \log(n - 2)$, and we will see that this bound is close to the actual value. At this moment, we are not able to establish the exact value, but what we need here is only a suitable lower bound.

It should be obvious that we will be done when we establish the following two facts:

(i) $\mathrm{gc}(A_n) > \log(n - 1)$.

(ii) $\mathrm{gc}(A_{n+1}) \leq \mathrm{gc}(A_n) + 1$

First let us note that if $\mathrm{Aut}(S) = A_n$ then S has a superedge involving more than $n - 2$ vertices. Indeed, if the number of vertices (letters) involved in an edge s' is $n - 2$ or less, and say $s' = s(x_1, \ldots, x_{n-2})$, then for every permutation $\tau \in S_n$ there exists a permutation $\sigma \in A_n$ such that, under the induced action of S_n on superedges, $\tau(s') = \sigma(s')$. Hence, the set S' of all edges $s \in S$ such that $s = \sigma(s')$ for some $\sigma \in S_n$ is invariant under any permutation of vertices. It follows that removing from S all the edges that are in S' leaves us with a supergraph S'' whose automorphism group is the same as that of S. Hence, we may assume that each edge in S involves at

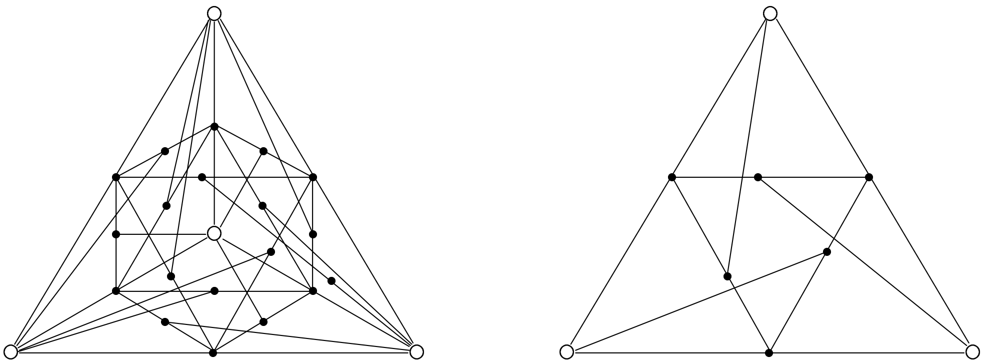

Figure 5: Supergraph with automorphism group A_4 and its "face".

least $n-1$ vertices. Now, since the maximal number of vertices involved in an edge of order k is 2^{k-1}, (i) easily follows.

Going one step further we infer that it is necessary that S has an edge (in $\geq n-1$ letters) such that no odd permutation in S_n fixes the edge. Otherwise, the argument in the previous paragraph works. Conversely, if there exists such an edge then it is not difficult to see that its orbit under the action of A_n forms a supergraph S with $\mathrm{Aut}(S) = A_n$.

Now, to prove (ii), suppose that $\mathrm{gc}(A_n) = k$. Then, there exists an edge $s = s(x_1, \ldots, x_m)$ of order k, with $m \leq n$, such that no odd permutation in S_n fixes it. It follows that the edge $s' = (x_{n+1}s)$ has the same property for permutations in S_{n+1}. Indeed, if $\tau \in S_{n+1}$ were an odd permutation fixing s', then $\tau(x_{n+1}) = x_{n+1}$, and τ' obtained from τ by restricting its domain to $\{x_1, \ldots, x_n\}$, would be odd and would fix s. It follows that the orbit of s' under the action of A_{n+1} forms a supergraph S' with $\mathrm{Aut}(S') = A_{n+1}$. Therefore, $\mathrm{gc}(A_{n+1}) \leq k+1$, as required. $\Box$

In Figure 5 a supergraph is given whose automorphism group is A_4. It can be viewed in three dimensions as a tetrahedron whose face is pictured beside. This observation makes it immediately clear that any 3-cycle preserves the structure of the supergraph, while a 2-cycle does not. It follows that the automorphism group is the alternating group. (I am grateful to Peter Jipsen for this observation; in fact, the decomposition in Figure 4 also follows this idea). It follows that $\mathrm{gc}(A_4) = 2$

Note that for each Mathieu group M, $\mathrm{gc}(M) \geq 3$, since they are 4- and 5-transitive (in such a case, if $((xy)(zt))$ is a superedge of a supergraph representing M, then each word of this type must be a superedge). We also remark that using A_n one can construct various groups G with $\mathrm{gc}(G)$ arbitrarily large. It is an open problem whether there are such groups not involving A_n in any way (see Section 4).

