The ℓ^p norm of the discrete Hilbert transform

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Joint work with **Rodrigo Bañuelos** (Purdue University)

Vilnius, July 5, 2018

Hilbert transforms

Definition

The continuous Hilbert transform is defined by

$$Hf(x) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{f(x-z)}{z}\,dz$$

for appropriate functions $f : \mathbb{R} \to \mathbb{R}$.

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Similarly, the discrete Hilbert transform is given by

$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{a_{n-k}}{k}$$

for appropriate doubly infinite sequences $(a_n : n \in \mathbb{Z})$.

Main theorem

Theorem (Rodrigo Bañuelos, MK)

For $p \in (1, \infty)$ we have

$$\|\mathcal{H}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}.$$

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- $\|H\|_{L^p \to L^p} = \max\{\tan(\frac{\pi}{2p}), \cot(\frac{\pi}{2p})\}$ (S. Pichorides, 1972).
- The more challenging problem, which asks for the norm of

$$\mathcal{H}_{\mathrm{RT}} a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{a_{n-k}}{k + 1/2},$$

remains open.

Będlewo

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- Fortunately, they forgot to mention that some experts considered it to be rather difficult.

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Continuous Hilbert transform

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Throughout the talk, we assume that $p \in (1, \infty)$.

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- $\|\mathcal{H}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}$ for every p (R. Bañuelos, MK)

Discrete Hilbert transforms

- A number of discrete Hilbert transforms exist, each of them is a Fourier multiplier:

operator symbol

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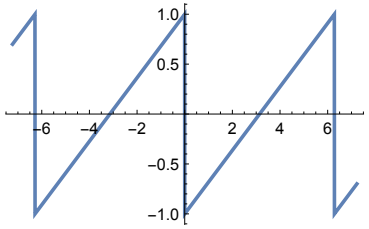

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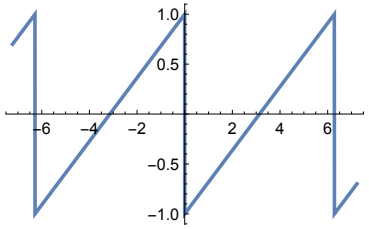

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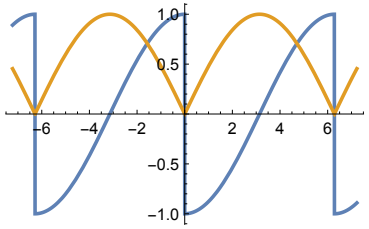

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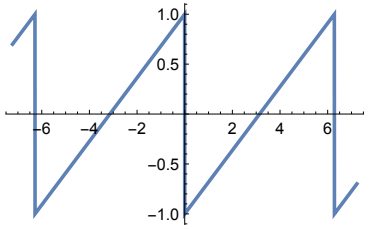

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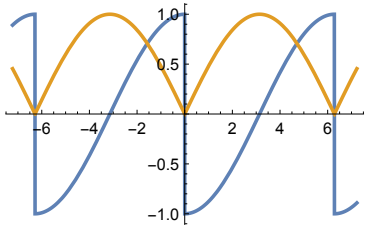

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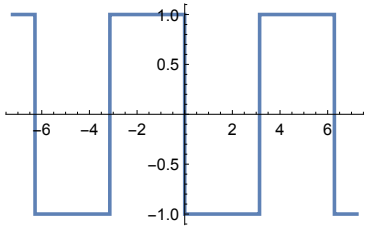

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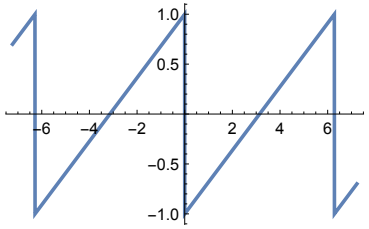

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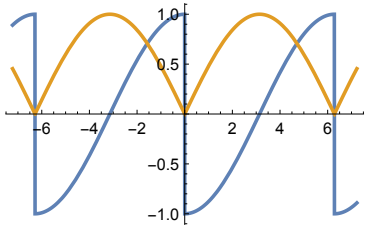

