The $\ell^p$ norm of the discrete Hilbert transform

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Joint work with Rodrigo Bañuelos (Purdue University)

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Hilbert transforms

**Definition**

The continuous Hilbert transform is defined by

\[
Hf(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(x - z)}{z} \, dz
\]

for appropriate functions \( f : \mathbb{R} \to \mathbb{R} \).
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**Definition**

Similarly, the **discrete Hilbert transform** is given by

\[ \mathcal{H}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k} \]

for appropriate doubly infinite sequences \( (a_n : n \in \mathbb{Z}) \).
Theorem (Rodrigo Bañuelos, MK)

For $p \in (1, \infty)$ we have

$$\|\mathcal{H}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}.$$
Main theorem

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- $\|H\|_{L^p \to L^p} = \max\{\tan\left(\frac{\pi}{2p}\right), \cot\left(\frac{\pi}{2p}\right)\}$ (S. Pichorides, 1972).
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For $p \in (1, \infty)$ we have

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- $||H||_{L^p \to L^p} = \max\{\tan\left(\frac{\pi}{2p}\right), \cot\left(\frac{\pi}{2p}\right)\}$ (S. Pichorides, 1972).
- The more challenging problem, which asks for the norm of

$$\mathcal{H}_{RT}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{a_{n-k}}{k + 1/2},$$

remains open.
Będlewo

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- Rodrigo Bañuelos and Eero Saksman invited me to join their fireside chat, and told me about the problem.
- Fortunately, they forgot to mention that some experts considered it to be rather difficult.
Continuous Hilbert transform

- The operator

\[ Hf(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(x - y)}{y} \, dy \]

is a Fourier multiplier: \( \hat{H}f = \hat{H} \cdot \hat{f} \), with symbol

\[ \hat{H}(\xi) = -i \text{ sign } \xi. \]
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- \( \|H\|_{L^p \to L^p} < \infty \) for \( p \in (1, \infty) \) (M. Riesz, 1928)
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Throughout the talk, we assume that \( p \in (1, \infty) \).
90 years of history in a nutshell

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- \[ \| \mathcal{H} \|_{\ell^p \to \ell^p} = \| H \|_{L^p \to L^p} \text{ when } p = 2^k \text{ or } p = \frac{2^k}{2^k - 1}, \]
  where \( k = 1, 2, \ldots \) (I.E. Verbitsky; E. Laeng, 2007)
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- $\|H\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}$ when $p = 2^k$ or $p = \frac{2^k}{2^k - 1}$, where $k = 1, 2, \ldots$ (I.E. Verbitsky; E. Laeng, 2007)
- $\|H\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}$ for every $p$ (R. Bañuelos, MK)
Discrete Hilbert transforms

- A number of discrete Hilbert transforms exist, each of them is a Fourier multiplier:

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\mathcal{H}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{a_{n-k}}{k} \quad \leftrightarrow \quad \text{symbol}
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\[ \mathcal{H}_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k} \]

\[ \mathcal{H}_{RT} n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{a_{n-k}}{k + 1/2} \]
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<table>
<thead>
<tr>
<th>Operator</th>
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<tbody>
<tr>
<td>$\mathcal{H} a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus {0}} \frac{a_{n-k}}{k}$</td>
<td><img src="image1.png" alt="Graph 1" /></td>
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\mathcal{H}_{ADP} a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{k a_{n-k}}{k^2 - 1/4} \quad \leftrightarrow \quad -i \cos(t/2) \text{sign } t.
\]
Approximation of the continuous Hilbert transform

- The continuous transform $H$ can be approximated by the discrete transform $\mathcal{H}$. 

