Point process models and local asymptotics in statistics

III – Example

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9 Inference from jumps ≥ 1 of a stable increasing process

We continue the example of section 4, all notations as there: S is a stable increasing process of some index $0 < \alpha < 1$ and some weight parameter $\xi > 0$. We observe all jumps $\geq \varepsilon$ in the trajectory of S up to time t, with particular choice $\varepsilon := 1$:

$$X_t \;=\; \sum_{\substack{0<s\leq t \\ \Delta S_s \geq 1}} \Delta S_s \;=\; \int_0^t \int_{[1,\infty)} z\, \mu(ds, dz) \quad , \quad t \geq 0$$

where $\mu(ds, dz)$ is Poisson random measure on $(0,\infty)\times[1,\infty)$ with intensity

$$\nu^{\alpha,\xi}(ds, dz) \;=\; \xi\, ds \;\; \alpha z^{-\alpha-1} 1_{\{z\geq 1\}}\, dz \;=\; \xi\, ds \;\; k_\alpha(z)\, dz$$

for some $0 < \alpha < 1$ and $\xi > 0$. Due to $\varepsilon = 1$, $k_\alpha(\cdot)$ is a probability density, and the counting process

$$N = (N_t)_{t\geq 0} \quad , \quad N_t \;:=\; \mu\left((0,t]\times[1,\infty)\right)$$

is Poisson with parameter ξ. Consider also the process

$$\overline{N} = (\overline{N}_t)_{t\geq 0} \quad , \quad \overline{N}_t \;:=\; \int_0^t \int_{\{z\geq 1\}} \log(z)\, \mu(ds, dz) \; .$$

Aims : Show that LAN holds at every point (α, ξ) as $n \to \infty$,

characterize sequences of estimators for (α, ξ) which as $n \to \infty$ achieve the local asymptotic minimax bound (and thus are also regular and efficient in the sense of Hájek).

In this example, it is easy to find maximum likelihood estimators (MLE) :

the log-likelihood ratios are (section 4, special case $\varepsilon = 1$)

$$\log\left([\prod_{0<s\le t} \frac{\widetilde{\alpha}\widetilde{\xi}}{\alpha\xi}\,(\Delta X_s)^{\alpha-\widetilde{\alpha}}]\ \exp\{\,-t(\,\widetilde{\xi}-\xi\,)\}\right) \;=\; \log(\frac{\widetilde{\alpha}\widetilde{\xi}}{\alpha\xi})\,N_t \;+\; (\alpha-\widetilde{\alpha})\,\overline{N}_t \;-\; (\widetilde{\xi}-\xi)\,t$$

so deriving with respect to $\widetilde{\alpha}$ or to $\widetilde{\xi}$ we obtain MLE's explicitly

$$\widehat{\alpha}_t \;:=\; \frac{N_t}{\overline{N}_t} \quad,\quad \widehat{\xi}_t \;:=\; \frac{N_t}{t} \quad,\quad \tau_1 \le t < \infty\ .$$

Rescaling time and writing $\mathbb{F}^n := (\mathcal{F}_{tn})_{t\ge0}$, the following are (local, at least) $(Q^{(\alpha,\xi)}, \mathbb{F}^n)$-martingales:

$$\frac{1}{\sqrt{n}}\left(\overline{N}_{tn} - \frac{\xi}{\alpha}tn\right)_{t\ge0} \;=\; \frac{1}{\sqrt{n}}\int_0^{\bullet n}\int_{\{z\ge1\}} \log(z)\,(\mu-\nu^{\alpha,\xi})(ds,dz)$$

$$\frac{1}{\sqrt{n}}\,(N_{tn} - \xi tn)_{t\ge0} \;=\; \frac{1}{\sqrt{n}}\int_0^{\bullet n}\int_{\{z\ge1\}} (\mu-\nu^{\alpha,\xi})(ds,dz)\ .$$

Also the difference of both is a (local, at least) $(Q^{(\alpha,\xi)}, \mathbb{F}^n)$-martingale:

$$\frac{1}{\sqrt{n}}\left(N_{tn} - \alpha\,\overline{N}_{tn}\right)_{t\ge0} \;=\; \frac{1}{\sqrt{n}}\int_0^{\bullet n}\int_{\{z\ge1\}} (1-\alpha\log z)\,(\mu-\nu^{\alpha,\xi})(ds,dz)\ .$$

Below, B denotes 2-dimensional standard Brownian motion, and D is the canonical path space of cadlag functions $[0,\infty)\to\mathbb{R}^2$.

