Discrete Hilbert transform

Mateusz Kwaśnicki

Wrocław University of Science and Technology

`mateusz.kwasnicki@pwr.edu.pl`

Joint work with **Rodrigo Bañuelos** (Purdue University)

Ryll-Nardzewski Day
12 June 2023

Hilbert transform

Definition (Hilbert)

The continuous Hilbert transform is defined by

$$Hf(x) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{f(x-s)}{s}\,ds$$

for appropriate functions $f : \mathbb{R} \to \mathbb{R}$.

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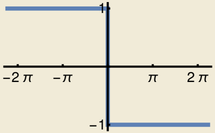
Theorem (Hilbert)

If $\mathcal{F}f$ denotes the Fourier transform of f, then

$$\mathcal{F}[Hf](\xi) = (-i\,\mathrm{sign}\,\xi)\mathcal{F}f(\xi)$$

for $\xi \in \mathbb{R}$.



Naive discrete Hilbert transform

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$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k}$$

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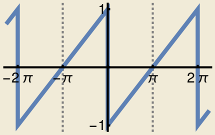
Theorem (Fourier?)

If $\mathcal{F}a$ denotes the Fourier series with coefficients a_n, then

$$\mathcal{F}[\mathcal{H}a_n](\xi) = (-i \operatorname{sign} \xi)(1 - \tfrac{1}{\pi}|\xi|)\mathcal{F}[a_n](\xi)$$

for $\xi \in (-\pi, \pi)$.



Kak–Hilbert transform

Definition (Ferrand and Duffin)

The Kak–Hilbert transform is given by

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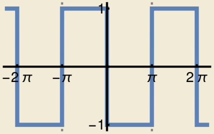

Riesz–Titchmarsh transform

Definition (Titchmarsh)

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If $\mathcal{F}[a_n]$ denotes the Fourier series with coefficients a_n, then

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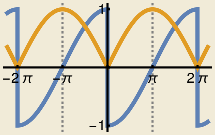

Arcozzi–Domelevo–Petermichl transform

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The Arcozzi–Domelevo–Petermichl transform is given by

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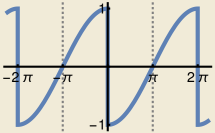
Theorem (Fourier?)

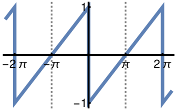
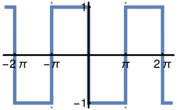
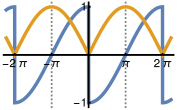
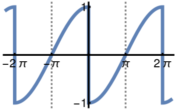
If $\mathcal{F}[a_n]$ denotes the Fourier series with coefficients a_n, then

$$\mathcal{F}[\mathcal{ADP} a_n](\xi) = (-i \operatorname{sign} \xi) \cos \tfrac{\xi}{2}\, \mathcal{F}[a_n](\xi)$$

for $\xi \in (-\pi, \pi)$.



Too many discrete analogues of the Hilbert transform

operator		Fourier symbol
$\mathcal{H}a_n = \dfrac{1}{\pi} \displaystyle\sum_{k \in \mathbb{Z} \setminus \{0\}} \dfrac{a_{n-k}}{k}$	$\leftsquigarrow\!\!\rightsquigarrow$	
$\mathcal{K}a_n = \dfrac{2}{\pi} \displaystyle\sum_{k \in 2\mathbb{Z}+1} \dfrac{a_{n-k}}{k}$	$\leftsquigarrow\!\!\rightsquigarrow$	
$\mathcal{R}a_n = \dfrac{1}{\pi} \displaystyle\sum_{k \in \mathbb{Z}} \dfrac{a_{n-k}}{k + \frac{1}{2}}$	$\leftsquigarrow\!\!\rightsquigarrow$	
$\mathcal{ADP}a_n = \dfrac{1}{\pi} \displaystyle\sum_{k \in \mathbb{Z}} \dfrac{k\, a_{n-k}}{k^2 - \frac{1}{4}}$	$\leftsquigarrow\!\!\rightsquigarrow$	

Elementary reductions

Question (Hilbert)