It is easy to establish the graphical complexity of the simplest permutation groups on an n-element set X. The full symmetric group S_n, the trivial group I_n, the reflection group R_n generated by a reflection permutation (that given by $\rho(i) = n+1-i$), the cyclic groups C_n generated by an n-cycle, and the dihedral groups D_n are all represented by graphs or directed graphs (cf. e.g. [19]), and therefore their graphical complexity does not exceed 2. For regular permutation groups the same is proved in [42].

In order to establish the complexity of more complex groups we adopt the approach introduced in [27] and look for closeness of the complexity classes on the most natural constructions. As we will see the classes based on supergraphs are much more "compatible" with these constructions, then classes considered so far.

Recall that by the *direct sum* $G_1 \oplus G_2$ of permutation groups $G_1 = (X_1, G_1)$ and $G_2 = (X_2, G_2)$ we mean the direct product of groups G_1 and G_2 acting on the *disjoint union* of sets X_1 and X_2 in the natural way (thus the product is determined up to permutation isomorphism). In the literature, this construction is more often called the *direct product*, but we prefer the first name (following [31]); this name is certainly more appropriate from the point of view of concrete permutation groups (to distinguish it from the direct product with product action, which leads to different permutation groups). It was established in [34], that the direct sum of two automorphism groups of simple graphs is the automorphism group of a simple graph, unless both the groups are permutation isomorphic, transitive, and *unique* (in the sense that there is a unique graph representing the group, up to graph isomorphism; see also [35] for a very interesting developement of this topic). A similar result were established in [19] for automorphism groups of edge-colored graphs; exception are edge-colored counterparts of unique graphs. On the other hand it was also observed in [19] that the class of automorphism groups of directed graphs is closed on the direct sum.

The reason is somehow explained in the result below. It can be interpreted so, that the complexity of the construction of the direct sum itself is 2: once edges of order 2 are accessible, the class is closed on the direct product.

Theorem 3 *For every pair of permutation groups G_1 and G_2, the complexity* $\mathrm{gc}(G_1 \oplus G_2) \leq \max\{\mathrm{gc}(G_1), \mathrm{gc}(G_2), 2\}$.

Proof. Let S_1 and S_2 be supergraphs representing G_1 and G_2 on disjoint sets of vertices X_1 and X_2, with $|X_1| = n$ and $|X_2| = m$, respectively. We form a supergraph S joining each vertex $x \in X_1$ with any vertex $y \in X_2$ by a directed edge $\langle x, y\rangle$, i.e, a superedge $x(xy)$. Then, obviously, $\mathrm{Aut}(S) \supseteq G_1 \oplus G_2$. It should be obvious that due to using directed edges we have also $\mathrm{Aut}(S) \subseteq S_n \oplus S_m$, and then, it is easy to see that also $\mathrm{Aut}(S) \subseteq G_1 \oplus G_2$ holds. Obviously, the graphical complexity of S satisfies the required conditions.□

This result suggest that also the problem of complexity of groups generated by a single permutation (i.e. those that are cyclic as abstract groups) should be easy. In fact, slight modification of the construction used above yields immediately

Theorem 4 *All permutation groups G generated by a single permutation have the complexity* $\mathrm{gc}(G) \leq 2$. *In fact, all these groups are automorphism groups of directed graphs.*

Proof. If $h = c_1c_2 \ldots c_n$ is an arbitrary permutation given in the form of a product of disjoint cycles, then the group generated by h is a direct sum of groups generated, respectively, by cycles $c_1, c_2, \ldots, c_n$, up to some parallel actions caused by common divisors of the lengths of the cycles. Each such a group is represented by a directed cycle of suitable length. All we need to do is to join all these cycles by as few edges as possibly, just to ensure that the action in any of the cycles forces a suitable action in other cycles.