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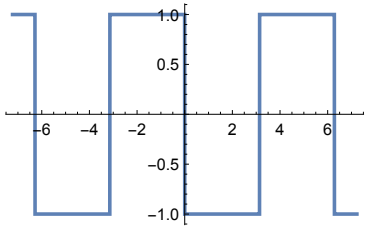

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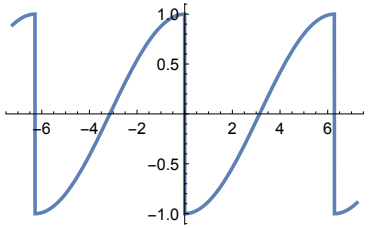

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$\mathcal{H} a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k}$	$\leftrightsquigarrow$	$-\, i(1 - \lvert t \rvert / \pi) \operatorname{sign} t;$
$\mathcal{H}_{\mathrm{RT}} a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{a_{n-k}}{k + 1/2}$	$\leftrightsquigarrow$	$-\, i e^{it/2} \operatorname{sign} t;$
$\mathcal{H}_{\mathrm{K}} a_n = \frac{2}{\pi} \sum_{k \in 2\mathbb{Z}+1} \frac{a_{n-k}}{k}$	$\leftrightsquigarrow$	$-\, i \operatorname{sign} t;$
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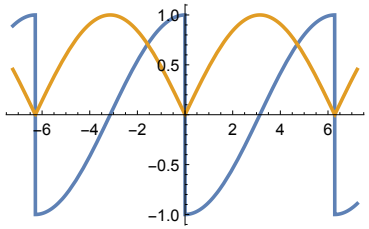
- A similar argument applies to $\mathcal{H}_{\mathrm{RT}}$, $\mathcal{H}_{\mathrm{K}}$ and $\mathcal{H}_{\mathrm{ADP}}$.
- This was first observed by E.C. Titchmarsh in 1926, as a part of his erroneous proof of equality of norms.

Riesz–Titchmarsh $\leftrightsquigarrow$ Kak–Hilbert

$$\mathcal{H}_{\mathrm{RT}} a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{a_{n-k}}{k + 1/2} \quad \leftrightsquigarrow$$



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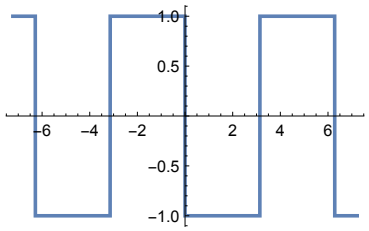

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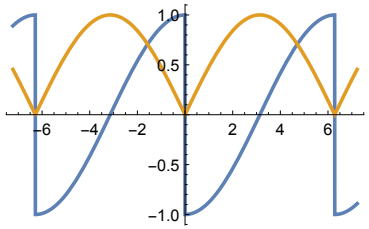

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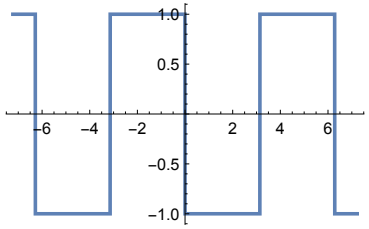

- The operators $\mathcal{H}_{\mathrm{RT}}$ and $\mathcal{H}_{\mathrm{K}}$ are equivalent:

$$\mathcal{H}_{\mathrm{K}} a_n = b_n \iff \begin{cases} \mathcal{H}_{\mathrm{RT}}[a_{2n}] = [b_{2n+1}], \\ \mathcal{H}_{\mathrm{RT}}[a_{2n-1}] = [b_{2n}]. \end{cases}$$

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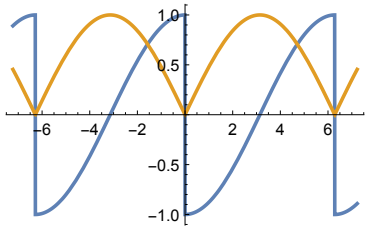

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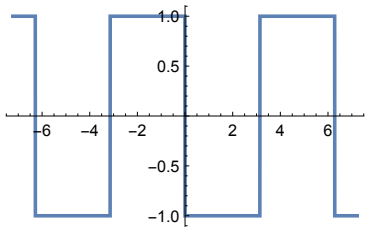

