Intrinsic ultracontractivity for isotropic stable processes in unbounded domains

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Based on:

- MK, *Intrinsic ultracontractivity for stable semigroups on unbounded open sets*, preprint;
- K. Bogdan, T. Kulczycki, MK, *Estimates and structure of α-harmonic functions*, Probab. Theory Rel. Fields 140 (2008).

Basic definitions

- $d = 1, 2, 3, \ldots$
- $D \subseteq \mathbf{R}^d$ — open (unbounded)
- (X_t) — isotropic α-stable process in $\mathbf{R}^d$, $\alpha \in (0, 2)$

$$\mathbf{E}^0 e^{i\,\xi\,X_t} = e^{-t\,|\xi|^\alpha}$$

$$\mathbf{E}^x f(X_t) = \int p_t(y-x)\,f(y)\,dy$$

$$p_t(y-x) \asymp C\ \min\left(t^{\frac{d}{\alpha}}, \frac{t}{|y-x|^{d+\alpha}}\right)$$

$$\nu_x(y) = \frac{\mathcal{A}_{d,-\alpha}}{|y-x|^{d+\alpha}}$$

$$\nu_x(E) = \int_E \nu_x(y)\,dy$$

Definition

For $p : \mathbf{R}_+ \to \mathbf{R}_+$, define horn-shaped region (HSR):

$$D_p = \left\{ (x_1, \tilde{x}) \in \mathbf{R}^d \ : \ x_1 > 0\,,\ |\tilde{x}| < p(x_1) \right\}\ .$$

Definition

HSR D_p is nondegenerate if for some $\kappa > 0$,

$$p(u) > 0 \quad \Longrightarrow \quad \exists x \ : \ (u, 0, \ldots, 0) \in B(x, \kappa\, p(u)) \subseteq D_p\,.$$

Killed process

- τ_D — first exit time from D,

$$\tau_D = \inf \{t \geq 0 \ : \ X_t \notin D\}$$

- (P_t^D) — semigroup of (X_t) killed upon exiting D,

$$\begin{aligned} P_t^D f(x) &= \mathbf{E}^x \Big(f(X_t) \ ; \ t < \tau_D \Big) \\ &= \int_D p_t^D(x,y)\, f(y)\, dy \qquad\qquad f \in L^2(D) \end{aligned}$$

- p_t^D positive, continuous and bounded on $D \times D$
- $\nu_x(D^c)$ — killing intensity

Theorem (MK)

$$P_t^D \text{ are compact} \quad \Longleftrightarrow \quad \lim_{|x|\to\infty} \mathbf{E}^x \tau_D = 0\,.$$

Remark: This holds true also for the Brownian motion

Theorem (MK)

$$P_t^D \text{ are compact} \quad \Longleftarrow \quad \lim_{|x|\to\infty} \nu_x(D^c) = \infty\,.$$

Example

For nondegenerate HSR $D = D_p$,

$$P_t^D \text{ are compact} \quad \Longleftrightarrow \quad \lim_{u\to\infty} p(u) = 0\,.$$

From now on we assume that P_t^D are compact

There exists a complete orthonormal sequence (φ_n):

$$P_t^D \varphi_n = e^{-\lambda_n t} \varphi_n \qquad 0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots \to \infty$$

Theorem (MK)

$$\varphi_1(x) \asymp \frac{C(D)\, \mathbf{E}^x \tau_D}{(1+|x|)^{d+\alpha}}\,.$$

Remark: $\mathbf{E}^x \tau_D$ is local:

$$\mathbf{E}^x \tau_D \asymp C(D)\, \mathbf{E}^x \tau_{D \cap B(x,1)}$$

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Theorem (MK)

$$\varphi_1(x) \asymp \frac{C(D)\, \mathbf{E}^x \tau_D}{(1+|x|)^{d+\alpha}}\,.$$

Proposition (MK)

For a HSR $D = D_p$ with sufficiently smooth p,

$$\mathbf{E}^x \tau_D \asymp C(D) \left(f(x_1)^2 - |\tilde{x}|^2 \right)^{\frac{\alpha}{2}}\,.$$

Definition

(P_t^D) is said to be intrinsically ultracontractive (IU) if

$$p_t^D(x,y) \le C(D,t)\,\varphi_1(x)\,\varphi_1(y)\,.$$

(P_t^D) is IU if

- D is bounded with smooth boundary (Chen, Song, 1997)
- D is bounded (Kulczycki, 1998)