Is there a constant C such that if b_n is the transform of a_n, then

$$\|b_n\|_p \leqslant C\|a_n\|_p.$$

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$$\mathcal{ADP}a_n = \tfrac{1}{2}(\mathcal{R}a_n + \mathcal{R}a_{n-1}).$$

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$\mathcal{R} \Leftrightarrow \mathcal{K}$

$$b_n = \mathcal{K}a_n \iff \begin{cases} b_{2n+1} = \mathcal{R}[a_{2n}], \\ b_{2n} = \mathcal{R}[a_{2n-1}]. \end{cases}$$

Slightly less elementary reduction

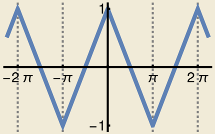
$\mathcal{K} \Rightarrow \mathcal{H}$

We have

$$b_n = \mathcal{J}[\mathcal{K}a_n] \iff \begin{cases} b_{2n} = \mathcal{H}a_n, \\ b_{2n+1} = 0, \end{cases}$$

where

$$\mathcal{J}a_n = \frac{4}{\pi^2} \sum_{k \in 2\mathbb{Z}+1} \frac{a_{n-k}}{k^2}$$



has norm 1 (convolution with a probability kernel).

Much less elementary reduction

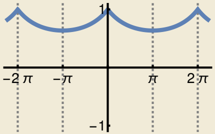
$\mathcal{ADP} \Rightarrow \mathcal{H}$

We have

$$\mathcal{H}a_n = \mathcal{J}[\mathcal{ADP}a_n],$$

where

$$\mathcal{J}a_n = \frac{1}{2\pi^2}\sum_{k\in\mathbb{Z}}\left(\psi_1\left(\tfrac{1}{4}+\tfrac{n}{2}\right)-\psi_1\left(\tfrac{3}{4}+\tfrac{n}{2}\right)\right)a_{n-k}$$



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Here $\psi_1 = (\log\Gamma)''$ is the trigamma function.

Approximation

$\mathcal{H} \Rightarrow H$

For appropriate functions $f : \mathbb{R} \to \mathbb{R}$ we have

$$Hf(x) = \lim_{\delta \to 0^+} \delta\, \mathcal{H}[f(n\delta)]$$

with $n = \lfloor \frac{x}{\delta} \rfloor$. Thus,

$$\|\mathcal{H} a_n\|_p \leqslant C\|a_n\|_p$$

implies

$$\|Hf\|_p \leqslant C\|f\|_p.$$

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Question (Riesz, Titchmarsh)

Are they all equal?

L^p bounds for the Hilbert transform

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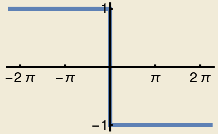

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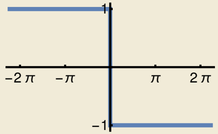

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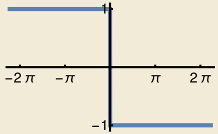

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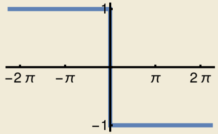

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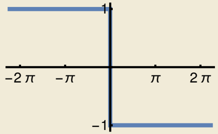

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- $\|H\|_{L^p \to L^p} = \max\{\tan(\frac{\pi}{2p}), \cot(\frac{\pi}{2p})\}$ (Pichorides and Cole)

  ($p = 2, 4, 8, 16, \dots$: Gohberg–Krupnik)

L^p bounds for the discrete analogues (1/3)

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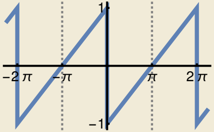

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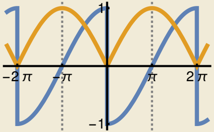

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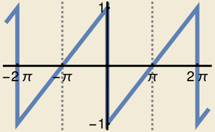

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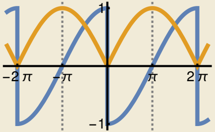

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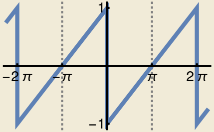

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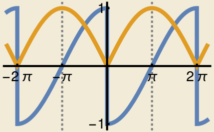