This was first observed by E.C. Titchmarsh in 1926, as a part of his erroneous proof of equality of norms.
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- A similar argument applies to $\mathcal{H}_{RT}$, $\mathcal{H}_K$ and $\mathcal{H}_{ADP}$. 
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Riesz–Titchmarsh $\leftrightarrow$ Kak–Hilbert

$$\mathcal{H}_{RT} a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{a_{n-k}}{k + 1/2}$$

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$$\mathcal{H}_{RT} a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{a_{n-k}}{k + 1/2} \iff$$

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- The operators $\mathcal{H}_{RT}$ and $\mathcal{H}_K$ are equivalent:

$$\mathcal{H}_K a_n = b_n \iff \begin{cases} \mathcal{H}_{RT} [a_{2n}] = [b_{2n+1}], \\ \mathcal{H}_{RT} [a_{2n-1}] = [b_{2n}]. \end{cases}$$
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- In particular, $\| \mathcal{H}_{RT} \|_{\ell^p \rightarrow \ell^p} = \| \mathcal{H}_{K} \|_{\ell^p \rightarrow \ell^p}$. 
Riesz–Titchmarsh $\rightsquigarrow$ Arcozzi–Domelevo–Petermichl

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- The operator $\mathcal{H}_{ADP}$ can be expressed in terms of $\mathcal{H}_{RT}$:

\[ \mathcal{H}_{ADP} a_n = \frac{1}{2} (\mathcal{H}_{RT} a_n + \mathcal{H}_{RT} a_{n-1}) \].
Riesz–Titchmarsh $\mapsto$ Arcozzi–Domelevo–Petermichl

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- In particular, $\|\mathcal{H}_{ADP}\|_{\ell^p \to \ell^p} \leq \|\mathcal{H}_{RT}\|_{\ell^p \to \ell^p}$. 
Kak–Hilbert $\sim\rightarrow$ Hilbert

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- The operator $\mathcal{H}$ can be expressed in terms of $\mathcal{H}_K$:

\[
\mathcal{H}a_n = \frac{1}{2} \mathcal{H}_K a_n + \frac{4}{\pi^2} \sum_{k \in 2\mathbb{Z}+1} \frac{1}{k^2} \mathcal{H}_K a_{n-k}.
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- In particular, $\|\mathcal{H}\|_{\ell^p \to \ell^p} \leq \|\mathcal{H}_K\|_{\ell^p \to \ell^p}$. 
Which discretisation is the right one?

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\|H\|_{L^p \to L^p} \leq \begin{cases} 
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- $\mathcal{H}$, $\mathcal{H}_{RT}$, $\mathcal{H}_K$ and $\mathcal{H}_{ADP}$ are contractions on $\ell^2$. 
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- $H$, $H_{\text{RT}}$, $H_K$ and $H_{\text{ADP}}$ are contractions on $\ell^2$.
- Only $H_{\text{RT}}$ and $H_K$ are unitary on $\ell^2$. 
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- Only $\mathcal{H}_{RT}$ and $\mathcal{H}_K$ are unitary on $\ell^2$.
- The conjecture $\|\mathcal{H}_{RT}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}$ is stronger and more interesting than $\|\mathcal{H}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}$. 

Our method completely fails for $\mathcal{H}_{RT}$.

Our result is likely the second one of its kind, where the $L^p$ norms of a singular integral operator and its $\ell^p$ discretisation are proved to be equal. A similar statement for second-order Riesz transforms was proved by K. Domolevo and S. Petermichl in 2014.
Which discretisation is the right one?

\[ \| H \|_{L^p \to L^p} \leq \left\{ \| \mathcal{H} \|_{L^p \to L^p}, \| \mathcal{H}_{ADP} \|_{L^p \to L^p} \right\} \leq \| \mathcal{H}_{RT} \|_{L^p \to L^p} = \| \mathcal{H}_K \|_{L^p \to L^p}. \]

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- The conjecture \( \| \mathcal{H}_{RT} \|_{L^p \to L^p} = \| H \|_{L^p \to L^p} \) is stronger and more interesting than \( \| \mathcal{H} \|_{L^p \to L^p} = \| H \|_{L^p \to L^p} \).
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- \( \mathcal{H} \), \( \mathcal{H}_{RT} \), \( \mathcal{H}_K \) and \( \mathcal{H}_{ADP} \) are contractions on \( \ell^2 \).
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- The conjecture \( \|\mathcal{H}_{RT}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p} \) is stronger and more interesting than \( \|\mathcal{H}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p} \).
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Hilbert transform and harmonic functions