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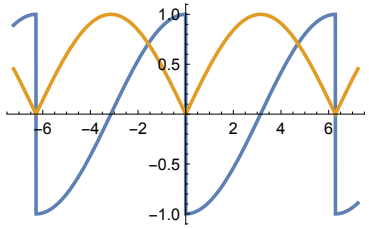

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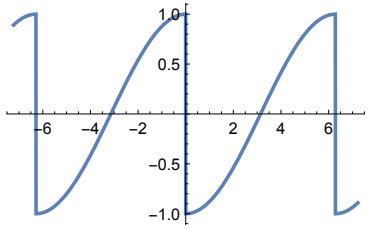

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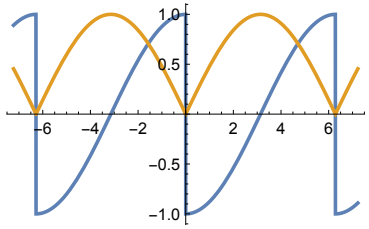

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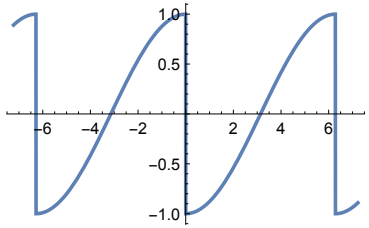

- The operator $\mathcal{H}_{\mathrm{ADP}}$ can be expressed in terms of $\mathcal{H}_{\mathrm{RT}}$:

$$\mathcal{H}_{\mathrm{ADP}}\, a_n = \tfrac{1}{2}\big(\mathcal{H}_{\mathrm{RT}}\, a_n + \mathcal{H}_{\mathrm{RT}}\, a_{n-1}\big).$$

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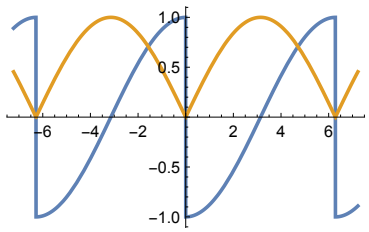

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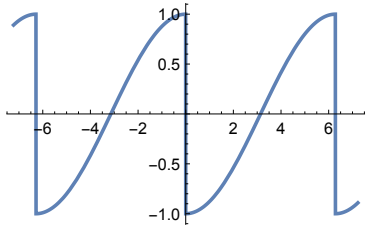


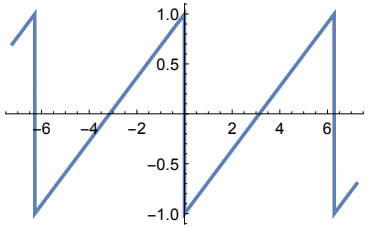
- The operator $\mathcal{H}_{\mathrm{ADP}}$ can be expressed in terms of $\mathcal{H}_{\mathrm{RT}}$:

$$\mathcal{H}_{\mathrm{ADP}}\, a_n = \tfrac{1}{2}\big(\mathcal{H}_{\mathrm{RT}}\, a_n + \mathcal{H}_{\mathrm{RT}}\, a_{n-1}\big).$$

- In particular, $\|\mathcal{H}_{\mathrm{ADP}}\|_{\ell^p\to\ell^p} \leqslant \|\mathcal{H}_{\mathrm{RT}}\|_{\ell^p\to\ell^p}$.

Kak–Hilbert $\rightsquigarrow$ Hilbert

$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k} \quad \leftrightsquigarrow$$



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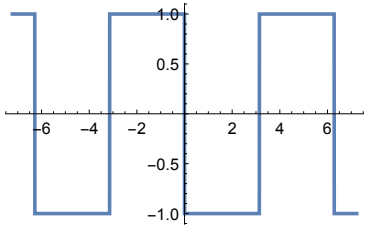

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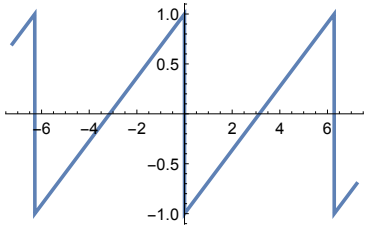

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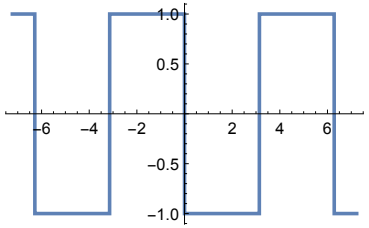