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- $\|\mathcal{R}\|_{\ell^2 \to \ell^2} = 1$ and $\mathcal{R}$ is unitary (Titchmarsh)

L^p bounds for the discrete analogues (1/3)

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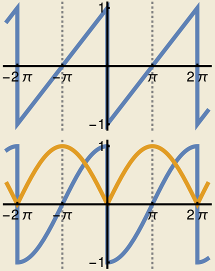

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- $\mathcal{R}$ extends continuously to L^p for $p \in (1, \infty)$ (Titchmarsh and M. Riesz)

L^p bounds for the discrete analogues (2/3)

$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k} \quad \leftrightsquigarrow$$

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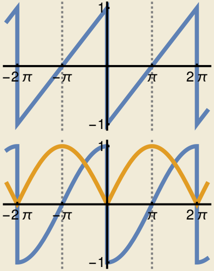

- $\|\mathcal{R}\|_{\ell^p \to \ell^p} \geqslant \|H\|_{L^p \to L^p}$ (Titchmarsh)

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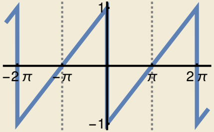

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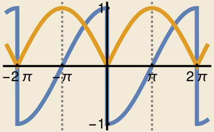

- $\|\mathcal{R}\|_{\ell^p \to \ell^p} \geqslant \|H\|_{L^p \to L^p}$ (Titchmarsh)
- $\|\mathcal{R}\|_{\ell^p \to \ell^p} \leqslant \|H\|_{L^p \to L^p}$, incorrect proof (Titchmarsh)

Titchmarsh, *Reciprocal formulae involving series and integrals*

The paper appeared in *Mathematische Zeitschrift* 25 in 1926.
The next issue contained the following letter.

Correction.

Von

E. C. Titchmarsh.

I. In paragraph 4 of my paper on 'Reciprocal formulae involving series and integrals' (Math. Zeitschr. 25 (1926), pp. 321—347), the proof that $N_p \leq N_p'$ is incorrect, and should be deleted. This does not affect anything else in the paper.

II. In obtaining the inequality which follows formula (2.32), we have assumed that (4a) as well as (3a) holds for the particular value of p taken. This merely involves a slight rearrangement of the proof.

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Czesław Ryll-Nardzewski was born on 7 October 1926.

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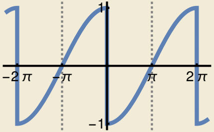

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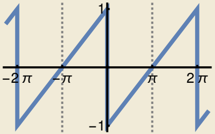

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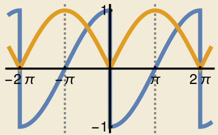

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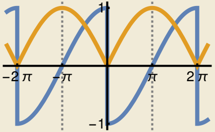

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Motivation: discrete analogues in harmonic analysis
(Magyar–Stein–Waigner, Pierce)

Będlewo

I learned about the problem at the Probability and Analysis conference in Będlewo (15–19 May 2017).

source: IMPAN
impan.pl

Będlewo

During a BBQ dinner, with free beer and a bonfire, Rodrigo Bañuelos and Eero Saksman invited me to join their fireside chat, and told me about it.

source: SACNAS
sacnas.org

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helsinki.fi

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They forgot to mention that it was a 90-year-old conjcecture.

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Main results

Theorem (Bañuelos–K)

For $p \in (1, \infty)$ we have

$$\|\mathcal{H}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}.$$

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For $p = 2, 4, 6, 8, \ldots$ we have

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Hilbert transform and harmonic functions

- For $y > 0$ define the Poisson integrals

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x-s)\,\frac{y}{s^2+y^2}\,ds,$$
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- The boundary values of u and v are given by

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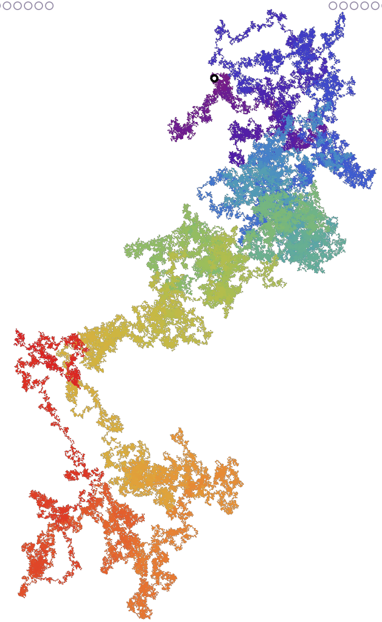
Harmonic functions and martingales

- Let B_t be the 2-D standard Brownian motion.



