Boundary Harnack inequality for stable processes on the Sierpiński gasket

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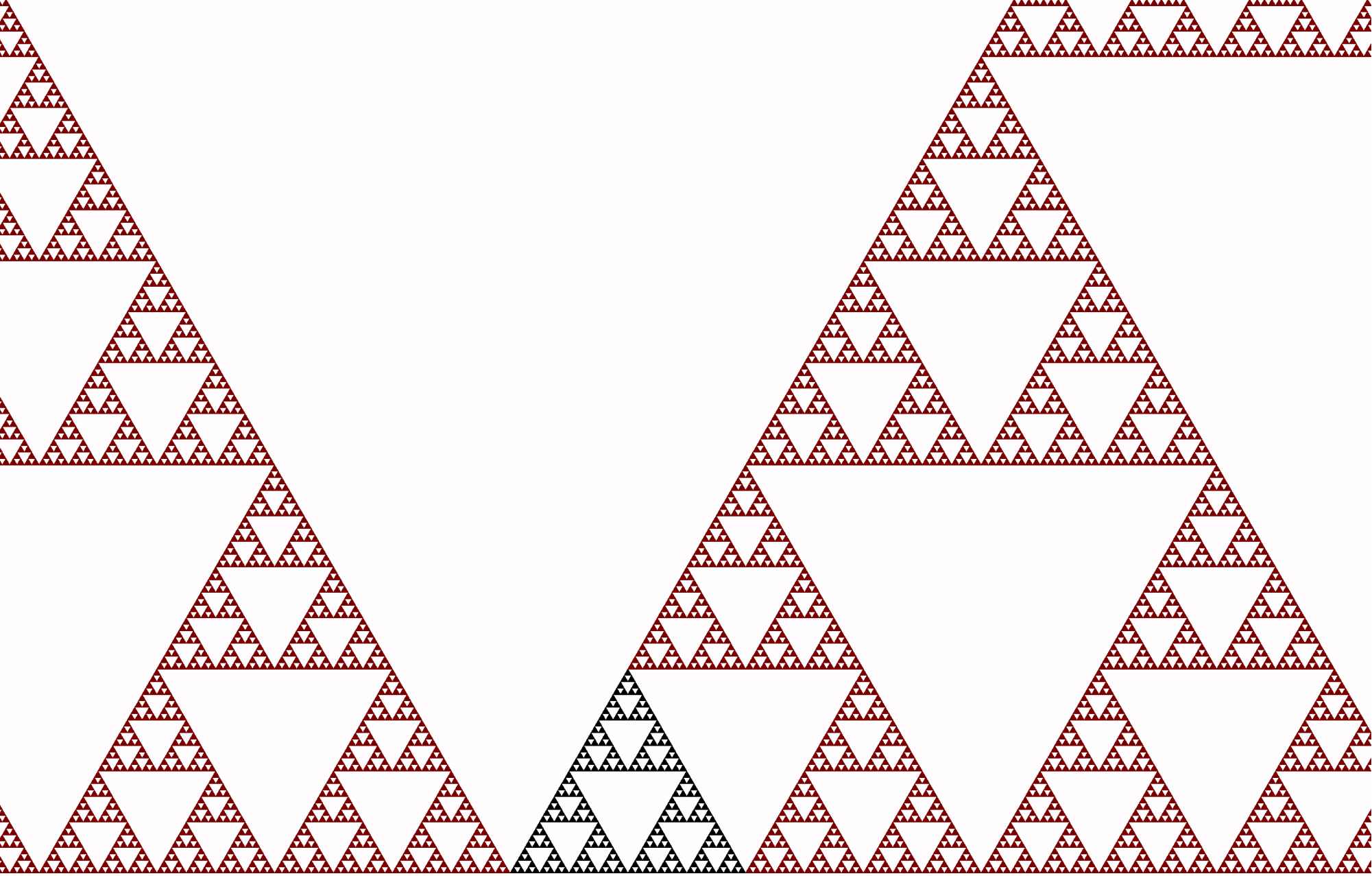
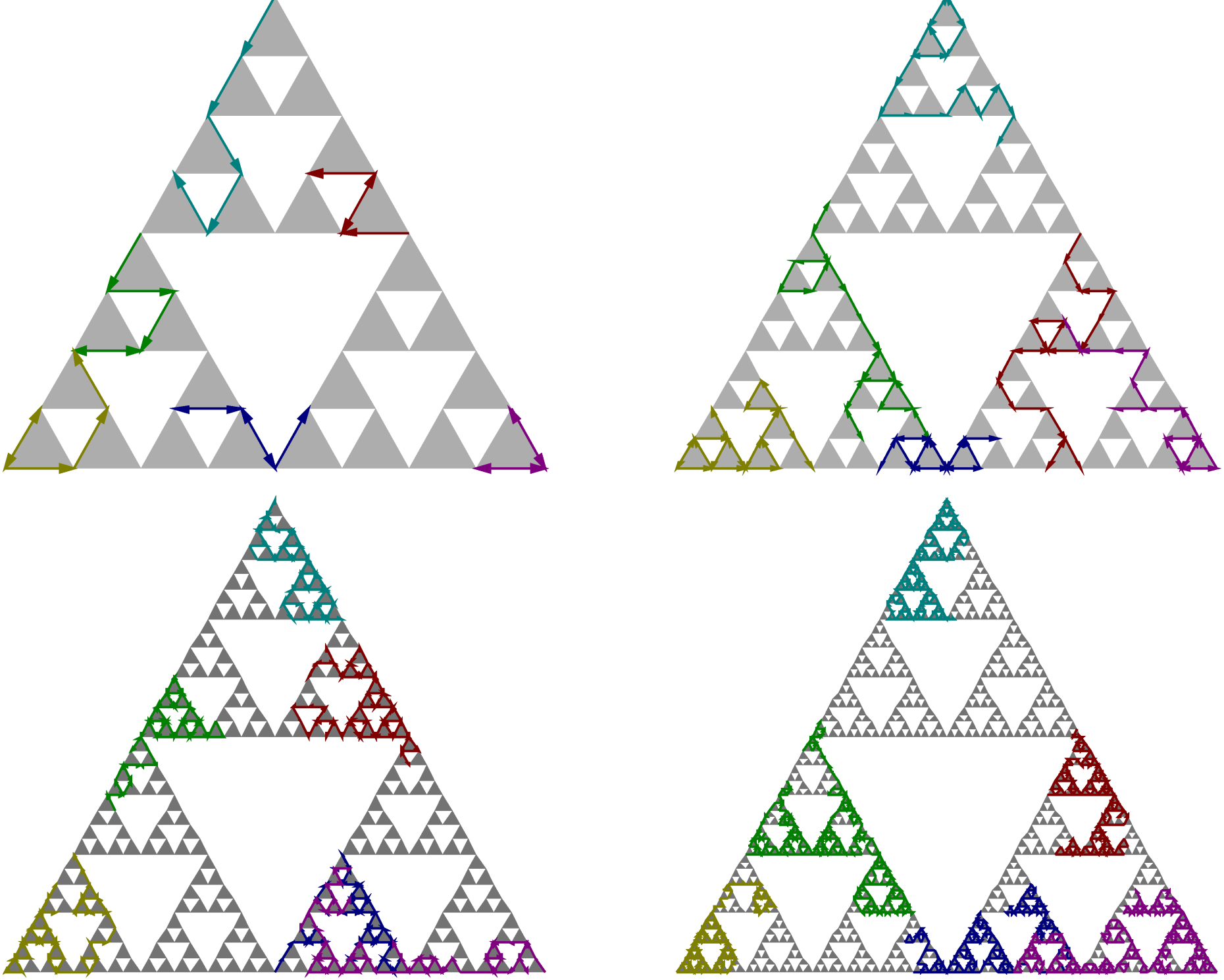
Based on:

- K. Kaleta, MK, *Boundary Harnack inequality for α-harmonic functions on the Sierpiński gasket*, preprint



Sierpiński gasket (SG)



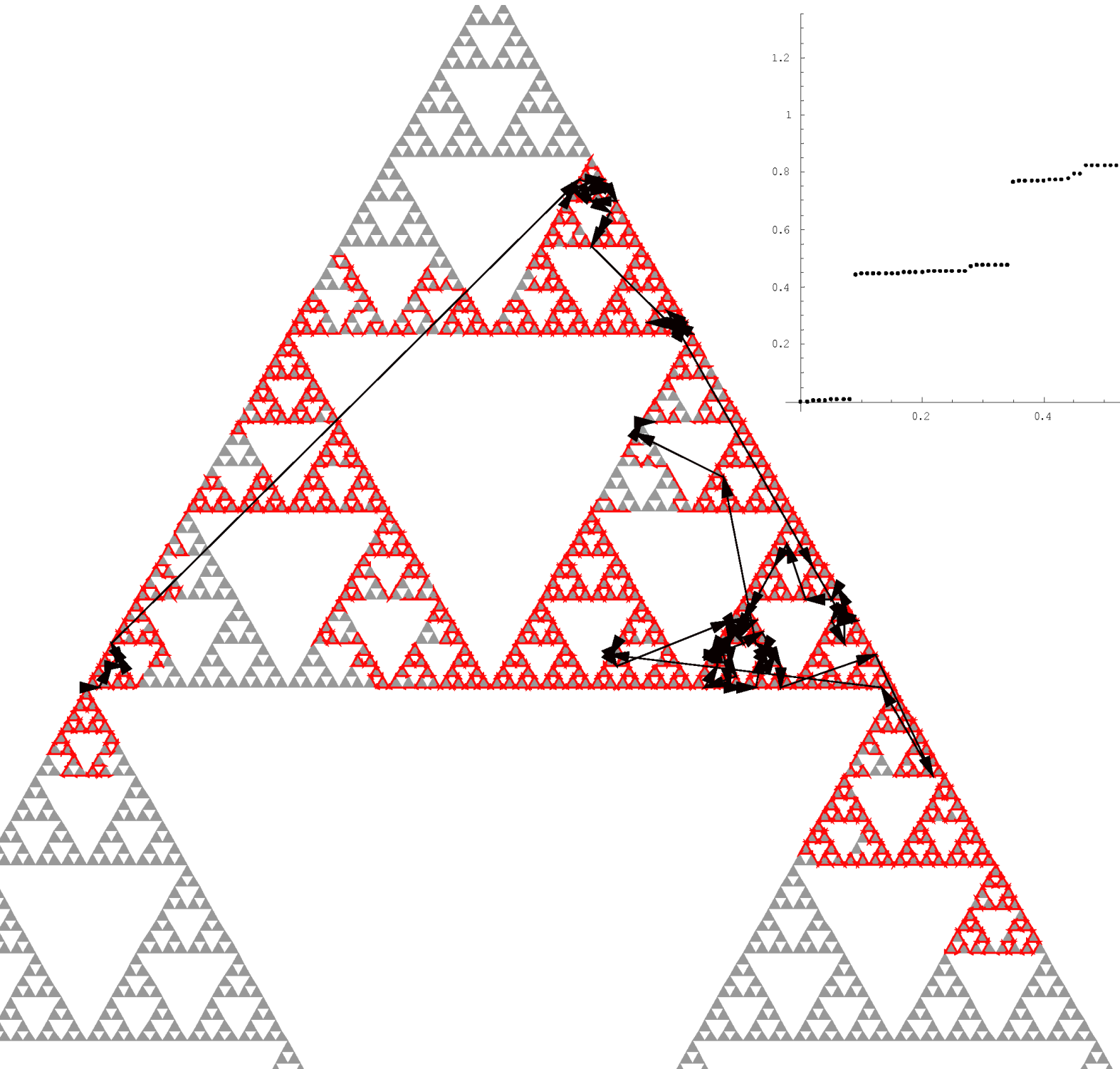
- (β_t) — diffusion process on SG
- $d_w = \frac{\log 5}{\log 2} \approx 2.322$ — walk dimension
- (β_t) is d_w-stable