- The operator $\mathcal{H}$ can be expressed in terms of $\mathcal{H}_{\mathrm{K}}$:

$$\mathcal{H}a_n = \frac{1}{2} \mathcal{H}_{\mathrm{K}} a_n + \frac{4}{\pi^2} \sum_{k \in 2\mathbb{Z}+1} \frac{1}{k^2} \mathcal{H}_{\mathrm{K}} a_{n-k}.$$

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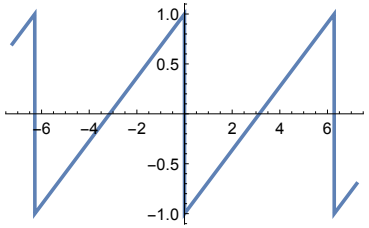

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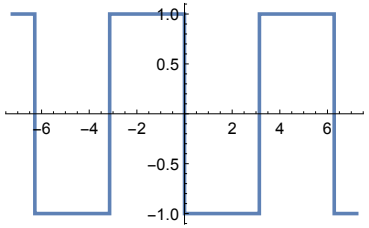

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Which discretisation is the right one?

$$\|H\|_{L^p \to L^p} \leqslant \left\{ \begin{array}{c} \|\mathcal{H}\|_{\ell^p \to \ell^p} \\ \|\mathcal{H}_{\mathrm{ADP}}\|_{\ell^p \to \ell^p} \end{array} \right\} \leqslant \|\mathcal{H}_{\mathrm{RT}}\|_{\ell^p \to \ell^p} = \|\mathcal{H}_{\mathrm{K}}\|_{\ell^p \to \ell^p}.$$

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- Our result is likely the second one of its kind, where the L^p norms of a singular integral operator and its ℓ^p discretisation are proved to be equal. A similar statement for second-order Riesz transforms was proved by K. Domolevo and S. Petermichl in 2014.

Hilbert transform and harmonic functions

- Let $f \in L^p$. For $y > 0$ we define the Poisson integrals

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x-z) \, \frac{y}{z^2+y^2} \, dz,$$

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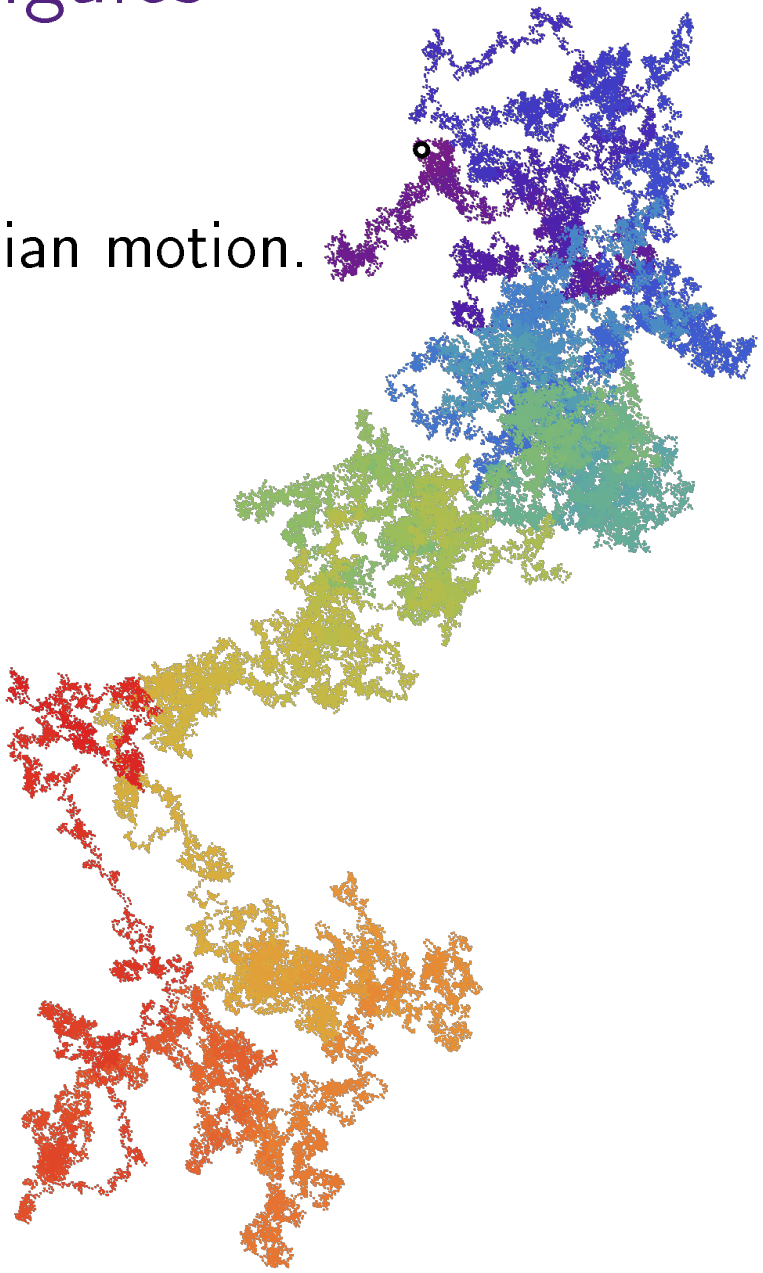
- Define $u(x,0) = f(x)$, $v(x,0) = Hf(x)$.