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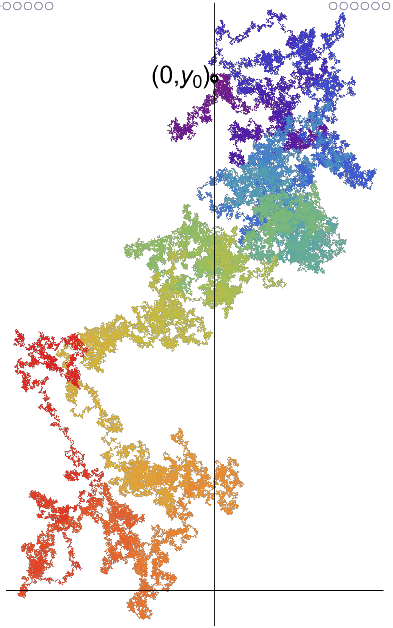

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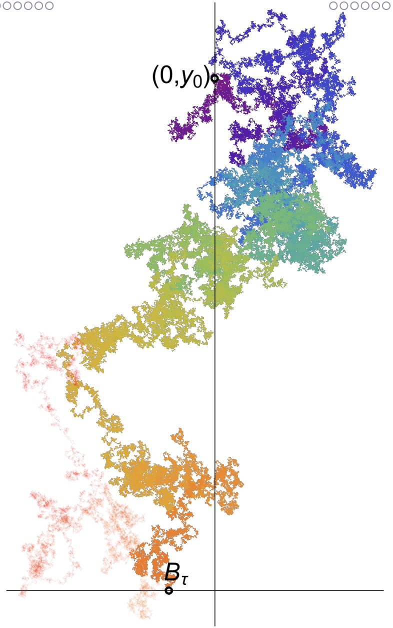

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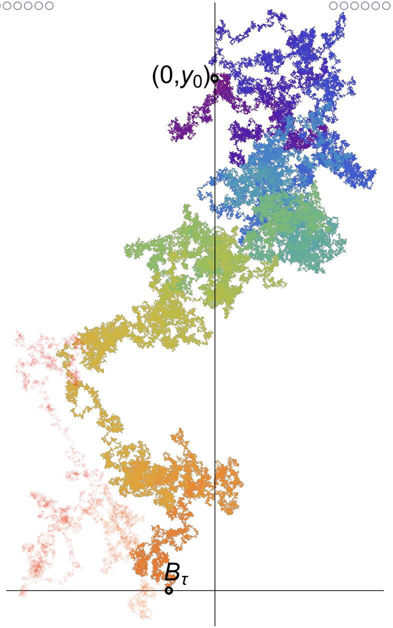

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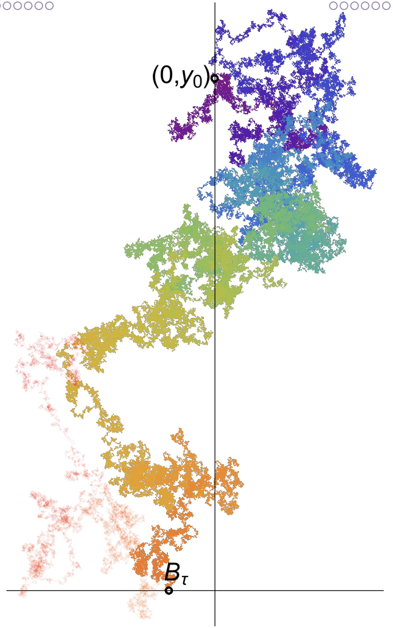
- Indeed:

$$dM_t = \nabla u(B_t) \cdot dB_t,$$
$$d[M]_t = |\nabla u(B_t)|^2 dt.$$



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and

$$d[M, N]_t = \nabla u(B_t) \cdot \nabla v(B_t) dt = 0dt$$

for $t < \tau$.

