The interplay between spectral theory and Lévy processes

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Starting point

If L is a self-adjoint operator on a finitely-dimensional vector space, then

$$\langle Lu, v\rangle = \sum_{j=1}^{N} \lambda_j \langle u, \varphi_j\rangle \langle \varphi_j, v\rangle \tag{EE}$$

for a complete orthogonal set of eigenvectors φ_j:

$$L\varphi_j = \lambda_j \varphi_j$$

EE stands for ‘eigenfunction expansion’.

$$L = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \\ \vdots \\ \overline{\varphi_n} \end{pmatrix}$$

Hilbert–Schmidt theory

If L is a compact self-adjoint operator on a Hilbert space, then

$$\langle Lu, v\rangle = \sum_{j=1}^{\infty} \lambda_j \langle u, \varphi_j\rangle \langle \varphi_j, v\rangle \tag{EE}$$

for a complete orthogonal set of eigenvectors φ_j:

$$L\varphi_j = \lambda_j \varphi_j$$

EE stands for 'eigenfunction expansion'.

$$L = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \bar{\varphi}_1 \\ \bar{\varphi}_2 \\ \vdots \end{pmatrix}$$

Spectral theorem

If L is a self-adjoint operator on a Hilbert space, then

$$\langle Lu, v\rangle = \int_Z \lambda d\langle E_\lambda u, v\rangle$$

for a resolution of identity E_λ.

Spectral theorem for Carleman's operators (Gårding, 1954)

If L is a self-adjoint Carleman operator on $L^2(X)$, then

$$\langle Lu, v\rangle = \int_Z \lambda_r \langle u, \varphi_r\rangle\langle\varphi_r, v\rangle dr \tag{GEE}$$

for a set of generalised eigenfunctions φ_r:

$$L\varphi_r(x) = \lambda_r\varphi_r(x)$$

Note: typically $\varphi_r \notin L^2(X)$

Carleman operators have 'nice' kernels:

$$Lu(x) = \int K(x,y)u(y)dy$$

with $\|K(x,\cdot)\|_2 < \infty$ for almost all x.

Non-normal case

If L is an arbitrary operator on a finitely-dimensional vector space, then

L can be written in a Jordan normal form.

Optimistic scenario

If we are lucky:

$$L = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} \overline{\psi_1} \\ \overline{\psi_2} \\ \vdots \\ \overline{\psi_n} \end{pmatrix}$$

$$\langle Lu, v\rangle = \sum_{j=1}^{N} \lambda_j \langle u, \psi_j\rangle\langle \varphi_j, v\rangle \qquad \text{(EE)}$$

for a complete set of eigenvectors φ_j and co-eigenvectors ψ_j:

$$L\varphi_j = \lambda_j \varphi_j, \qquad L^*\psi_j = \overline{\lambda}_j \psi_j$$

F. Riesz's theory

If L is a compact operator on a Hilbert space, then

L can be 'written' in a Jordan normal form.

Optimistic scenario

If we are lucky:

$$L = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \overline{\psi_1} \\ \overline{\psi_2} \\ \vdots \end{pmatrix}$$

$$\langle Lu, v \rangle = \sum_{j=1}^{\infty} \lambda_j \langle u, \psi_j \rangle \langle \varphi_j, v \rangle \tag{EE}$$

for a complete set of eigenvectors φ_j and co-eigenvectors ψ_j:

$$L\varphi_j = \lambda_j \varphi_j, \qquad L^* \psi_j = \overline{\lambda}_j \psi_j$$

Overly optimistic scenario?

If L is an appropriate operator on $L^2(X)$, we hope for

$$\langle Lu, v\rangle = \int_Z \lambda_r \langle u, \psi_r\rangle\langle\varphi_r, v\rangle dr \tag{GEE}$$

for a set of generalised eigenfunctions φ_r and generalised co-eigenfunctions ψ_r:

$$L\varphi_r = \lambda_r\varphi_r, \qquad L^*\psi_r = \overline{\lambda}_r\psi_r$$