Lemma 1 : For all $0 < \alpha < 1$ and $\xi > 0$, we have weak convergence under $Q^{(\alpha,\xi)}$ (in D, as $n \to \infty$)

$$S(n,(\alpha,\xi)) \;:=\; \frac{1}{\sqrt{n}} \begin{pmatrix} N_{tn} - \alpha\, \overline{N}_{tn} \\ N_{tn} - \xi\, tn \end{pmatrix}_{t \ge 0} \quad \xrightarrow{w} \quad \xi^{\frac{1}{2}}\, B\,.$$

Proof : First, integration by parts successively in $k \in \mathbb{N}_0$ grants

$$(+) \qquad \int_{\{z \ge 1\}} \log^k(z)\, \alpha z^{-\alpha-1}\, dz \;=\; \frac{k!}{\alpha^k}$$

for all $0 < \alpha < 1$, $\xi > 0$. The components of $S(n,(\alpha,\xi))$ are locally square integrable martingales. Using (+) we calculate angle brackets

$$\left\langle \frac{1}{\sqrt{n}}(N_{\bullet n} - \alpha\, \overline{N}_{\bullet n})\,,\, \frac{1}{\sqrt{n}}(N_{\bullet n} - \alpha\, \overline{N}_{\bullet n}) \right\rangle_t \;=\; \frac{1}{n}\int_0^{tn}\int_{\{z\ge 1\}} (1-\alpha \log z)^2\, \nu^{\alpha,\xi}(ds,dz) \;=\; \xi\, t$$

$$\left\langle \frac{1}{\sqrt{n}}(N_{\bullet n} - \alpha\, \overline{N}_{\bullet n})\,,\, \frac{1}{\sqrt{n}}(N_{\bullet n} - \xi \bullet n) \right\rangle_t \;=\; \frac{1}{n}\int_0^{tn}\int_{\{z\ge 1\}} (1-\alpha \log z)\, \nu^{\alpha,\xi}(ds,dz) \;=\; 0$$

$$\left\langle \frac{1}{\sqrt{n}}(N_{\bullet n} - \xi \bullet n\,,\, \frac{1}{\sqrt{n}}(N_{\bullet n} - \xi \bullet n) \right\rangle_t \;=\; \frac{1}{n}\int_0^{tn}\int_{\{z\ge 1\}} \nu^{\alpha,\xi}(ds,dz) \;=\; \xi\, t$$

whence

$$\langle\, S_n(\alpha,\xi)\, \rangle_t \;=\; \xi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t$$

for $0 \le t < \infty$. Thus weak convergence in D under $Q^{(\alpha,\xi)}$ as $n \to \infty$ holds in virtue of the martingale convergence theorem (corollary VIII.3.24 in Jacod-Shiryaev 1987). □

Since we deal with PRM, we could have formulated a 'elementary' proof, via classical central limit theory: independence assumptions in the definition of PRM show that martingale increments as above reduce to independent random variables.

From now on we write

$$\vartheta \;:=\; \begin{pmatrix} \alpha \\ \xi \end{pmatrix} \quad \in \quad \Theta \;:=\; (0,1)\times(0,\infty)\,.$$

Fix a reference point $\vartheta \in \Theta$ and define local scale at ϑ by

$$\delta_n(\vartheta) \;:=\; \frac{1}{\sqrt{n}} \begin{pmatrix} \alpha & 0 \\ 0 & \xi \end{pmatrix} \quad , \quad \delta_n = \delta_n(\vartheta) \downarrow 0 \quad \text{as } n \to \infty\,.$$

Introduce local parameter $h = \binom{h_1}{h_2}$ at ϑ, with h ranging over open sets

$$\Theta_{\vartheta,n} := \{h \in \mathbb{R}^2 : \vartheta + \delta_n h \in \Theta\} \quad \uparrow \quad \mathbb{R}^2 \quad \text{as } n \to \infty.$$

At a fixed reference point $\vartheta \in \Theta$, at stage n of the asymptotics:

- reparametrize neighbourhoods of ϑ, replacing $\binom{\widetilde{\alpha}}{\widetilde{\xi}}$ in earlier notation by

$$\vartheta \;+\; \delta_n(\vartheta)\, h \;=\; \begin{pmatrix} \alpha(1+\frac{h_1}{\sqrt{n}}) \\ \xi(1+\frac{h_2}{\sqrt{n}}) \end{pmatrix} \quad , \quad h \;\in\; \Theta_{\vartheta,n} \;=\; \ldots\mathbb{R}^2\ldots$$

  and view the local parameter h as new parametrization

- change time from t to tn, i.e. consider the filtration $\mathbb{F}^n := (\mathcal{F}_{tn})_{t\geq 0}$

and study the statistical model in shrinking neighbourhoods of the reference point ϑ.