- Let $f \in L^p$. For $y > 0$ we define the Poisson integrals

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - z) \frac{y}{z^2 + y^2} \, dz,$$

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- Then $u$ and $v$ are conjugate harmonic functions:
  \[ \Delta u = \Delta v = 0, \quad \nabla v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla u. \]
Hilbert transform and harmonic functions

- Let \( f \in L^p \). For \( y > 0 \) we define the Poisson integrals

\[
\begin{align*}
  u(x, y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - z) \frac{y}{z^2 + y^2} \, dz, \\
  v(x, y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - z) \frac{z}{z^2 + y^2} \, dz.
\end{align*}
\]

- Then \( u \) and \( v \) are conjugate harmonic functions:

\[
\Delta u = \Delta v = 0, \quad \nabla v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla u.
\]

- The boundary values of \( u \) and \( v \) are given by

\[
\begin{align*}
  f(x) &= \lim_{y \to 0^+} u(x, y), \\
  Hf(x) &= \lim_{y \to 0^+} v(x, y)
\end{align*}
\]

(the limits exist in \( L^p \) and almost everywhere).
Hilbert transform and harmonic functions

• Let $f \in L^p$. For $y > 0$ we define the Poisson integrals

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - z) \frac{y}{z^2 + y^2} \, dz,$$

$$v(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x - z) \frac{z}{z^2 + y^2} \, dz.$$

• Then $u$ and $v$ are conjugate harmonic functions:

$$\Delta u = \Delta v = 0, \quad \nabla v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla u.$$

• The boundary values of $u$ and $v$ are given by

$$f(x) = \lim_{y \to 0^+} u(x, y), \quad Hf(x) = \lim_{y \to 0^+} v(x, y)$$

(the limits exist in $L^p$ and almost everywhere).

• Define $u(x, 0) = f(x), \ v(x, 0) = Hf(x)$. 
Harmonic functions and martingales

- Let $B_t$ be the 2-D standard Brownian motion.
Harmonic functions and martingales

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- Since $u$ is a harmonic function in $\mathbb{R} \times (0, \infty)$, the process
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  is a martingale.
- Indeed: by the Itô formula, for $t < \tau$ we have
  \[ dM_t = \nabla u(B_t) \cdot dB_t, \]
  \[ d[M]_t = |\nabla u(B_t)|^2 dt. \]
Hilbert transform and martingales

- We have defined two conjugate harmonic functions: $u(x, y)$ and $v(x, y)$, with boundary values $f(x)$ and $Hf(x)$, respectively.
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  and
  $$d[M, N]_t = \nabla u(B_t) \cdot \nabla v(B_t) dt = 0 dt$$
  for $t < \tau$. 
Burkholder’s inequality

Theorem (R. Bañuelos, G. Wang, 1995)

If $M_t$ and $N_t$ are martingales and

- $N_t$ is differentially subordinate to $M_t$:
  \[ d[N]_t \leq d[M]_t; \]

- $M_t$ and $N_t$ are orthogonal:
  \[ d[M, N]_t = 0dt, \]

then

\[ \mathbb{E}|N_\infty - N_0|^p \leq (C_p)^p \mathbb{E}|M_\infty - M_0|^p, \]

with $C_p = \max\{\tan\left(\frac{\pi}{2p}\right), \cot\left(\frac{\pi}{2p}\right)\}$. 
Summary

- We begin with $f \in L^p$. 

- Then we define two conjugate harmonic functions $u$ and $v$, with boundary values $f$ and $Hf$.

- The corresponding martingales $M_t = u(B_{\min\{t, \tau\}})$ and $N_t = v(B_{\min\{t, \tau\}})$.