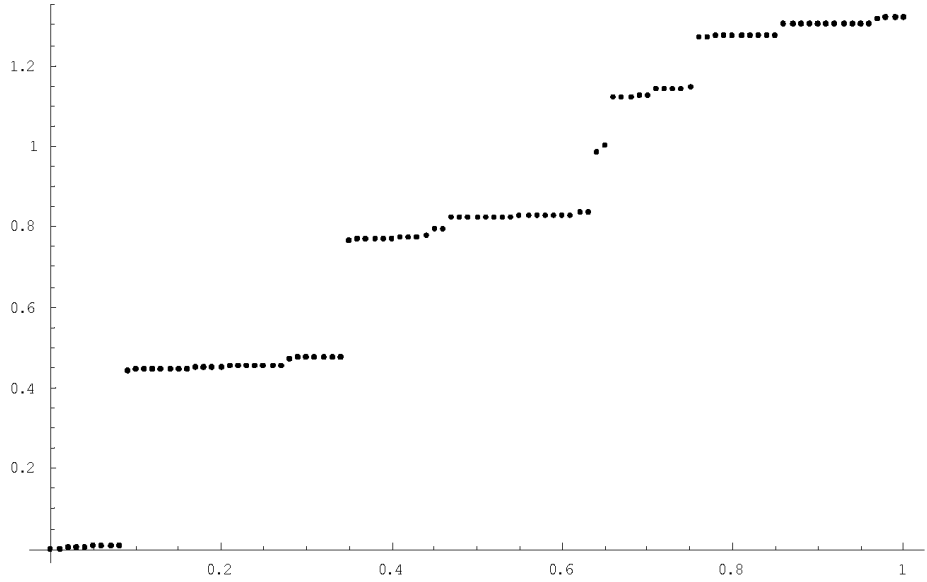
$$\left(2\beta_t\right) \stackrel{D}{=} \left(\beta_{2^{d_w} t}\right)$$

- (η_t) — $\frac{\alpha}{d_w}$-stable subordinator
  (increasing jump process, $\alpha \in (0, d_w)$)

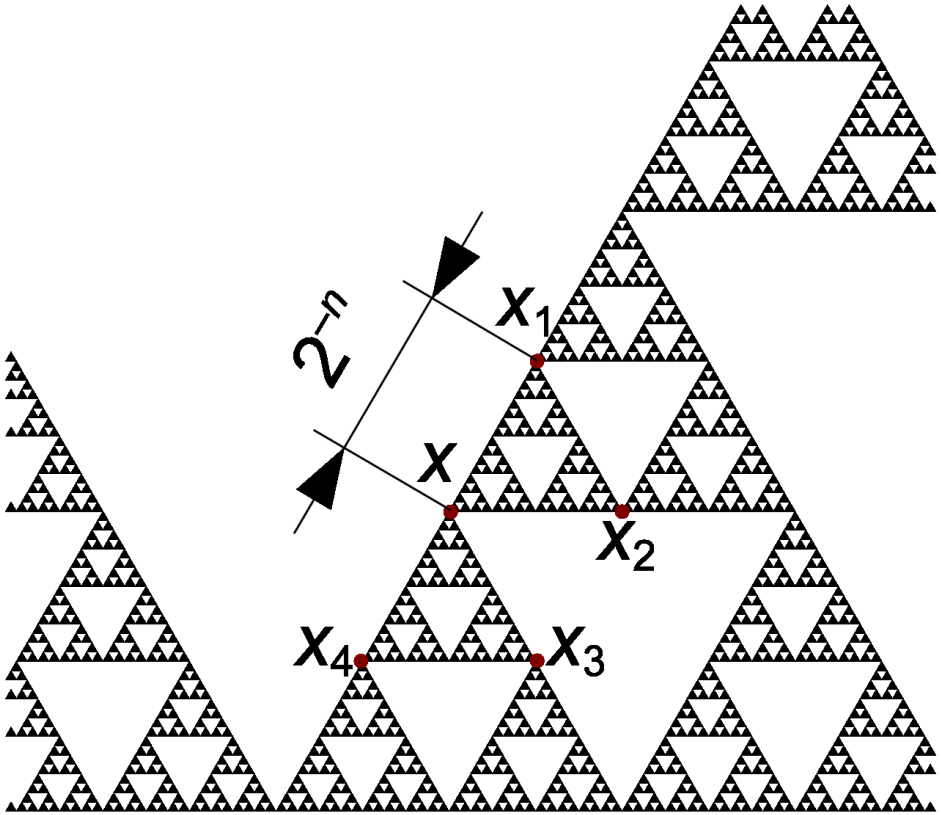
- $(X_t) = (\beta_{\eta_t})$ — α-stable process