We thus consider a sequence of filtered local models at ϑ

$$\mathcal{E}_n(\vartheta) \;:=\; \left(\Omega\,,\,\mathbb{F}^n\,,\,\left\{Q^{(\vartheta+\delta_n(\vartheta)h)} : h \in \Theta_{\vartheta,n}\right\}\right) \quad , \quad n \to \infty$$

where log-likelihood ratio processes take the form ($0 \leq t < \infty$)

$$(*) \qquad \underbrace{\log L_{tn}^{(\vartheta+\delta_n h)/\vartheta}}_{=\log L_{tn}^{(\widetilde{\alpha},\widetilde{\xi})/(\alpha,\xi)}} \;=\; \underbrace{\log(1+\frac{h_1}{\sqrt{n}})}_{=\log\frac{\widetilde{\alpha}}{\alpha}}\, N_{tn} \;+\; \underbrace{\log(1+\frac{h_2}{\sqrt{n}})}_{=\log\frac{\widetilde{\xi}}{\xi}}\, N_{tn} \;-\; \underbrace{h_1\frac{\alpha}{\sqrt{n}}}_{=\widetilde{\alpha}-\alpha}\, \overline{N}_{tn} \;-\; \underbrace{h_2\frac{\xi}{\sqrt{n}}}_{=\widetilde{\xi}-\xi}\, tn\,.$$

Using expansions

$$\log(1+z) \;=\; z - \frac{1}{2}z^2 + o(z^2) \quad \text{as} \quad z \to 0$$

in $(*)$ and arranging terms

$$\begin{aligned} \log L_{tn}^{(\vartheta+\delta_n h)/\vartheta} \;&=\; h_1\frac{1}{\sqrt{n}}(N_{tn}-\alpha\overline{N}_{tn}) \;+\; h_2\frac{1}{\sqrt{n}}(N_{tn}-\xi tn) \;-\; \frac{1}{2}(h_1^2+h_2^2)\,\frac{1}{n}N_{tn} \;+\; \ldots \\ &=\; h_1\frac{1}{\sqrt{n}}(N_{tn}-\alpha\overline{N}_{tn}) \;+\; h_2\frac{1}{\sqrt{n}}(N_{tn}-\xi tn) \;-\; \frac{1}{2}(h_1^2+h_2^2)\,\xi\, t \;+\; \ldots \end{aligned}$$

up to remainder terms which are negligible under $Q^{(\vartheta)}$ as $n \to \infty$. Here a score martingale at ϑ appears

$$S(n,\vartheta)_t \;:=\; \frac{1}{\sqrt{n}}\begin{pmatrix} N_{tn}-\alpha\,\overline{N}_{tn} \\ N_{tn}-\xi\, tn \end{pmatrix} \quad , \quad t \geq 0$$

together with a process information at ϑ

$$J(n,\vartheta)_t \;:=\; \langle S(n,\vartheta)\rangle_t \;=\; \xi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t$$

and we know about weak convergence of the score martingale under $Q^{(\vartheta)}$, by lemma 1.

Lemma 2 : ('2nd Le Cam lemma') At every reference point $\vartheta \in \Theta$, with local scale $\delta_n(\vartheta) = \frac{1}{\sqrt{n}} \begin{pmatrix} \alpha & 0 \\ 0 & \xi \end{pmatrix}$ and local parameter $h \in \ldots \mathbb{R}^2 \ldots$ as above, we have local asymptotic normality

$$\log L_{\bullet n}^{(\vartheta+\delta_n h)/\vartheta} \quad = \quad h^\top S(n,\vartheta) \; - \; \frac{1}{2}\, h^\top J(n,\vartheta)\, h \; + \; R(n,\vartheta)$$

where under $Q^{(\vartheta)}$

$$\begin{cases} S(n,\vartheta) \quad \longrightarrow \quad \xi^{\frac{1}{2}}\, B \qquad \text{weakly in } D \text{ as } n \to \infty \,, \\ J(n,\vartheta) \; = \; \left\langle \xi^{\frac{1}{2}}\, B \right\rangle \qquad \text{for all } n \,, \\ \text{paths of } R(n,\vartheta) \text{ vanish uniformly on compact time intervals as } n \to \infty \,. \end{cases}$$