Markov chains

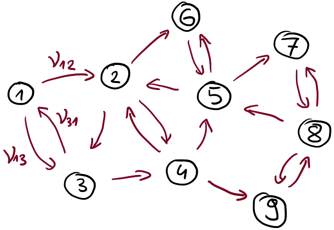
The generator of a continuous-time Markov chain:

$$L = \begin{pmatrix} -\nu_1 & \nu_{12} & \nu_{13} & \cdots & \nu_{1n} \\ \nu_{21} & -\nu_2 & \nu_{23} & \cdots & \nu_{2n} \\ \nu_{31} & \nu_{32} & -\nu_3 & \cdots & \nu_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_{n1} & \nu_{n2} & \nu_{n3} & \cdots & -\nu_n \end{pmatrix}$$

with $\nu_{ij} \geqslant 0$ and $\nu_i = \sum_{j \neq i} \nu_{ij}$.

Its transition probabilities:

$$P_t = \exp(tL)$$

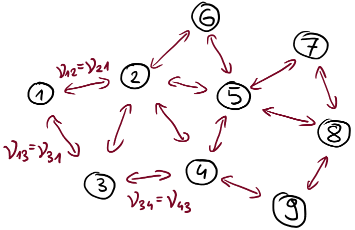


Symmetric Markov chains

For a symmetric Markov chain:

$$\langle Lu, v\rangle = \sum_{j=1}^{N} (-\lambda_j)\langle u, \varphi_j\rangle\langle \varphi_j, v\rangle$$
$$\langle P_t u, v\rangle = \sum_{j=1}^{N} e^{-t\lambda_j}\langle u, \varphi_j\rangle\langle \varphi_j, v\rangle \qquad \text{(EE)}$$

with $\lambda_j \geqslant 0$.



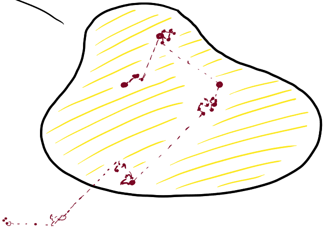
Markov processes

The generator of a Markov process, e.g.:

$$Lu(x) = \Delta u(x) \quad \text{(Laplace operator)}$$

or

$$Lu(x) = \int (u(y) - u(x))\nu(x,y)dy$$



Its transition operators:

$$P_t u(x) = \exp(tL)u(x)$$

$$= \int p_t(x,y)u(y)dy$$

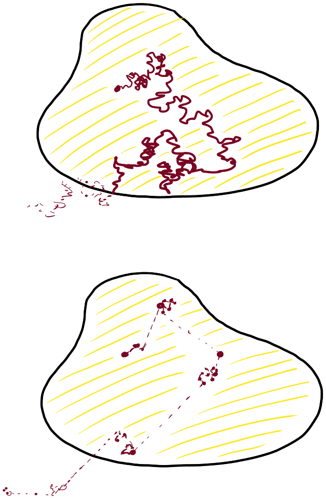
Symmetric Markov processes

If P_t are compact operators:

$$\langle Lu, v\rangle = \sum_{j=1}^{\infty} (-\lambda_j)\langle u, \varphi_j\rangle\langle \varphi_j, v\rangle$$

$$\langle P_t u, v\rangle = \sum_{j=1}^{\infty} e^{-t\lambda_j}\langle u, \varphi_j\rangle\langle \varphi_j, v\rangle \qquad \text{(EE)}$$

with $\lambda_j \geqslant 0$.



Symmetric Markov processes (Getoor, 1959)

If P_t are Carleman operators:

$$\begin{aligned} \langle Lu, v\rangle &= \int_Z (-\lambda_r)\langle u, \varphi_r\rangle\langle \varphi_r, v\rangle dr \\ \langle P_t u, v\rangle &= \int_Z e^{-t\lambda_r}\langle u, \varphi_r\rangle\langle \varphi_r, v\rangle dr \end{aligned} \tag{GEE}$$

with $\lambda_r \geqslant 0$.

Non-symmetric Markov processes

For a non-symmetric Markov process, if P_t are compact operators:

we only know what follows from F. Riesz's theory.

Optimistic scenario

If we are lucky:

$$\langle Lu, v\rangle = \sum_{j=1}^{\infty} (-\lambda_j)\langle u, \psi_j\rangle\langle \varphi_j, v\rangle$$

$$\langle P_t u, v\rangle = \sum_{j=1}^{\infty} e^{-t\lambda_j}\langle u, \psi_j\rangle\langle \varphi_j, v\rangle$$

with $\operatorname{Re} \lambda_j \geqslant 0$.