$$\left(2X_t\right) \stackrel{D}{=} \left(X_{2^{\alpha} t}\right)$$




1.2
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1

Δ — Kigami Laplacian on SG, infinitesimal generator of (β_t)



Limit of rescaled discrete Laplacians

$$\Delta_n f(x) = \frac{f(x_1) + f(x_2) + f(x_3) + f(x_4)}{4} - f(x)$$

- Δ — Laplacian on SG, infinitesimal generator of (β_t) (local, self-adjoint, negative definite)

- $-(-\Delta)^{\alpha/d_w}$ — fractional Laplacian, inf. gen. of (X_t) (non-local, self-adjoint, negative definite)

$$
\begin{aligned}
-(-\Delta)^{\alpha/d_w} f(x) &= \lim_{t \searrow 0} \frac{\mathbf{E}^x f(X_t) - f(x)}{t} \\
&= PV \int (f(y) - f(x))\, \nu(x,y)\, \mu(dy) \\
\nu(x,y)\, \mu(dy) &= \lim_{t \searrow 0} \frac{\mathbf{P}^x(X_t \in dy)}{t} \\
&\asymp C\, |x-y|^{-d-\alpha}\, \mu(dy)
\end{aligned}
$$

- Δ — Laplacian on SG, infinitesimal generator of (β_t) (local, self-adjoint, negative definite)

- Δ^{α/d_w} — fractional Laplacian, inf. gen. of (X_t) (non-local, self-adjoint, negative definite)

$$\Delta^{\alpha/d_w} f(x) = \lim_{t \searrow 0} \frac{\mathbf{E}^x f(X_t) - f(x)}{t}$$

$$= PV \int (f(y) - f(x))\, \nu(x,y)\, \mu(dy)$$

$$\nu(x,y)\, \mu(dy) = \lim_{t \searrow 0} \frac{\mathbf{P}^x(X_t \in dy)}{t}$$

$$\asymp C\, |x-y|^{-d-\alpha}\, \mu(dy)$$

$D \subseteq \mathrm{SG}$ — open

Definition

A function $f : D \to [0, \infty)$ is said to be harmonic in D if

$$\Delta f(x) = 0 \qquad \text{for } x \in D$$

Equivalently,

$$f(x) = \mathbf{E}^x f(\beta(\tau_B)) \qquad \text{for } x \in B,\ \overline{B} \subseteq D$$

(τ_B — first exit time from B)

$D \subseteq \mathrm{SG}$ — open

Definition

A function $f : \mathrm{SG} \to [0, \infty)$ is said to be α-harmonic in D if

$$\Delta^{\alpha/d_w} f(x) = 0 \qquad \text{for } x \in D$$

Equivalently,

$$f(x) = \mathbf{E}^x f(X(\tau_B)) \qquad \text{for } x \in B,\ \overline{B} \subseteq D$$