- Since $M_{\infty} = u(B_{\tau}) = f(B_{\tau})$ and $N_{\infty} = v(B_{\tau}) = Hf(B_{\tau})$,

  Burkholder's inequality implies that $E |Hf(B_{\tau}) - v(0, y_0)|^p \leq (Cp)^PE |f(B_{\tau}) - u(0, y_0)|^p$.

- We now pass to the limit as $y_0 \to \infty$. 

Summary

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- We now pass to the limit as $y_0 \to \infty$. 

Since $B_{\tau}$ has a Cauchy distribution on $\mathbb{R} \times \{0\}$, we have

\[
\int_{-\infty}^{\infty} |Hf(x) - v(0, y_0)|^p \frac{y_0}{x^2 + y_0^2} \, dx 
\]

\[
\leq (C_p)^p \int_{-\infty}^{\infty} |f(x) - u(0, y_0)|^p \frac{y_0}{x^2 + y_0^2} \, dx.
\]
Pichorides estimate

- Since $B_\tau$ has a Cauchy distribution on $\mathbb{R} \times \{0\}$, we have

$$
\int_{-\infty}^\infty |Hf(x) - \nu(0, y_0)|^p \frac{y_0^2}{x^2 + y_0^2} \, dx \\
\leq (C_p)^p \int_{-\infty}^\infty |f(x) - u(0, y_0)|^p \frac{y_0^2}{x^2 + y_0^2} \, dx.
$$

- We multiply both sides by $y_0$ and pass to the limit as $y_0 \to \infty$ to get the Pichorides bound

$$
\|Hf\|_{L^p}^p \leq (C_p)^p \|f\|_{L^p}^p.
$$
Conditioned process

- The Brownian motion $B_t$ hits the entire boundary $\mathbb{R} \times \{0\}$ as it exists the half-plane $\mathbb{R} \times (0, \infty)$.
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Conditioned process

- The Brownian motion $B_t$ hits the entire boundary $\mathbb{R} \times \{0\}$ as it exists the half-plane $\mathbb{R} \times (0, \infty)$.
- We need a diffusion $X_t$ which only hits the discrete subset of the boundary: $\mathbb{Z} \times \{0\}$.
- The process $X_t$ can be defined as the limit as $\varepsilon \to 0^+$ of the Brownian motion $B_t$ conditioned on the event

$$B_\tau \in \left( \bigcup_{k \in \mathbb{Z}} (k - \varepsilon, k + \varepsilon) \right) \times \{0\}.$$
Stochastic differential equation

- More rigorously: $X_t$ is defined as the Brownian motion $B_t$ conditioned in the sense of Doob by the harmonic function

\[
h(x, y) = \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{y}{(x - k)^2 + y^2} = \frac{\sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi x)}.
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- The corresponding generator is

$$\frac{1}{2} h^{-1} \Delta (hu) = \frac{1}{2} \Delta u + \frac{\nabla h \cdot \nabla u}{h}.$$
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\]

- Equivalently: \( X_t \) is a solution of the SDE

\[
dX_t = dB_t + \frac{\nabla h(X_t)}{h(X_t)} \, dt.
\]
Martingale

- Consider a sequence $a_n$ in $\ell^p$. 
Martingale

- Consider a sequence $a_n$ in $\ell^p$.
- The $X_t$-harmonic extension of this sequence is defined by

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- We extend $u$ to a continuous function on $\mathbb{R} \times [0, \infty)$ so that $u(n, 0) = a_n$. 

\[ \text{Main result} \quad \text{Some history} \quad \text{Continuous transform} \quad \text{Discrete transform} \quad \text{Comments} \]
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- The formula $M_t = u(X_{\min\{t, \tau\}})$ defines a martingale.
- There is no notion of a conjugate $X_t$-harmonic function!
Martingale transform