Non-symmetric case

For a general non-symmetric Markov process:

we know virtually nothing.

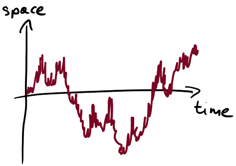
Self-adjoint example

The 1-D Brownian motion:

$$Lu(x) = u''(x)$$

Plancherel's formula:

$$\langle Lu, v\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-\xi^2)\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$$



Equivalently:

$$\langle Lu, v\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-\xi^2)\langle u, \varphi_\xi\rangle\langle \varphi_\xi, v\rangle d\xi \qquad \text{(GEE)}$$

with $\varphi_\xi(x) = e^{i\xi x}$.

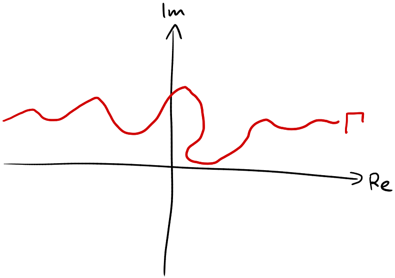
Non-uniqueness of GEE

By contour deformation:

$$\langle Lu, v\rangle = \frac{1}{2\pi}\int_\Gamma (-\xi^2)\langle u, \psi_\xi\rangle\langle\varphi_\xi, v\rangle d\xi \tag{GEE}$$

with $\varphi_\xi(x) = e^{i\xi x}$, $\psi_\xi(x) = e^{i\bar{\xi}x}$, as long as Γ goes 'from $-\infty$ to $+\infty$'.

It is clear that $\Gamma = (-\infty, \infty)$ is 'optimal'.



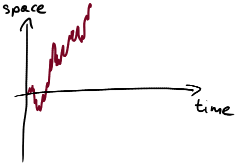
Normal example

The 1-D Brownian motion with drift:

$$Lu(x) = u''(x) + 2bu'(x)$$

Plancherel's formula:

$$\langle Lu, v\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-\xi^2 + 2bi\xi)\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$$



Equivalently:

$$\langle Lu, v\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-\xi^2 + 2bi\xi)\langle u, \varphi_\xi\rangle\langle\varphi_\xi, v\rangle d\xi \tag{GEE}$$

with $\varphi_\xi(x) = e^{i\xi x}$.

Non-uniqueness of GEE

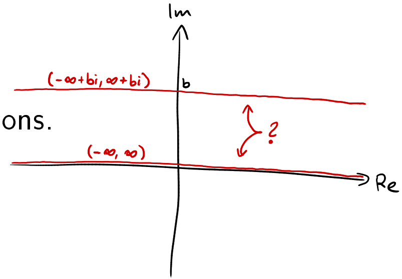
By contour deformation:

$$\langle Lu, v\rangle = \frac{1}{2\pi}\int_\Gamma (-\xi^2 + 2bi\xi)\langle u, \psi_\xi\rangle\langle \varphi_\xi, v\rangle d\xi \qquad \text{(GEE)}$$

with $\varphi_\xi(x) = e^{i\xi x}$, $\psi_\xi(x) = e^{i\bar{\xi}x}$, as long as Γ goes ‘from $-\infty$ to $+\infty$’.

The choice of Γ is no longer clear:

- $\Gamma = (-\infty, \infty)$ leads to $\psi_\xi = \varphi_\xi$ bounded;
- $\Gamma = (-\infty + bi, \infty + bi)$ leads to real-valued expressions.



Non-normal example

The killed 1-D Brownian motion with drift in $(0,\infty)$:

$$Lu(x) = u''(x) + 2bu'(x) \qquad x \in (0,\infty)$$

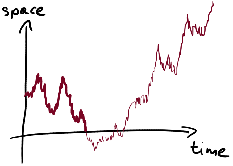
with Dirichlet boundary condition $u(0) = 0$.