Burkholder's inequality

Theorem (Bañuelos–Wang)

If M_t and N_t are martingales and

- N_t is differentially subordinate to M_t:
$$d[N]_t \leqslant d[M]_t;$$
- M_t and N_t are orthogonal:
$$d[M,N]_t = 0dt,$$

then

$$\mathbb{E}|N_\tau - N_0|^p \leqslant (C_p)^p\, \mathbb{E}|M_\tau - M_0|^p,$$

with $C_p = \max\{\tan(\frac{\pi}{2p}), \cot(\frac{\pi}{2p})\}$.

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- Pass to the limit as $y_0 \to \infty$ to get

$$\|Hf\|_p^p \leqslant (C_p)^p \, \|f\|_p^p.$$

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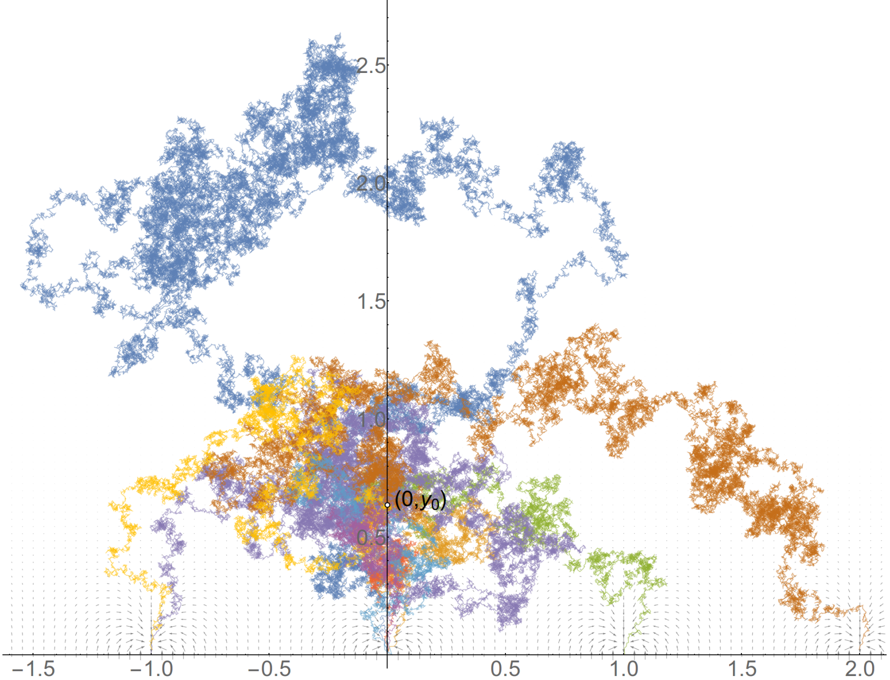
- Construct Z_t by conditioning the Brownian motion so that

$$B_\tau \in \bigcup_{k\in\mathbb{Z}} (k-\varepsilon, k+\varepsilon) \times \{0\},$$

and passing to the limit as $\varepsilon \to 0^+$. (Doob)



2.5
2.0
1.5
1.0
0.5
(0,y0)
−1.5
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  (Initially I made a sign error and I got $\tilde{\mathcal{H}} = \mathcal{H} \ldots$)

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- Solution: Prove that

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  (Had I not sent an enthusiastic email to Rodrigo before noticing the error,
  I would have never found enough motivation to do that.)

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- Steps (3) and (4) do not preserve the ℓ^2 norm.
- Therefore, no similar argument can be given for the unitary operator $\mathcal{R}$.