Let us denote the directed cycles corresponding to cyclic permutations $c_1, \ldots, c_n$ by $C^1, \ldots, C^n$. For each $i = 1, \ldots, n-1$ we join the cycles C^i and C^{i+1} by directed edges so that the automorphism group of the resulting directed graph is one generated by a single permutation $h' = c_ic_{i+1}$; it is enough to take one directed edge joining C^i with C^{i+1}, and add all the images of this edge under the permutation h'. (In Figure 6 the situation is pictured for $h' = (10, 11, \ldots, 15)(0, 1, \ldots, 9)$; to make the picture clear and

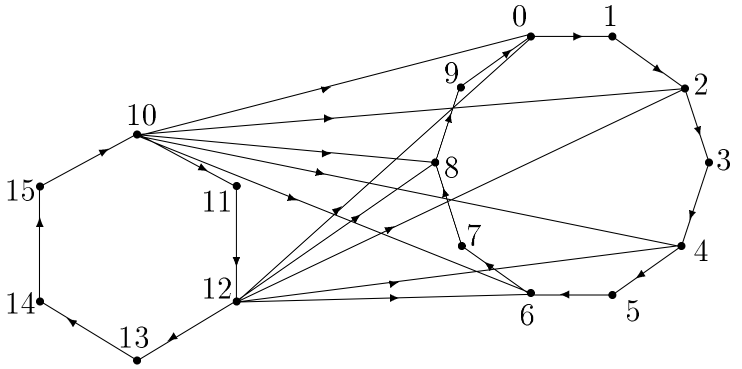

Figure 6: A part of a graph representing $C_6 \oplus C_{10}$.

show the pattern, edges outgoing from vertices 11,13,14, and 15 are omitted). This construction extends obviously on n cycles. □

The next natural construction to consider is that of intersection of permutation groups.

Theorem 5 *For every finite set of permutation groups $G_i = (X, G_i)$ on a set X, the complexity* $\mathrm{gc}(\bigcap G_i) \leq 1 + \max\{\mathrm{gc}(G_i)\}$.

Proof. Let S_i be the supergraph representing G_i of the lowest possible order, i.e., $\mathrm{Aut}(S_i) = G_i$, and the order of S_i is $\mathrm{gc}(G_i)$. Denote $G = \bigcap G_i$ and $m = \max\{\mathrm{gc}(G_i)\}$. We wish to construct a supergraph S with $\mathrm{Aut}(S) = G$ and order not exceeding $m + 1$. We cannot simply take $S = \bigcup S_i$, since at first, there can be comparable IC-words in $\bigcup S_i$, and at second, there may be new permutations preserving S, that do not preserve all S_i individually.

To overcome this obstacle, our construction consists of a few steps.

First we show that we may assume that S_i are pairwise disjoint (as sets of IC-words). Suppose that $I = S_k \cap S_j \neq \emptyset$ for some $k \neq j$. Then, obviously, I is closed under the action of $G = \bigcap G_i$, as both $S_k \setminus I$ and $S_j \setminus I$ are. Moreover, $G_k = \mathrm{Aut}(S_k \setminus I) \cap \mathrm{Aut}(I)$, and $G_j = \mathrm{Aut}(S_j \setminus I) \cap \mathrm{Aut}(I)$. It follows that if, in the family G_i, we replace G_k and G_j by the groups $\mathrm{Aut}(S_k \setminus I)$, $\mathrm{Aut}(S_j \setminus I)$, and $\mathrm{Aut}(I)$, then the intersection of all the groups in the family remains G. To keep the family S_i as one consisting of supergraphs representing groups G_i, at the same time, we replace supergraphs S_k and S_j in the family S_i

by $S_k \setminus I$, $S_j \setminus I$, and I. Then, the maximal order of a supergraph in the family $\max\{ord(S_i)\}$ remains equal to m. Applying such a replacement as long as $I = S_k \cap S_j \neq \emptyset$ for some $k \neq j$, we finally obtain a family S_i of disjoint supergraphs representing a family G_i of groups whose intersection is $\bigcap G_i = G$, and $\max\{ord(S_i)\} = m$.

For the next step consider the orbits of sets S_i under G. Each such an orbit can be treated as a single supergraph whose automorphism group contains $\mathrm{Aut}(G_i)$, and which is more important, the intersection of these automorphism groups is contained in $\mathrm{Aut}(G_i)$, and contains G. Therefore, replacing each S_i by its G-orbits, we obtain a family S_i of disjoint supergraphs with the properties as above, and with an additional property, that every supergraph is superedge-transitive. In particular, in every supergraph S_i, all the superedges have the same pattern (as words).

Now, let S_i' be the set of edges obtained from S_i by replacing each edge $t \in S_i$ by the set of edges $w_r = x_r x_r \dots x_r(x_r t)$, for all $1 \leq i \leq |X|$, where the number of occurrences of x_r is chosen so that w_r is of order m. Observe that, since the edges in S_i have the same pattern, the edges in S' are non-comparable, of the same order m, and $\mathrm{Aut}(S_i') = \mathrm{Aut}(S_i)$. In particular, $\bigcap \mathrm{Aut}(S_i) = G$.