The solution of the eigenvalue problem:

$$L\varphi = (-\xi^2 + 2ib\xi)\varphi$$

is given by

$$\varphi_\xi(x) = e^{i\xi x} - e^{i(-\xi+2ib)x}$$



Non-normal GEE

After an elementary calculation:

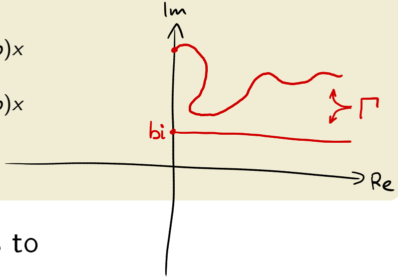
$$\langle Lu, v\rangle = \frac{1}{\pi}\int_\Gamma (-\xi^2 + 2bi\xi)\langle u, \psi_\xi\rangle\langle \varphi_\xi, v\rangle d\xi \qquad \text{(GEE)}$$

with

$$\varphi_\xi(x) = e^{i\xi x} - e^{i(-\xi+2ib)x}$$

$$\psi_\xi(x) = e^{i\bar{\xi}x} - e^{i(-\bar{\xi}-2ib)x}$$

as long as Γ goes 'from a point on $i\mathbb{R}$ to $+\infty$'.



The choice of Γ clear again: $\Gamma = (bi, \infty + bi)$ leads to

- φ_ξ, ψ_ξ as small as possible,
- all expressions real-valued.

Goal

Study generalised eigenfunction expansions for generators L of other Markov processes

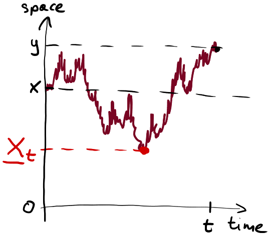
Applications so far:

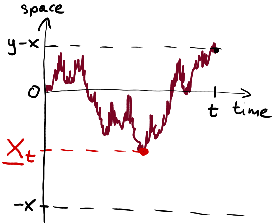
- expression for the heat kernel in $(0, \infty)$:

$$p_t^+(x,y) = \int_0^\infty \lambda_r \psi_r(x) \varphi_r(y) dr \qquad \text{(GEE)}$$

- supremum and infimum functionals:

$$\mathbb{P}(\underline{X}_t > -x) = \int_0^\infty p_t^+(x,y) dy.$$





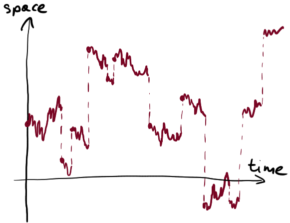
Lévy process

A 1-D Lévy process is a translation-invariant Markov process on $\mathbb{R}$.

Lévy operators

A 1-D Lévy operator is the generator of a 1-D Lévy process:

$$Lu(x) = au''(x) + ibu'(x) + \int_{-\infty}^{\infty} (u(y) - u(x) - (\dots))\nu(y-x)dy$$



Lévy–Khinchin theorem

A Lévy operator L is a Fourier multiplier:

$$\widehat{Lu}(\xi) = -f(\xi)\hat{u}(\xi)$$

where the characteristic exponent is given by:

$$f(\xi) = a\xi^2 - ib\xi + \int_{-\infty}^{\infty} (1 - e^{i\xi z} - (\dots))\nu(z)dz$$

Transition operators $P_t = \exp(tL)$ are Fourier multipliers with symbol $e^{-tf(\xi)}$.

Bernstein's theorem

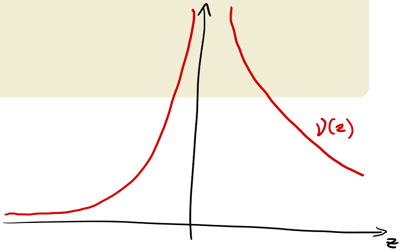
The following are equivalent:

- ν is completely monotone (or CM): $(-1)^n \nu(z) \geqslant 0$ for $z > 0$;
- ν is the Laplace transform of a non-negative measure.

CM jumps

A Lévy process has CM jumps if

$$\nu(z) \text{ and } \nu(-z) \text{ are CM.}$$



Rogers functions

A Rogers function is a holomorphic function in $\{\operatorname{Re}\xi > 0\}$ such that $\operatorname{Re}\dfrac{f(\xi)}{\xi} \geqslant 0$.

