Practice problems #3: implicit functions and constrained extrema

(1) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$F(x,y,z) = xyz - (x+y)e^z = 0.$$

(2) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$F(x,y,z) = \sin(xy) + \sin(xz) + \sin(yz) = 0.$$

(3) Argue that for every x and y there is a unique z such that

$$F(x,y,z) = \arctan(x+z) + e^{y+z} - xy = 1.$$

Note that $F(0,0,0) = 1$, so that $z = 0$ if $x = y = 0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $x = y = 0$.

(4) If $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ satisfy $x = u + v + w$, $y = u^2 + v^2 + w^2$, $z = u^3 + v^3 + w^3$, show that

$$\frac{\partial u}{\partial x} = \frac{vw}{(u-v)(u-w)}.$$

(5) Find the maximum and the minimum of the function $f(x,y) = x^2y$ subject to the constraint $g(x,y) = x^2 + y^2 - 3 = 0$

(6) Find the maximum and the mimimum value of the function $f(x,y) = xy$ on the curve $3x^2 + y^2 = 6$.

(7) Find the maximum and the mimimum value of the function $f(x,y,z) = x^2 - y^2$ on the surface $x^2 + 2y^2 + 3z^2 = 1$.

(8) Find the maximum and the mimimum value of the function $f(x,y) = x^3 - 3xy^2 + 4x^2 + 4y^2 - 3x$ in the unit circle $x^2 + y^2 \leqslant 1$.

(9) Find the maximum value of $f(x,y) = -x\log x - y\log y$ subject to the constraint $g(x,y) = x + y - 1 = 0$. Note that here $x, y > 0$. How would you show that the value that you found is indeed the maximum?

(10) Find the maximum value of $f(x,y,z) = -x\log x - y\log y - z\log z$ subject to the constraint $g(x,y,z) = x + y + z - 1 = 0$. Note that here $x, y, z > 0$. How would you show that the value that you found is indeed the maximum?