It may still happen that some edges in different S_i' and S_j' have the same pattern, and taking the union may result in appearing new automorphisms. Therefore for the last step we make use of the possibility of increasing the order as follows.

Let $S_1, S_2, \dots, S_r$ denote now all the supergraphs in the recently obtained family. We define by induction, $S_1' = S_1$, and for $k > 1$, S_k' as the set consisting of all edges (st) with $s \in S_{k-1}$ and $t \in S_k$. Then, each edge in S_k' is of order $m+1$, whenever $k > 1$, and since S_k are pairwise disjoint, the edges in different S_i' and S_j' have different patterns, and moreover, for $k > 1$, $\mathrm{Aut}(S_k') = \mathrm{Aut}(S_{k-1}) \cap \mathrm{Aut}(S_k)$.

Let $S = \bigcup S_k'$. It follows, that $\mathrm{Aut}(S) = \bigcap \mathrm{Aut}(S_k') = G$ and the order of S is $m+1$, as required. □

Obviously, the automorphism group of an edge-colored graph is the intersection of the automorphism groups of graphs given by particular colors. Such groups can be also characterized as invariance groups of families of binary symmetric relations, and in this connection are called 2^*-*closed*. The automorphism groups of an edge-colored directed graphs are invariance groups of families of binary relations, and called 2-*closed* ([16, 31, 45]). For these classes we have the following corollary.

Corollary 6 *For each* 2^**-closed group* G *the complexity* $\mathrm{gc}(G) \leq 2$*, and for each* 2*-closed group* G *the complexity* $\mathrm{gc}(G) \leq 3$. $\Box$

4. Research problems

There are many open problems arising naturally in this connection. We point out here only one general question that seems most interesting, leads to many particular problems, and is strictly connected with other characteristics, by which one tries to compare the "complexity" of permutation groups.

In [19, 20] we consider the classes $GR(k)$ of permutation groups represented by k-colored graphs (more precisely, by k-edge-colored complete graphs). The main open problem here is whether for every $k > 1$ there are permutations groups in $GR(k)$ which are not in $GR(k-1)$. We have an example of a group which in this sense requires $k = 6$ colors. The situation is analogous for the classes $DGR(k)$ of permutation groups represented by k-colored digraphs. In contrast, by Theorem 2, the situation seems quite different for the classes $SGR(k)$ of permutation groups represented by supergraphs of order k. Yet, also for these classes, one can formulate an analogue of the problem above.

Indeed, observe that the construction in the proof of Theorem 2 is based on alternating groups A_n. The fact that they are $(n-2)$-transitive causes that one needs at least $(n-1)$-tuples or corresponding superedges of high order to represent such groups. Now, a question arises whether this is the only reason for this phenomenon, i.e. whether every permutation group of high graphical complexity must involve somehow an alternating group, or on the contrary, there exist other groups with arbitrarily large graphical complexity. A similar question may be asked for arity of representing relations: does there exist a constant k such that every permutation group not involving an alternating group can be represented as the symmetry group of a k-ary relation? Of course, to solve the problem one must also specify what means here "not involving". It seems that solving the first problem, which seems more accessible, may help to solve the second. Any answer to these questions, yes or no, would be of a great value, giving better insight into possible combinatorial complexity of permutation groups. For now, we see no good reasons to state any definite conjecture in this matter.

For more particular problems, rather than giving a list of problems, we discuss four groups of problems, sometimes suggesting more particular ones. The partition is not very strict. The problems in different groups are closely related. We tried to suggest only such problems that seem accessible. While we do not think that any of them is easy, it is not excluded that one or two of the suggested particular problems easily follow from known results, we have overlooked. For some of the problems, especially those concerning primitive groups, an application of advanced methods in permutation group theory, like O'Nan-Scott Theorem, may prove to be successful.

4.1. Highly complex permutation groups. The main general problem is whether there exist at all more complex permutation group not involving alternating groups. Any proof of Theorem 2 not using alternating groups would be considered as a solution to this problem.

A more particular and precise question is whether there are transitive (primitive) permutation groups G, other than alternating groups with arbitrarily large $\mathrm{gc}(G)$? It is not clear at the present stage whether the fact that a group has a subgroup isomorphic to A_n (but does not contain A_n) yield any lower bound for $\mathrm{gc}(G)$?