Equivalently: $\dfrac{f(\xi)}{\xi}$ is a Nevanlinna–Pick function.

Theorem (Rogers, 1983)

For a Lévy process, the following are equivalent:

- it has CM jumps;
- $f(\xi)$ extends to a Rogers function.

Spine

The spine of a Rogers function $f(\xi)$ is the curve

$$\Gamma = f^{-1}((0,\infty)) = \{\xi : f(\xi) \in (0,\infty)\}$$

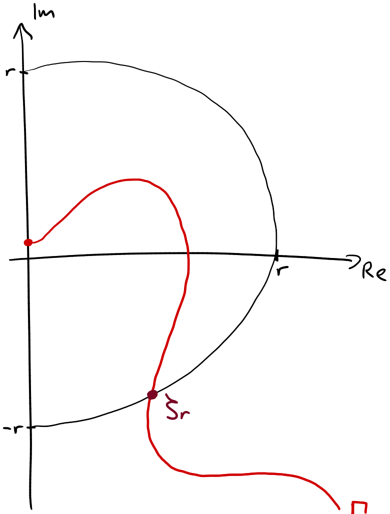
Lemma (K, 2019, 2023$^+$)

The spine intersects centred circles at most once:

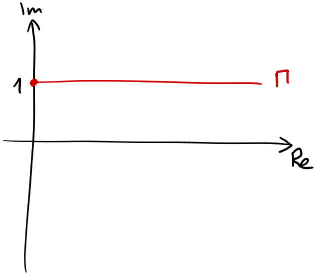
$$\Gamma = \{\zeta_r : r \in Z\}$$

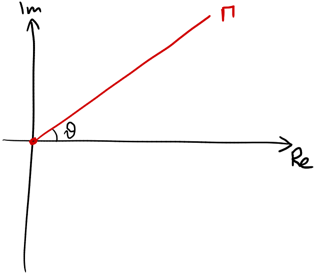
with $|\zeta_r| = r$ and $Z \subseteq (0,\infty)$. Furthermore:

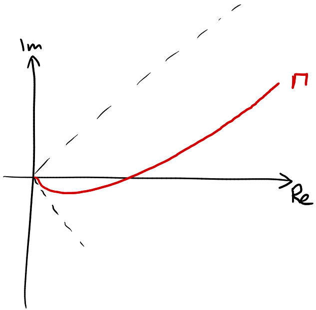
- $r \mapsto \zeta_r$ is $\frac{1}{30}$-Hölder continuous.
- $r \mapsto \lambda_r = f(\zeta_r)$ is $\frac{1}{3}$-Hölder continuous.



Sample spines:

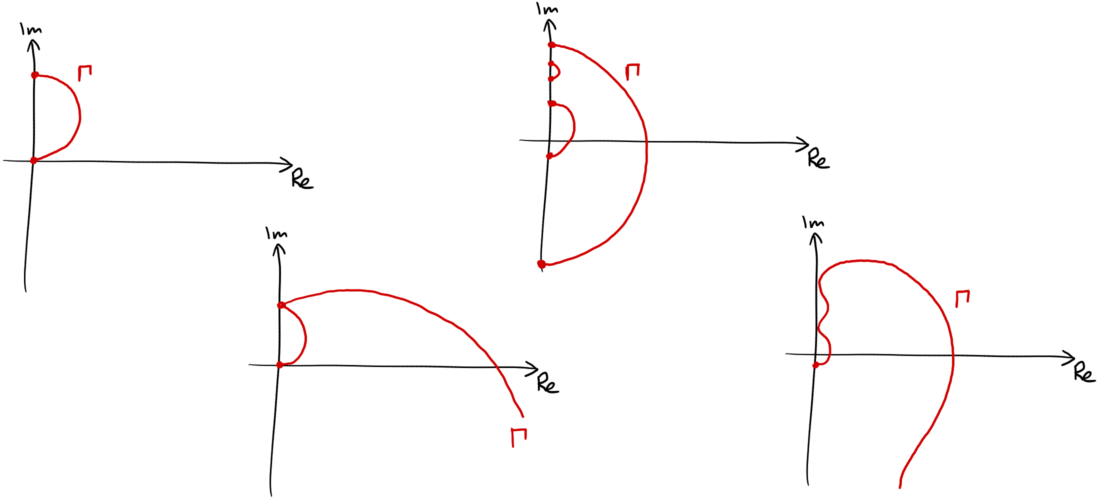






BM + drift	stable	mixed stable
$f(\xi) = \xi^2 - 2i$	$f(\xi) = a\xi^\alpha$	$f(\xi) = a\xi^\alpha + b\xi^\beta$
$\zeta_r = \sqrt{r^2 - 1} + i$	$\zeta_r = re^{i\vartheta}$	$\zeta_r \sim re^{i\vartheta}$
$\lambda_r = r^2 + 1$	$\lambda_r = \lvert a\rvert r^\alpha$	$\zeta_r \sim \lvert a\rvert r^\alpha + \lvert b\rvert r^\beta$

Sample spines for various meromorphic Rogers functions:



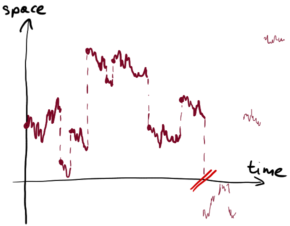
Lévy operators in half-line

A Lévy operator L restricted to $(0, \infty)$:

$$\langle L^+ u, v \rangle = \int_0^\infty Lu(x)\,\overline{v(x)}\,dx$$

Proabilistically: killing the process as soon as it exits $(0, \infty)$.

Transition operators: $P_t^+ = \exp(tL^+)$.



Theorem (K, 2011; K–Małecki–Ryznar, 2013)

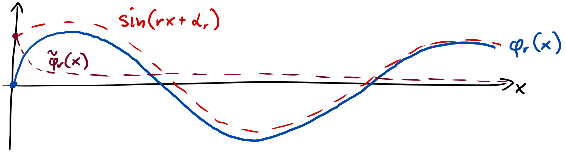
For a symmetric Lévy process with CM jumps and $u, v \in C_c((0, \infty))$:

$$\langle P_t^+ u, v \rangle = \frac{2}{\pi} \int_0^\infty e^{-tf(r)} \langle u, \varphi_r \rangle \langle \varphi_r, v \rangle dr \qquad \text{(GEE)}$$

where

$$\varphi_r(x) = \sin(rx + \alpha_r) - \tilde{\varphi}_r(x)$$

with explicit α_r and ‘explicit’ CM correction $\tilde{\varphi}_r(x)$.



Theorem (K, 2019, 2023$^+$)

For a Lévy process with CM jumps such that:

$$\limsup_{r\to\infty} |\operatorname{Arg} \zeta_r| < \frac{\pi}{2}$$

and admissible u and v we have:

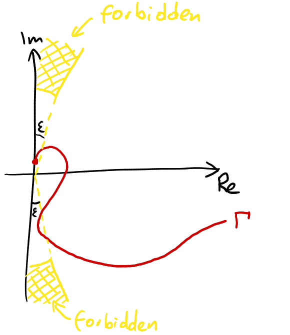
$$\langle P_t^+ u, v \rangle = \frac{2}{\pi} \int_Z e^{-t\lambda_r} \langle u, \psi_r \rangle \langle \varphi_r, v \rangle |\zeta_r'| dr \qquad \text{(GEE)}$$

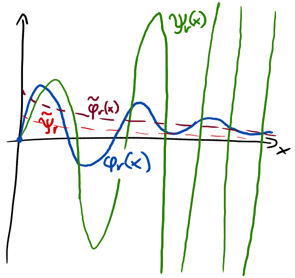
where

$$\varphi_r(x) = e^{-x \operatorname{Im} \zeta_r} \sin(x \operatorname{Re} \zeta_r + \alpha_r) - \tilde{\varphi}_r(x)$$

$$\psi_r(x) = e^{x \operatorname{Im} \zeta_r} \sin(x \operatorname{Re} \zeta_r + \beta_r) - \tilde{\psi}_r(x)$$

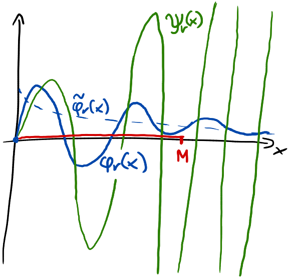
with explicit α_r, β_r and 'explicit' CM corrections $\tilde{\varphi}_r(x)$, $\tilde{\psi}_r(x)$.