- Since $M_t = u(X_{\min\{t, \tau\}})$ and

\[
dX_t = dB_t + \frac{\nabla h(X_t)}{h(X_t)} dt,
\]

\[
d[X]_t = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) dt,
\]

\[
\nabla u \cdot \frac{\nabla h}{h} + \frac{1}{2} \Delta u = 0,
\]
Martingale transform

- Since $M_t = u(X_{\min\{t,\tau\}})$ and

$$dX_t = dB_t + \frac{\nabla h(X_t)}{h(X_t)} \, dt,$$

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<table>
<thead>
<tr>
<th>Main result</th>
<th>Some history</th>
<th>Continuous transform</th>
<th>Discrete transform</th>
<th>Comments</th>
</tr>
</thead>
</table>

**Summary — changes!**

- We begin with $f \in L^p$, $a_n \in \ell^p$
Summary — changes!

- We begin with $f \in L^p$, $a_n$ in $\ell^p$.
- Then we define two conjugate harmonic functions $u$ and $v$, with boundary values $f$ and $Hf$, an $X_t$-harmonic function $u$. 
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- We begin with $f \in L^p$ and $a_n$ in $\ell^p$.

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  \[ dN_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla u(X_t) \cdot dB_t \]

• Since \( M_\infty = u(B_\tau) = f(B_\tau) \) and \( N_\infty = v(B_\tau) \),
Burkholder’s inequality implies that

\[
E|Hf(B_\tau) - \nu(0, y_0)|^p \leq (C_p)^p E|f(B_\tau) - u(0, y_0)|^p.
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Summary — changes!

- We begin with $a_n$ in $\ell^p$

- Then we define $u$ as an $X_t$-harmonic function

- ...and two martingales $M_t = u(\chi_{\min\{t, \tau\}})$ and

$$dN_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla u(X_t) \cdot dB_t$$

- Since $M_\infty = u(\chi_{\tau}) = a_{X_\tau}$, Burkholder's inequality implies that

$$\mathbb{E}|N_\tau - N_0|^p \leq (C_p)^p \mathbb{E}|a_{X_\tau} - u(0, y_0)|^p.$$
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- Then we define an $X_t$-harmonic function $u$.
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Since $M_\infty = u(X_\tau) = aX_\tau$, Burkholder’s inequality implies that

$$\mathbb{E}|N_\tau - N_0|^p \leq (C_p)^p \mathbb{E}|aX_\tau - u(0, y_0)|^p.$$

By Jensen’s inequality,

$$\mathbb{E}|\mathbb{E}(N_\tau - N_0 | X_\tau)|^p \leq (C_p)^p \mathbb{E}|aX_\tau - u(0, y_0)|^p.$$
Estimate of the $\ell^p$ norm of some transform

- As in the continuous case, by passing to the limit as $y_0 \to \infty$, we find that

$$
\| b_n \|_{\ell^p}^p \leq (C_p)^p \| a_n \|_{\ell^p}^p,
$$

where

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b_n = \lim_{y_0 \to \infty} \mathbb{E}(N_\tau - N_0 | X_\tau = (n, 0)).
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- Surprise: after tedious calculations, we obtain
  \[
  b_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 k^2) \sinh^2 y} \, dy\right).
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Estimate of the $\ell^p$ norm of some transform

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  (It is easy to err and drop a minus sign, and get $b_n = \mathcal{H}a_n$).

- If we write $b_n = \tilde{\mathcal{H}}a_n$, then we have $\|\tilde{\mathcal{H}}\|_{\ell^p \to \ell^p} \leq C_p$. 
Convolution

- The operator $\tilde{H}$ is a convolution operator with kernel

$$\tilde{h}_n = \frac{1}{\pi n} \left( 1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 n^2) \sinh^2 y} \, dy \right) 1_{\mathbb{Z}\backslash\{0\}}(n).$$
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- To complete the proof, it suffices to give a probability sequence $\varrho_n$ such that

$$\mathcal{H}a_n = \sum_{k \in \mathbb{Z}} \varrho_k \tilde{\mathcal{H}}a_{n-k}.$$
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  \]