Factorization

- Replace $\mathcal{H}$ by an equivalent operator, denoted again $\mathcal{H}$, analogous to $\mathcal{K}$:

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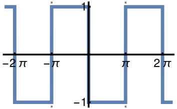

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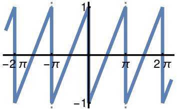

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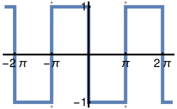

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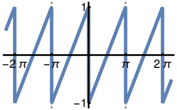

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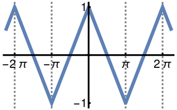

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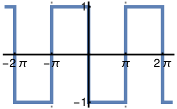

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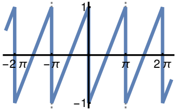

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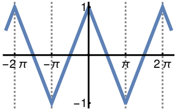
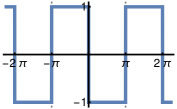


- $\mathcal{J}$ is a convolution operator with a probability kernel.

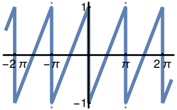
Factorization

- Replace $\mathcal{H}$ by an equivalent operator, denoted again $\mathcal{H}$, analogous to $\mathcal{K}$:

$$\mathcal{K}a_n = \frac{2}{\pi} \sum_{k \in 2\mathbb{Z}+1} \frac{a_{n-k}}{k} \quad \leftrightsquigarrow$$



$$\mathcal{H}a_n = \frac{2}{\pi} \sum_{k \in 2\mathbb{Z}\setminus\{0\}} \frac{a_{n-k}}{k} \quad \leftrightsquigarrow$$



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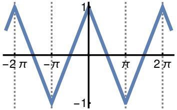

- $\mathcal{J}$ is a convolution operator with a probability kernel.
- We have $\mathcal{H}a_n = \mathcal{J}[\mathcal{K}a_n]$.

Product rule

Lemma (Titchmarsh)

We have

$$\mathcal{K}a_n \cdot \mathcal{K}b_n = \mathcal{K}[\mathcal{H}a_n \cdot b_n] + \mathcal{K}[a_n \cdot \mathcal{H}b_n] + \mathcal{I}[a_n \cdot b_n].$$

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- Compare with the cotangent of sum formula

$$\cot\alpha \cot\beta = \cot(\alpha+\beta)\cot\alpha + \cot(\alpha+\beta)\cot\beta + 1.$$

$p \rightsquigarrow 2p$

- By the product rule:

$$(\mathcal{K}a_n)^2 = 2\mathcal{K}[\mathcal{H}a_n \cdot a_n] + \mathcal{I}[a_n^2].$$

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- $p = 2 \;\rightsquigarrow\; p = 4 \;\rightsquigarrow\; p = 8 \;\rightsquigarrow\; \dots$
- Note: we can replace $\|\mathcal{H}\|_{\ell^p\to\ell^p} = \cot\frac{\pi}{2p}$ by $\|\mathcal{H}\|_{\ell^p\to\ell^p} \leqslant \|\mathcal{K}\|_{\ell^p\to\ell^p}$.

$p \rightsquigarrow 3p$ (1/2)

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$$\begin{aligned}(\mathcal{K}a_n)^3 &= 2\mathcal{K}a_n \cdot \mathcal{K}[\mathcal{H}a_n \cdot a_n] + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2] \\ &= 2\mathcal{K}[(\mathcal{H}a_n)^2 \cdot a_n] + 2\mathcal{K}[a_n \cdot \mathcal{H}[\mathcal{H}a_n \cdot a_n]] \\ &\qquad + 2\mathcal{I}[\mathcal{H}a_n \cdot a_n^2] + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2].\end{aligned}$$

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$$\big(\|\mathcal{K}\|_{\ell^p\to\ell^p}\big)^3 \leqslant 2\cot\frac{3\pi}{2p}\cot^2\frac{\pi}{p} + 2\cot\frac{3\pi}{2p}\cot\frac{\pi}{p}\cot\frac{\pi}{2p} + 2\cot\frac{\pi}{2p} + \|\mathcal{K}\|_{\ell^p\to\ell^p}.$$

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- After a short calculation, this implies that $\|\mathcal{K}\|_{\ell^p\to\ell^p} \leqslant \cot\frac{\pi}{2p}$.
- Note: we use $\|\mathcal{H}\|_{\ell^{p/2}\to\ell^{p/2}} = \cot\frac{\pi}{p}$ in an essential way.

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- Enumeration of all intermediate terms is a non-obvious task.
- To get things under control, we introduce frames, skeletons and buildings.