If $f(\xi) = c\xi^\alpha$ (and in many other examples), we have:

$$\varphi_r(x) \approx e^{-arx} \sin(brx + \alpha_r)$$
$$\psi_r(x) \approx e^{arx} \sin(brx + \beta_r)$$



If $a > 0$ and u, v are compactly supported, then

$$\langle u, \psi_r \rangle = O(\tfrac{1}{r} e^{arM}),$$
$$\langle \varphi_r, v \rangle = O(\tfrac{1}{r})$$

Hence, the integral in

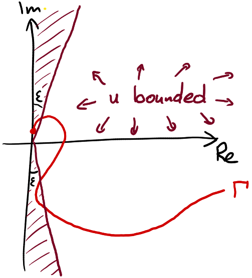
$$\langle P_t^+ u, v \rangle = \frac{2}{\pi} \int_Z e^{-t\lambda_r} \langle u, \psi_r \rangle \langle \varphi_r, v \rangle |\zeta_r'| dr \qquad \text{(GEE)}$$

need not even converge!

Admissible functions

A function u is admissible if

- it is a holomorphic function in $\{|\operatorname{Arg}\xi| < \frac{\pi}{2} - \varepsilon\}$;
- $|u(\xi)| \leqslant C \exp(-C|\xi| \log|\xi|)$ in this sector.

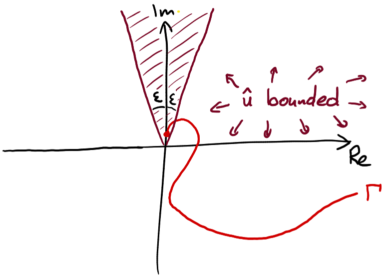


The Laplace transform of u is entire and

$$\left|\int_0^\infty e^{-\xi x} u(x) dx\right| \leqslant \frac{C}{1 + |\xi|}$$

in $\{|\operatorname{Arg}\xi| \leqslant \pi - \varepsilon\}$.

Dense in $L^2((0,\infty))$: $e^{-r\xi \log(1+\xi)}$ is admissible.



Corollary (K, 2019, 2023$^+$)

For $\beta > 1$ and a Lévy process with CM jumps such that:

$$\limsup_{r \to \infty} |\operatorname{Arg} \zeta_r| < \frac{\pi}{2\beta}$$

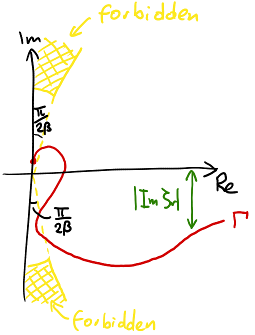
and

$$\int_Z e^{-t\lambda_r} e^{s|\operatorname{Im} \zeta_r|} |\zeta_r'| dr \leqslant A e^{s^\beta}$$

we have

$$p_t^+(x, y) = \frac{2}{\pi} \int_Z e^{-t\lambda_r} \psi_r(x) \varphi_r(y) |\zeta_r'| dr \qquad \text{(GEE)}$$

Note: not quite optimal for stable Lévy proc. (fractional derivatives)



Corollary (K, 2019, 2023$^+$)

For $\beta > 1$ and a Lévy process with CM jumps such that:

$$\liminf_{r\to\infty} \operatorname{Arg} \zeta_r > -\frac{\pi}{2\beta}$$

$$\inf\{\operatorname{Arg} f(\xi) : \xi \in (0, e^{i\alpha}\delta)]\} > 0$$

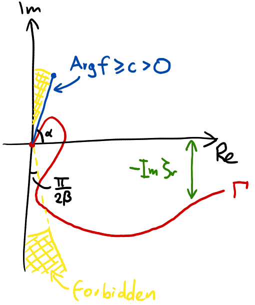
and

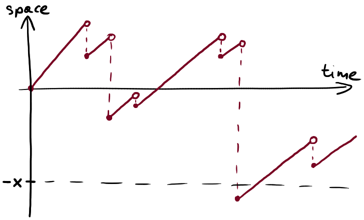
$$\int_Z e^{-t\lambda_r} e^{s \max\{0, -\operatorname{Im} \zeta_r\}} |\zeta_r'| dr \leqslant A e^{s^\beta}$$

we have

$$\mathbb{P}(\underline{X}_t > -x) = \frac{2}{\pi} \int_Z e^{-t\lambda_r} \psi_r(x) \hat{\varphi}_r(0) |\zeta_r'| dr \quad \text{(GEE)}$$