- The sequence $\varrho_n$ is found explicitly (in an integral form), after lengthy calculations.
Remarks about the proof

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- In the proof, the operator $\mathcal{H}$ is expressed as the composition of four operations:

  1. definition of the martingale: $a_n \leadsto M_t$;
  2. martingale transform: $M_t \leadsto N_t$;
  3. conditional expectation: $N_t \leadsto \tilde{\mathcal{H}}a_n$;

- Items (3) and (4) do not preserve the $\ell^2$ norm, and therefore no similar argument can be given for the unitary operators $H_{RT}$ and $H_{KT}$.

- The sequence $H_{ADP}a_n$ can be expressed as a convolution of $\tilde{\mathcal{H}}a_n$ with some sequence $\varrho_n$, but $\varrho_n$ contains negative entries. For this reason a similar argument leads to a weaker estimate of $\|H_{ADP}\|_{\ell^p} \rightarrow \ell^p$ than conjectured.
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• In the proof, the operator $\mathcal{H}$ is expressed as the composition of four operations:

1. definition of the martingale: $a_n \rightsquigarrow M_t$;
2. martingale transform: $M_t \rightsquigarrow N_t$;
3. conditional expectation: $N_t \rightsquigarrow \tilde{\mathcal{H}} a_n$;
4. convolution with $\varrho_n$: $\tilde{\mathcal{H}} a_n \rightsquigarrow \mathcal{H} a_n$. 
Remarks about the proof

- In the proof, the operator $\mathcal{H}$ is expressed as the composition of four operations:
  (1) definition of the martingale: $a_n \leadsto M_t$;
  (2) martingale transform: $M_t \leadsto N_t$;
  (3) conditional expectation: $N_t \leadsto \tilde{\mathcal{H}}a_n$;
  (4) convolution with $\varrho_n$: $\tilde{\mathcal{H}}a_n \leadsto \mathcal{H}a_n$.

- Items (3) and (4) do not preserve the $\ell^2$ norm, and therefore no similar argument can be given for the unitary operators $\mathcal{H}_{RT}$ and $\mathcal{H}_K$. 
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• The sequence $\mathcal{H}_{ADP}a_n$ can be expressed as a convolution of $\tilde{\mathcal{H}}a_n$ with some sequence $\varrho_n$, but $\varrho_n$ contains negative entries. For this reason a similar argument leads to a weaker estimate of $\|\mathcal{H}_{ADP}\|_{\ell^p \to \ell^p}$ than conjectured.
Open problems

Conjecture 1

We have \( \| \mathcal{H}_{RT} \|_{\ell^p \to \ell^p} = C_p \)
Open problems

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We have $\|\mathcal{H}_{RT}\|_{\ell^p \to \ell^p} = C_p$ or at least $\|\mathcal{H}_{ADP}\|_{\ell^p \to \ell^p} = C_p$.
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**Conjecture 2**

We have $\|\mathcal{H}_{RT}\|_{\ell^1 \to \ell^1,w} = \|H\|_{L^1 \to L^1,w}$.
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**Conjecture 1**

We have \( \| \mathcal{H}_{RT} \|_{\ell^p \to \ell^p} = C_p \) or at least \( \| \mathcal{H}_{ADP} \|_{\ell^p \to \ell^p} = C_p \).

**Conjecture 2**

We have \( \| \mathcal{H}_{RT} \|_{\ell^1 \to \ell^1,w} = \| H \|_{L^1 \to L^1,w} \).

**Conjecture 3**

The discrete Riesz transform

\[
\mathcal{R}_j a_n = c_d \sum_{k \in \mathbb{Z}^d \setminus \{(0,0,\ldots,0)\}} a_{n-k} \frac{k_j}{|k|^{d+1}}
\]

satisfies \( \| \mathcal{R}_j \|_{\ell^p \to \ell^p} = C_p \).