Harmonic functions and martingales

- Let B_t be the 2-D standard Brownian motion.



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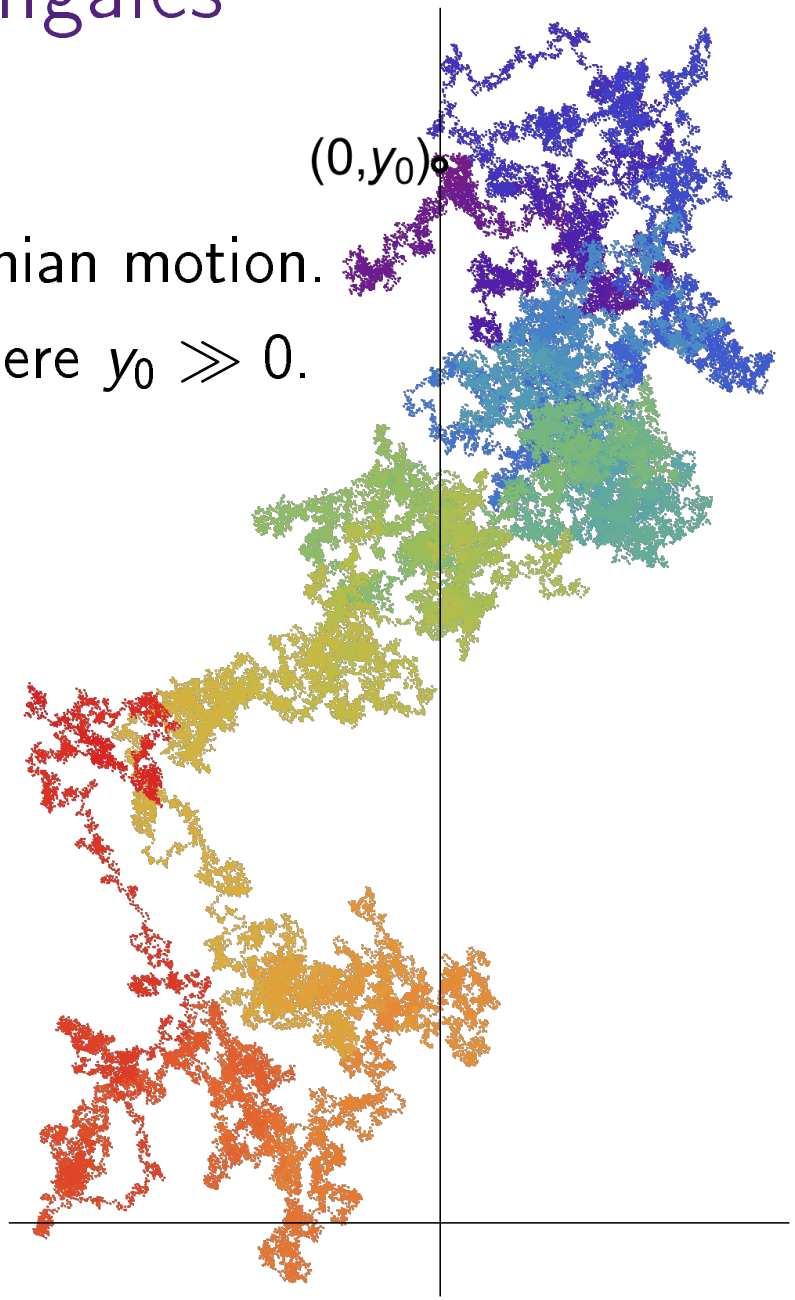