Note: not quite optimal for stable Lévy proc. (fractional derivatives)

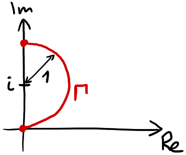




Example: classical risk process

A Lévy process X_t with exponentially distributed negative jumps compensated by a positive drift (a martingale).

Then (up to scaling):



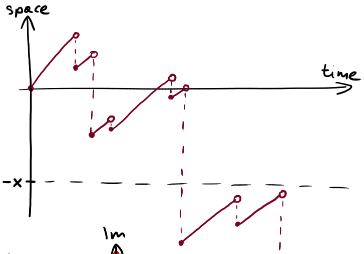
$$f(\xi) = -i\xi + \frac{\xi}{\xi - i} = \frac{-i\xi^2}{\xi - i}$$

so that Γ is a semi-circle $|\xi - i| = 1$ and we find that

$$\mathbb{P}^0(\underline{X}_t > -x) = \frac{2}{\pi} \int_0^2 e^{-r^2(t+x/2)} \sin(\tfrac{1}{2} r x \sqrt{4 - r^2} + 2 \arcsin(\tfrac{1}{2} r)) \, \frac{1}{r} \, dr$$

In terms of variable $\alpha = 2 \arcsin(\frac{1}{2} r)$:

$$\mathbb{P}^0(\underline{X}_t > -x) = \frac{1}{\pi} \int_0^\pi e^{-(1-\cos\alpha)(2t+x)} \sin(x \sin\alpha + \alpha) \, \frac{\sin\alpha}{1 - \cos\alpha} \, d\alpha$$

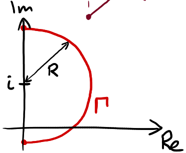


Example: classical risk process with small drift

A Lévy process X_t with exponentially distributed negative jumps with small positive drift (a supermartingale).

Then for some $R > 1$ (up to scaling):

$$f(\xi) = -i\xi + R\frac{\xi}{\xi - i} = \frac{-i\xi^2 + (R^2 - 1)\xi}{R(\xi - i)}$$



so that Γ is a semi-circle $|\xi - i| = R$ and

$$\mathbb{P}^0(\underline{X}_t > -x) = \frac{2}{\pi}\int_0^\pi e^{-(1+R^2-2R\cos\alpha)t-(1-R\cos\alpha)x} \times$$

$$\times \sin(Rx\sin\alpha + \alpha)\,\frac{R^2\sin\alpha}{1 + R^2 - 2R\cos\alpha}\,d\alpha$$

History

- $L = \partial^2$, $f(\xi) = \xi^2$: Brownian motion
  — classical (Fourier sine transform)
- $L = \partial^2 + 2b\partial$, $f(\xi) = \xi^2 - 2ib\xi$: Brownian motion with drift
  — also classical (Doob's h-transform)
- symmetric L: (complete) subordinate BM
  — K, 2011; K–Małecki–Ryznar, 2013
- $L = \partial^\beta(-\partial)^\gamma$, $f(\xi) = a\xi^\alpha$: stable Lévy processes
  — K–Kuznetsov, 2018
- general L — K, 2019; K, 2023$^+$

Elements of the proof:

- integral expression for

$$\int_0^\infty \int_0^\infty \int_0^\infty e^{-\tau t - \xi x - \eta y} p_t^+(x, y)\, dx dy dt$$

(Baxter–Donsker, Fristedt, Pecherski–Rogozin)

fluctuation theory

- inversion of Laplace transforms
- lots of contour deformations

(basic) complex analysis

- even more auxiliary estimates
- boundary geometry of level lines of 2-D harmonic functions

very technical calculations

- regularity of the Hilbert transform

(very basic) harmonic analysis