If (P_t^D) is IU, then

$$\sup_{x,y\in D}\left|\frac{p_t^D(x,y)}{e^{-\lambda_1 t}\,\varphi_1(x)\,\varphi_1(y)} - 1\right| \asymp C(D)\,e^{-(\lambda_2-\lambda_1)\,t} \qquad (t \ge 1)$$

Theorem (MK)

The following are equivalent:

(a) (P_t^D) is IU;

(b) $p_t^D(x,y) \le \dfrac{C(D,t)}{(1+|x|)^{d+\alpha}(1+|y|)^{d+\alpha}}$.

Corollary

$$D_1 \subseteq D_2\,,\ (P_t^{D_2}) \text{ is IU} \quad \Longrightarrow \quad (P_t^{D_1}) \text{ is IU}\,.$$

Corollary

$$(P_t^D) \text{ is IU} \quad \Longrightarrow \quad P_t^D \text{ are Hilbert-Schmidt operators}\,.$$

Theorem (MK)

The following are equivalent:

(a) (P_t^D) is IU;

(b) $p_t^D(x,y) \le \dfrac{C(D,t)}{(1+|x|)^{d+\alpha}(1+|y|)^{d+\alpha}}$;

(c) $\sup_x \mathbf{P}^x(\tau_{D\setminus \overline{B}(0,r)} > t) \le \dfrac{C(D,t)}{(1+r)^{d+\alpha}}$.

Remark: Condition (c) is local at infinity

Theorem (MK)

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Theorem (MK)

$$(P_t^D) \text{ is IU} \quad \Longleftarrow \quad \lim_{|x|\to\infty} \frac{\nu_x(D^c)}{\log|x|} = \infty\,,$$

$$(P_t^D) \text{ is IU} \quad \Longrightarrow \quad \lim_{|x|\to\infty} \mathrm{dist}(x,\partial D)\,(\log|x|)^{\frac{1}{\alpha}} = 0\,.$$

Corollary

For a nondegenerate HSR $D = D_p$,

$$(P_t^D) \text{ is IU} \quad \Longleftrightarrow \quad \lim_{u\to\infty} (\log u)^{\frac{1}{\alpha}} p(u) = 0\,.$$

Example

If $p(u) \sim u^{-q}$ $(q > 0)$, then (P_t^D) is IU.
If $p(u) \sim (\log u)^{-q}$, then

$$(P_t^D) \text{ is IU} \quad \Longleftrightarrow \quad q > \frac{1}{\alpha}\,.$$

Example

If $D = \bigcup_{n=1}^{\infty} B(x_n, r_n)$, (disjoint balls, $x_n = (u_n, 0, ..., 0)$), then

$$(P_t^D) \text{ is IU} \quad \Longleftrightarrow \quad \lim_{n\to\infty} r_n^{\alpha} \log u_n = 0\,.$$

Example

Let $k_n = 2^n$, $q_n = (n+1)^{-d-1}$ and

$$D = \left\{ x \in \mathbf{R}^d \ : \ |x| \in \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{k_n - 1} \left(n + \frac{m + q_n}{k_n}, n + \frac{m+1}{k_n} \right) \right\} ;$$

then (P_t^D) is IU, but $|D^c| < \infty$.

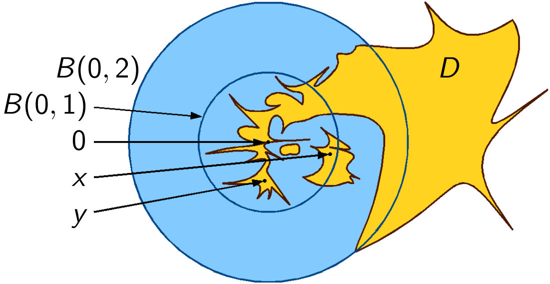
Boundary Harnack inequality (K. Bogdan, T. Kulczycki, MK)

If

- $f, g \geq 0$ and $f = g = 0$ on $B(0,2) \setminus D$,
- $f(x) = \mathbf{E}^x f(X(\tau_D))$, $g(x) = \mathbf{E}^x g(X(\tau_D))$;

then

$$\frac{f(x)}{g(x)} \leq C \, \frac{f(y)}{g(y)} \qquad \text{for } x, y \in B(0,1) \cap D.$$