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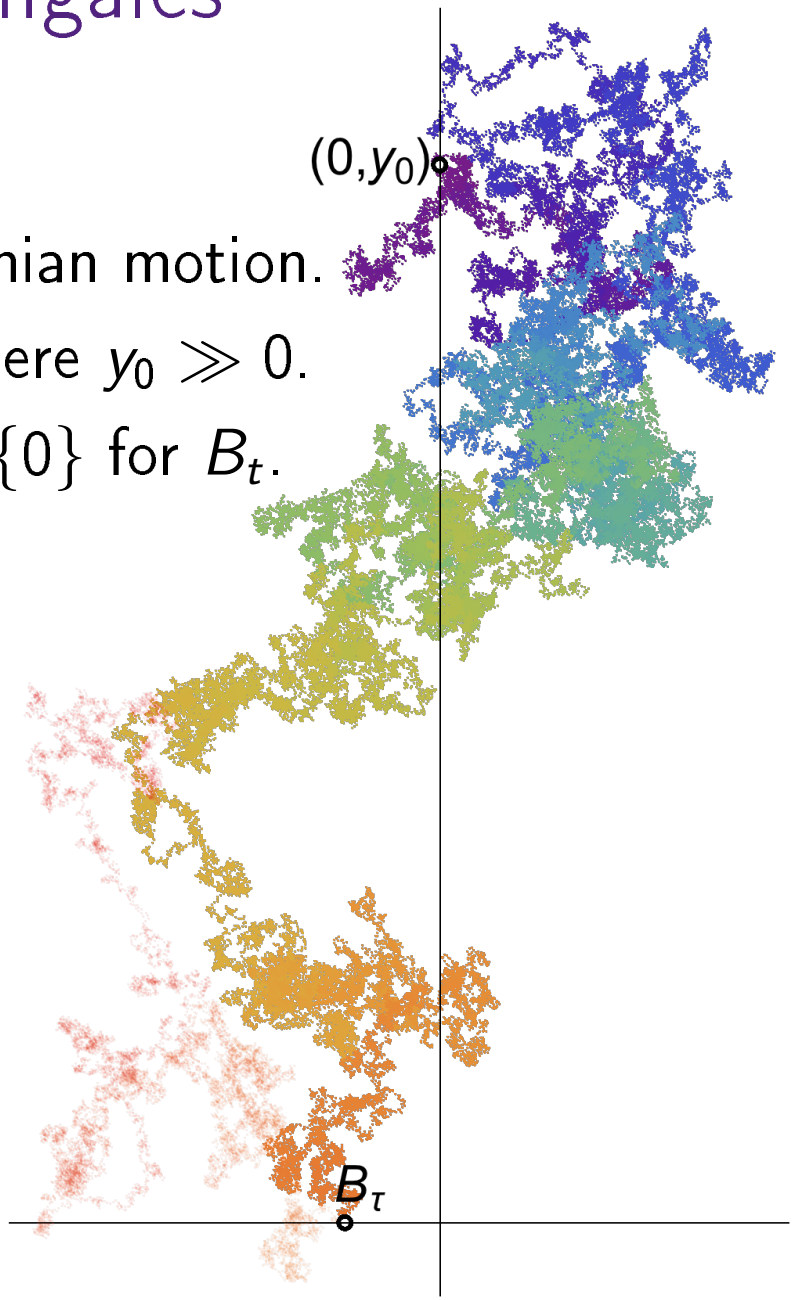

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- Indeed: by the Itô formula, for $t < \tau$ we have

$$dM_t = \nabla u(B_t) \cdot dB_t,$$
$$d[M]_t = |\nabla u(B_t)|^2 dt.$$

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and

$$d[M, N]_t = \nabla u(B_t) \cdot \nabla v(B_t) dt = 0 dt$$

for $t < \tau$.