We have seen that maximum likelihood estimators (MLE) are given by

$$\widehat{\vartheta}_v \; := \; \begin{pmatrix} \widehat{\alpha}_v \\ \widehat{\xi}_v \end{pmatrix} \quad , \quad \widehat{\alpha}_v = \frac{N_v}{\overline{N}_v} \quad , \quad \widehat{\xi}_v = \frac{N_v}{v} \,.$$

Here $\overline{N}_v \sim \frac{\xi}{\alpha} v$ and $N_v \sim \xi v$ $Q^{(\alpha,\xi)}$-almost surely as $v \to \infty$, whence consistency and

$$\frac{\sqrt{v}}{\alpha}\,(\,\widehat{\alpha}_v - \alpha\,) \; = \; \frac{v}{\alpha\,\overline{N}_v}\,\frac{1}{\sqrt{v}}\,(\,N_v - \alpha \overline{N}_v\,) \quad \sim \quad \frac{1}{\xi}\,\frac{1}{\sqrt{v}}\,(\,N_v - \alpha \overline{N}_v\,)$$

$$\frac{\sqrt{v}}{\xi}\left(\widehat{\xi}_v - \xi\right) \quad = \quad \frac{1}{\xi}\,\frac{1}{\sqrt{v}}\,(\,N_v - \xi v\,)$$

as $v \to \infty$. In time scale $\bullet n$, this is the assertion

$$\delta_n^{-1}(\vartheta)\left(\widehat{\vartheta}_{tn} - \vartheta\right) \quad = \quad \frac{1}{\xi\, t} \begin{pmatrix} \frac{1}{\sqrt{n}}\,(\,N_{tn} - \alpha \overline{N}_{tn}\,) \\ \frac{1}{\sqrt{n}}\,(\,N_{tn} - \alpha \overline{N}_{tn}\,) \end{pmatrix} \; + \; o_{Q^{(\vartheta)}}(1)$$

as $n \to \infty$, for every $0 < t < \infty$ fixed. We thus find that rescaled ML estimation errors behave as

$$Z(n,\vartheta)_t \; := \; J(n,\vartheta)_t^{-1} S(n,\vartheta)_t \quad , \quad 0 < t < \infty \quad , \quad n \to \infty$$

in the sequence of local models at ϑ.

Lemma 3 : At every $\vartheta \in \Theta$, as $n \to \infty$, rescaled ML estimation errors admit the representation

$$\delta_n^{-1}(\vartheta)\left(\widehat{\vartheta}_{tn} - \vartheta\right) \; = \; J(n,\vartheta)_t^{-1} S(n,\vartheta)_t \; + \; \widetilde{R}(n,\vartheta)_t \quad , \quad 0 < t < \infty$$

where paths of $\widetilde{R}(n,\vartheta)$ vanish uniformly on compact time intervals $\subset (0,\infty)$, under $Q^{(\vartheta)}$, as $n \to \infty$.

Note that it does not make sense to consider $t = 0$ ...

For inference about the unknown parameter $\vartheta \in \Theta$, Lemmata 2 and 3 allow to deal with

- deterministic observation schemes
- a broad class of random observation schemes.

<u>Deterministic observation schemes:</u>

at stage n of the asymptotics we observe up to time $n\,,\ n \to \infty$.

We discuss asymptotic optimality properties for estimators as $n \to \infty$.

Corollary 1 : The MLE sequence $(\widehat{\vartheta}_n)_n$ is regular and efficient in the sense of Hájek.