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$$\frac{f(x)}{g(x)} \leq C \frac{f(y)}{g(y)} \qquad \text{for } x, y \in B(0,1) \cap D.$$

Remarks:

- C **does not depend on** D
- No smoothness assumptions on ∂D

History:

- D Lipschitz, $C = C(D)$ (Bogdan, 1997)
- D arbitrary, $C = C(D)$ (Song, Wu, 1999)

Boundary Harnack inequality (K. Bogdan, T. Kulczycki, MK)

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- $f, g \geq 0$ and $f = g = 0$ on $B(0,2) \setminus D$,
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Key lemma (K. Bogdan, T. Kulczycki, MK)

Under the same hypotheses,

$$f(x) \asymp C \left(\mathbf{E}^x \tau_{D \cap B(0,2)} \right) \left(\int_{D \setminus B(0,1)} f(y) \, \nu_0(y) \, dy \right)$$

for $x \in B(0,1) \cap D$.

Definitions

For $x, y \in D$, $z \in D^c$, $w \in \partial D$,

- $G_D(x,y) = \int_0^\infty p_t^D(x,y)\,dt$ — Green function
- $P_D(x,z) = \int_D G_D(x,y)\,\nu_y(z)\,dy$ — 'Poisson kernel'
- $M_D(x,w) = \lim_{y\to w} \dfrac{G_D(x,y)}{G_D(x_0,y)}$ — Martin kernel

Theorem (K. Bogdan, T. Kulczycki, MK)

(i) $M_D(x,w)$ exists for all $w \in \partial D$;

(i) $M_D(\cdot,w)$ is α-harmonic in D $\iff$ $w \in \partial_M D$, where

$$\partial_M D = \left\{ w \in \partial D \ : \ \int_D \nu_y(w)\,\mathbf{E}^y \tau_D\,dy = \infty \right\} .$$

$$\partial_M D = \left\{ w \in \partial D \ : \ \int_D \nu_w(y)\, \mathbf{E}^y \tau_D \, dy = \infty \right\} .$$

Example

For $p : (0,1) \to [0,\infty)$ nondecreasing, define thorn:

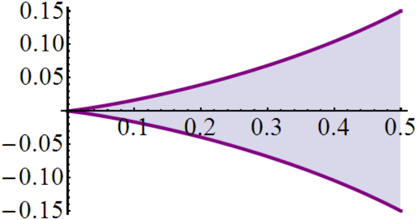
$$D_p = \left\{ (x_1, \tilde{x}) \in \mathbf{R}^d \ : \ 0 < x_1 < 1 \, , \ |\tilde{x}| < p(x_1) \right\} .$$

Then

$$0 \in \partial_M D_f \iff \int_0^1 \frac{(p(u))^{d+\alpha-1}}{u^{d+\alpha}} \, du = \infty \, .$$

For $p(u) \sim u |\log u|^{-q}$:

$$0 \in \partial_M D \iff q \le \frac{1}{d+\alpha-1}$$



Theorem (K. Bogdan, T. Kulczycki, MK)

If $f \geq 0$ is α-harmonic in D, then

$$f(x) = \int_{D^c} P_D(x,z)\, f(z)\, dz + \int_{\partial_M D} M_D(x,w)\, \mu(dw)\,.$$

The representation is unique.

Remarks:

- $|\partial_M D| = 0$
- $P_D(x,\cdot)$ is density of $\mathbf{P}^x$-distribution of $X(\tau_D)$ on $D^c \setminus \partial_M D$ (Ikeda, Watanabe: on $D^c \setminus \partial D$)

A different point of view

A function $f \geq 0$ is α-harmonic in D with outer charge λ if

$$f(x) = \int_{D^c} P_D(x,z)\,\lambda(dz) + \int_{\partial_M D} M_D(x,w)\,\mu(dw)$$
$$= P_D\lambda(x) + M_D\mu(x)\,.$$

(λ — a measure on $D^c \setminus \partial_M D$)

Boundary Harnack inequality

If

- $f = P_D\lambda + M_D\mu$, $g = P_D\lambda' + M_D\mu'$,
- $\lambda(B(0,2)) = \lambda'(B(0,2)) = \mu(B(0,2)) = \mu'(B(0,2)) = 0$,

then

$$\frac{f(x)}{g(x)} \leq C\,\frac{f(y)}{g(y)} \qquad \text{for } x, y \in B(0,1) \cap D.$$