Burkholder's inequality

Theorem (R. Bañuelos, G. Wang, 1995)

If M_t and N_t are martingales and

- N_t is differentially subordinate to M_t:
$$d[N]_t \leqslant d[M]_t;$$
- M_t and N_t are orthogonal:
$$d[M,N]_t = 0dt,$$

then

$$\mathbb{E}|N_\infty - N_0|^p \leqslant (C_p)^p\, \mathbb{E}|M_\infty - M_0|^p,$$

with $C_p = \max\{\tan(\frac{\pi}{2p}), \cot(\frac{\pi}{2p})\}$.

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- Since $M_\infty = u(B_\tau) = f(B_\tau)$ and $N_\infty = v(B_\tau) = Hf(B_\tau)$, Burkholder's inequality implies that

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- We now pass to the limit as $y_0 \to \infty$.

Pichorides estimate

- Since B_τ has a Cauchy distribution on $\mathbb{R} \times \{0\}$, we have

$$\int_{-\infty}^{\infty} |Hf(x) - v(0, y_0)|^p \frac{y_0}{x^2 + y_0^2}\, dx$$
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$$\leqslant (C_p)^p \int_{-\infty}^{\infty} |f(x) - u(0, y_0)|^p \frac{y_0^2}{x^2 + y_0^2}\, dx.$$

- We multiply both sides by y_0 and pass to the limit as $y_0 \to \infty$ to get the Pichorides bound

$$\|Hf\|_{L^p}^p \leqslant (C_p)^p \|f\|_{L^p}^p.$$

Conditioned process

- The Brownian motion B_t hits the entire boundary $\mathbb{R} \times \{0\}$ as it exists the half-plane $\mathbb{R} \times (0, \infty)$.

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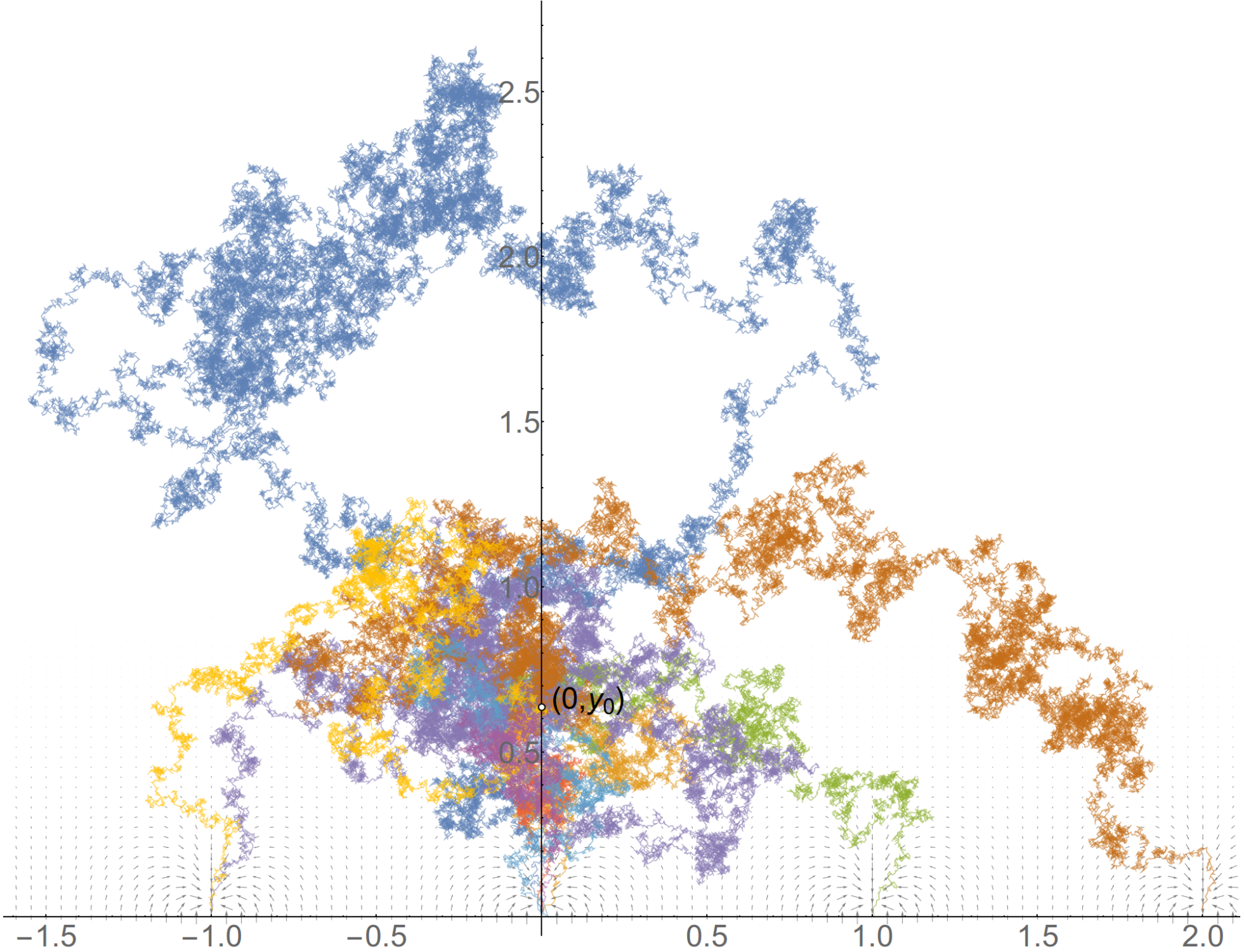
- We need a diffusion X_t which only hits the discrete subset of the boundary: $\mathbb{Z} \times \{0\}$.
- The process X_t can be defined as the limit as $\varepsilon \to 0^+$ of the Brownian motion B_t conditioned on the event

