Practice problems #4: gradient, divergence, curl

(1) Define

(a) $\vec{F} = \left[\dfrac{x^2 - y^2}{x^2 y}, \dfrac{y^2 - x^2}{x y^2}\right]$,

(b) $\vec{F} = [(\cos x - \sin x)(\cos y + \sin y), (\cos x + \sin x)(\cos y - \sin y)]$,

(c) $\vec{F} = [x e^y, y e^x]$,

(d) $\vec{F} = \left[\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2}\right]$.

Check whether $\vec{F}$ is a potential vector field and, if yes, find the corresponding potential.

(2) Find the potential of the vector field

(a) $\vec{F} = \left[\dfrac{yz}{1 + x^2y^2z^2}, \dfrac{xz}{1 + x^2y^2z^2}, \dfrac{xy}{1 + x^2y^2z^2}\right]$,

(b) $\vec{F} = [(x + y + z)^2 - x^2, (x + y + z)^2 - y^2, (x + y + z)^2 - z^2]$,

(c) $\vec{F} = [\cos(x + y) + \cos(x + z), \cos(x + y) + \cos(y + z), \cos(x + z) + \cos(y + z)]$,

(d) $\vec{F} = \left[\dfrac{x^2 - yz}{x^2 y}, \dfrac{y^2 - xz}{y^2 z}, \dfrac{z^2 - xy}{z^2 x}\right]$.

(3) Find $\operatorname{div} \vec{F}$ and $\operatorname{rot} \vec{F}$ if

(a) $\vec{F} = [x, y, z]$,

(b) $\vec{F} = [y, z, x]$,

(c) $\vec{F} = [x^2 y, x y^2, xyz]$,

(d) $\vec{F} = \left[\dfrac{x}{(x^2 + y^2 + z^2)^{3/2}}, \dfrac{y}{(x^2 + y^2 + z^2)^{3/2}}, \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}\right]$.

Whenever possible, find the corresponding potentials.

(4) Verify that $\operatorname{div}(\operatorname{rot} \vec{F}) = 0$ for any (twice continuously differentiable, three-dimensional) vector field $\vec{F}$.