Corollary 2 : Consider loss functions $\ell : \mathbb{R}^2 \to [0,\infty)$ continuous, subconvex and bounded. Then

a) for arbitrary sequences of $\mathcal{F}_n$-measurable estimators $(\widetilde{\vartheta}_n)_n$ for ϑ

$$\lim_{c\uparrow\infty}\ \limsup_{n\to\infty}\ \sup_{|h|\le c}\ E_{\vartheta+\delta_n(\vartheta)h}\left(\ell\left(\delta_n^{-1}(\vartheta)\left(\widetilde{\vartheta}_n-(\vartheta+\delta_n(\vartheta)h)\right)\right)\right)\quad\ge\quad E\left(\ell\left(\xi^{-\frac{1}{2}}B_1\right)\right)\ ;$$

b) the MLE sequence achieves this bound: for every $0<c<\infty$,

$$\lim_{n\to\infty}\ \sup_{|h|\le c}\ E_{\vartheta+\delta_n(\vartheta)h}\left(\ell\left(\delta_n^{-1}(\vartheta)\left(\widehat{\vartheta}_n-(\vartheta+\delta_n(\vartheta)h)\right)\right)\right)\quad=\quad E\left(\ell\left(\xi^{-\frac{1}{2}}B_1\right)\right)\ .$$

Random observation schemes: Let $\mathcal{T}$ denote the class of all strictly increasing sequences $(T_n)_n$ of $\mathbb{F}$-stopping times with the following properties i) and ii):

i) for $\vartheta \in \Theta$, there is some constant $0 < c(\vartheta) < \infty$ such that

$$c(\vartheta) \; = \; \lim_{n\to\infty} \frac{1}{n} T_n \qquad Q^{(\vartheta)}\text{-almost surely ;}$$

ii) for $\vartheta \in \Theta$, there is some compact $K(\vartheta)$ contained in $(0,\infty)$ and

a sequence $\sigma(n,\vartheta)$ of $\mathbb{F}^n$-stopping times taking values in $K(\vartheta)$, $n \ge 1$,

events $A_n(\vartheta) \in \mathcal{F}_{T_n}$, $n \ge 1$, such that $\liminf_{n\to\infty} A_n(\vartheta) = \Omega \quad Q^{(\vartheta)}$-almost surely

such that $Q^{(\vartheta)}$-almost surely

for every $n \ge 1$, T_n coincides with $\sigma(n,\vartheta)\, n$ in restriction to $A_n(\vartheta)$.

Then necessarily, also $\lim_{n\to\infty} \sigma(n,\vartheta) = c(\vartheta)$ exists $Q^{(\vartheta)}$-almost surely.

Examples: consider increasing integrable additive functionals $A = (A_t)_{t\ge 0}$ of X and define

$$T_n \; := \; \inf\{\, t > 0 : A_t \ge n \,\} \;\; , \;\; n \ge 1 \quad , \quad c(\vartheta) := [\lim_{t\to\infty} \tfrac{1}{t} A_t]^{-1} \text{ under } Q^{(\vartheta)} \;\; ;$$

in particular: $A_t := N_t$ with $c(\vartheta) = \frac{1}{\xi}$; $\quad A_t := \overline{N}_t$ with $c(\vartheta) = \frac{\alpha}{\xi}$.

Corollary 3 : For random observation schemes of class $\mathcal{T}$

at stage n of the asymptotics we observe up to time T_n , $n \to \infty$

a) we have LAN with central sequence $J(n,\vartheta)^{-1}_{\sigma(n,\vartheta)} S(n,\vartheta)_{\sigma(n,\vartheta)}$ under $Q^{(\vartheta)}$;

b) the MLE sequence $\widehat{\vartheta}_{T_n} = \binom{\widehat{\alpha}_{T_n}}{\widehat{\xi}_{T_n}}$, $n \ge 1$ is regular and efficient in the sense of Hájek ;

c) the local asymptotic minimax bound

$$\lim_{c\uparrow\infty} \limsup_{n\to\infty} \sup_{|h|\le c} E_{\vartheta+\delta_n(\vartheta)h} \left(\ell\left(\delta_n^{-1}(\vartheta) \left(\widetilde{\vartheta}_{T_n} - (\vartheta + \delta_n(\vartheta)h) \right)\right)\right) \quad \ge \quad E\left(\ell\left([\,\xi\, c(\vartheta)\,]^{-\frac{1}{2}}\, B_1 \right)\right)$$

holds for any sequence of $\mathcal{F}_{T_n}$-measurable estimators $\widetilde{\vartheta}_{T_n}$, $n \ge 1$, and for arbitrary loss functions $\ell : \mathbb{R}^2 \to [0,\infty)$ which are continuous, subconvex and bounded ;

d) the MLE sequence $\widehat{\vartheta}_{T_n} = \binom{\widehat{\alpha}_{T_n}}{\widehat{\xi}_{T_n}}$, $n \ge 1$, achieves this bound.