$$B_\tau \in \left(\bigcup_{k \in \mathbb{Z}} (k - \varepsilon, k + \varepsilon) \right) \times \{0\}.$$


2.5
2.0
1.5
1.0
0.5
$(0, y_0)$
−1.5
−1.0
−0.5
0.5
1.0
1.5
2.0

Stochastic differential equation

- More rigorously: X_t is defined as the Brownian motion B_t conditioned in the sense of Doob by the harmonic function

$$h(x,y) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{y}{(x-k)^2+y^2} = \frac{\sinh(2\pi y)}{\cosh(2\pi y)-\cos(2\pi x)}\,.$$

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- Equivalently: X_t is a solution of the SDE

$$dX_t = dB_t + \frac{\nabla h(X_t)}{h(X_t)}\,dt.$$

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- There is no notion of a conjugate X_t-harmonic function!

Martingale transform

- Since $M_t = u(X_{\min\{t,\tau\}})$ and

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$$\nabla u \cdot \frac{\nabla h}{h} + \tfrac{1}{2}\Delta u = 0,$$

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Estimate of the ℓ^p norm of *some* transform

- As in the continuous case, by passing to the limit as $y_0 \to \infty$, we find that

$$\|b_n\|_{\ell^p}^p \leqslant (C_p)^p \|a_n\|_{\ell^p}^p,$$

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- If we write $b_n = \tilde{\mathcal{H}} a_n$, then we have $\|\tilde{\mathcal{H}}\|_{\ell^p \to \ell^p} \leqslant C_p$.

Convolution

- The operator $\tilde{\mathcal{H}}$ is a convolution operator with kernel

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- Items (3) and (4) do not preserve the ℓ^2 norm, and therefore no similar argument can be given for the unitary operators $\mathcal{H}_{\mathrm{RT}}$ and $\mathcal{H}_{\mathrm{K}}$.
- The sequence $\mathcal{H}_{\mathrm{ADP}} a_n$ can be expressed as a convolution of $\tilde{\mathcal{H}} a_n$ with some sequence ϱ_n, but ϱ_n contains negative entries. For this reason a similar argument leads to a weaker estimate of $\|\mathcal{H}_{\mathrm{ADP}}\|_{\ell^p \to \ell^p}$ than conjectured.

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Conjecture 3

The discrete Riesz transform

$$\mathcal{R}_j a_n = c_d \sum_{k \in \mathbb{Z}^d \setminus \{(0,0,\ldots,0)\}} a_{n-k} \frac{k_j}{|k|^{d+1}}$$

satisfies $\|\mathcal{R}_j\|_{\ell^p \to \ell^p} = C_p$.