A really challenging still open problem is whether for every $k \geq 2$ there are groups represented by a k-edge-colored graph, but not represented by any m-edge-colored graph with $m < k$. There are groups represented by edge-colored graphs requiring at least 5 colors (see [19]). So far, this is the best result in this direction. For directed graphs no example is known requiring more than 2 colors apart from D mentioned after Corollary 6.

Note that generally it is not excluded that, like in case of multiply-transitivity, there exist some finite bounds here, with exceptions for some special groups involving alternating groups. Formulated precisely and proved, this would be also a remarkable result!

4.2. Constructions. As we have seen in the proof of Theorem 5 the number of colors in an edge-colored graphs corresponds somehow to the number of orbits in the representing supergraph. So in order to classify groups within the class of given $\mathrm{gc}(G) = k$ one may wish to consider the minimal number of orbits necessary for a supergraph S of order k to represent G, as an additional measure of complexity.

Write $\mathrm{gc}(G) = (k, r)$ to denote the fact that the least order of a supergraph representing G is k, and the lest number of orbits in a supergraph S of order k representing G is r. We may consider the *complexity classes* $\mathrm{GC}(k, r)$ defined as the set of all permutation groups represented by supergraphs of order k with r orbits (of *rank* k). A natural, but apparently very difficult question is if every class has new groups comparing with classes contained in it.

More accessible problems are those asking whether the classes $\mathrm{GC}(k, r)$ are closed on various constructions of permutation groups. The general results of [27] will be useful here. One should consider possibilities of distinguishing new special cases of general constructions defined in [27], that are natural or easy to handle, in addition to those pointed out in [27]. Also other constructions appearing in the literature, like wreath product with primitive action, induced actions on n-tuples [16], natural multi-component constructions [28], should be considered in this connection.

In search for new constructions, one should look for the smallest imprimitive group which has no natural construction considered so far. Generally, it would be valuable to have a direct, easy to use, classification of imprimitive and intransitive permutation groups on small sets (cf. Fig. 2 in [27]; note that the group $S_2 \wr S_2$ is missing there).

4.3. Natural classes. The complexity of natural classes of permutation groups should be investigated. These may include the automorphism groups of groups and other algebraic structures, the symmetry groups of configurations, the automorphism groups of lattices and ordered sets, and the groups induced on orbits by these groups. Have these and other classes bounds on their complexity? Are they closed on natural constructions?

The work of Godsil, Hetzel, Imrich, Nowitz and Watkins on graphical regular representation (see [18]) shows, for example, that for regular permutation groups $\mathrm{gc}(G) = 1$, unless G is an abelian groups of exponent greater than 2, a generalized dicyclic group, or one of 13 exceptional groups listed in [18]. For the mentioned regular groups $\mathrm{gc}(G) > 1$. What is the exact value in each case?

It would be useful for this research to collect other results in the literature that can be interpreted in terms of complexity. Our references list a number of papers concerning concrete permutation groups, that may come in handy.

It would be also useful to collect problems in other areas of mathematics and computer science that concern concrete permutation groups rather than abstract ones.

4.4. Directed supergraphs. Starting in the definition of supergraph from ordered pairs rather from unordered ones we obtain *directed supergraphs*. From some point of view they can seem even more natural structures than supergraphs. The symbolic representation consisting of sets of general groupoid words is unique and simpler, n-tuples have more natural interpretation, but graphical representation is a little more complicated (involving arrows and possible double edges).

We decided to choose undirected graphs as primary object (looking for the simplest possible graphical representation), but in the course of investigation considering directed graphs may prove to be useful or even more natural. Therefore all the problems and results should be always viewed also from this point of view.

We note that the results proved in this paper can be easily formulated and proved for directed supergraphs, as well. In particular, every permutation group is represented by a directed supergraph (the bound of the order may be get a little bit smaller), and one may consider directed version of graphical complexity denoted, say, by $\mathrm{dgc}(G)$. Obviously, $\mathrm{dgc}(G) \leq \mathrm{gc}(G)$, and it is not difficult to provide examples when the equality holds.

Generally it would be interesting to compare both the measures more precisely. Can $\mathrm{gc}(G)$ can be bounded in terms of $\mathrm{dgc}(G)$? When does the strict inequality holds?

To close the paper, let us also remark that the whole new area of research, which may turn out to be interesting, at least in some aspects, is that of **supergraphs** on their own. Many questions asked naturally for graphs may be asked for supergraphs, as well.

The list of references below contains also items to which we do not refer directly, but which concern concrete permutation groups and may be useful for studying the suggested problems. The author's preprints are available at *http://www.math.uni.wroc.pl/∼kisiel*.

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