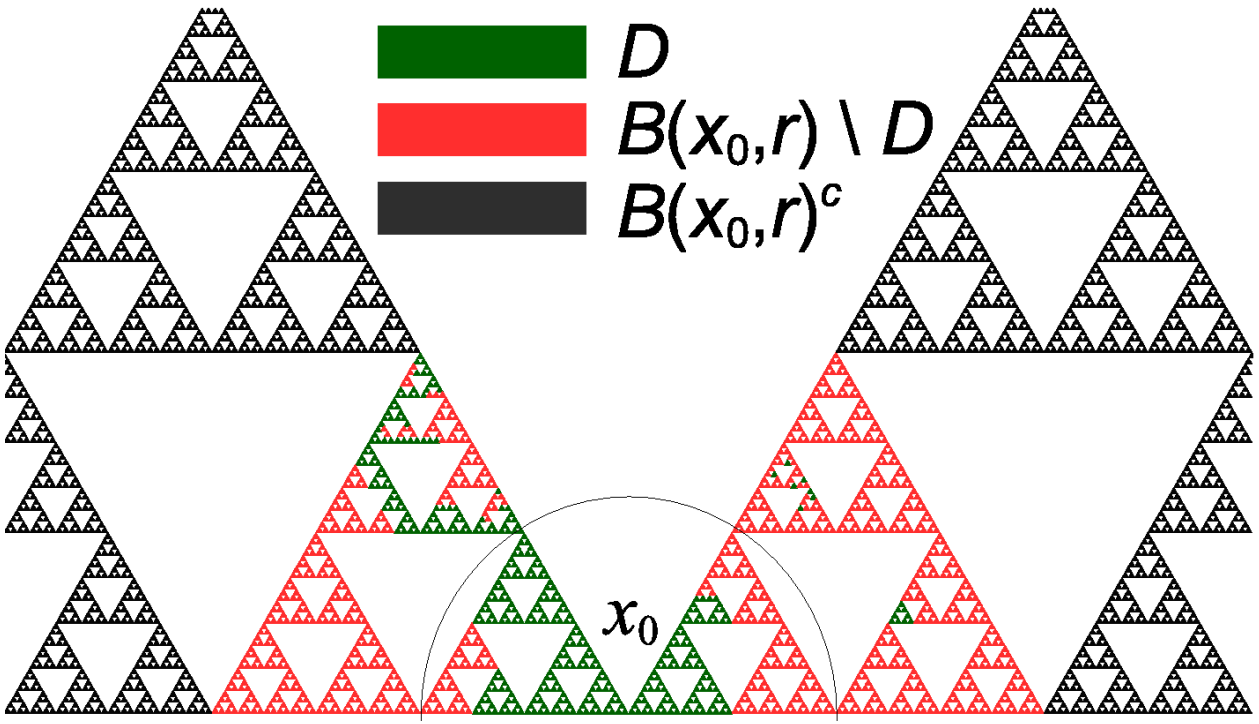
(τ_B — first exit time from B)

Uniform boundary Harnack inequality (UBHI)

Let $\alpha < 1$

$$\frac{f(x)}{g(x)} \asymp C(\alpha)\,\frac{f(y)}{g(y)} \qquad \text{for } x, y \in D \cap B\left(x_0, \frac{r}{2}\right)$$

for all x_0, r, D and all f, g nonnegative, α-harmonic in D and vanishing in $B(x_0, r) \setminus D$.



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UBHI in $\mathbf{R}^d$

K. Bogdan, T. Kulczycki, MK (2008)

Tools:

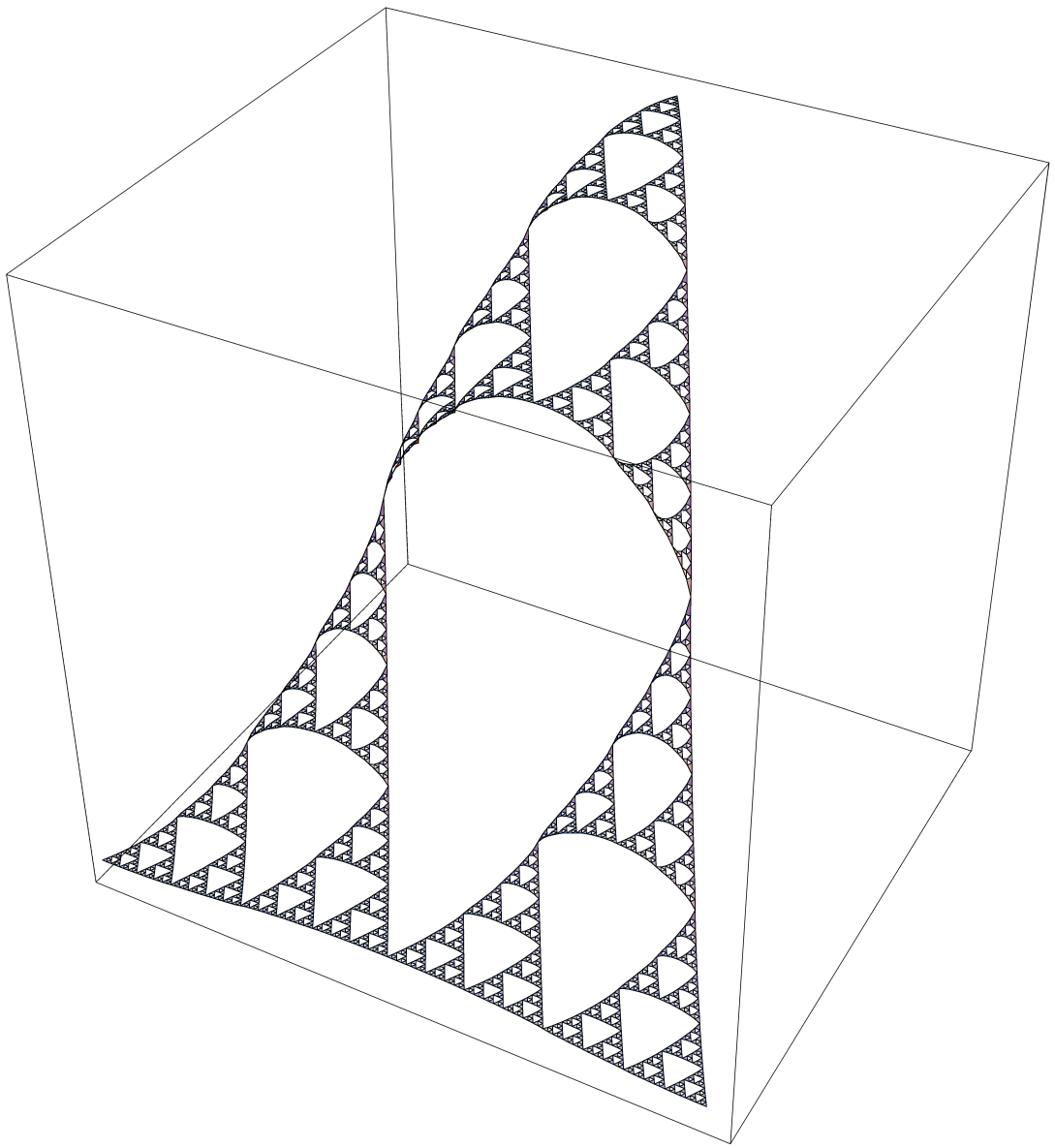
- cut-off function
- estimates on exit distribution from balls

Cut-off function

A function f such that

- $f(x) = 1$ for $x \in B(x_0, \frac{1}{2}r)$
- $f(x) = 0$ for $x \notin B(x_0, r)$
- $\Delta^{\alpha/d_w} f$ is bounded

Cut-off function



Splines on fractals (R.S. Strichartz, M. Usher)

Estimates on exit distribution from balls

(K. Bogdan, A. Stós, P. Sztonyk)

- Does (X_t) hit the boundary of $B(x_0, r)$ on exit?
  - $\partial B(x_0, r)$ is generically $(d-1)$-dimensional (*product formula*)
  - $(d-1)$-dimensional object is polar if $\alpha < 1$
  - (X_t) never hits polar sets

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  - (X_t) never hits polar sets

- Asymptotics of distribution density near boundary
  - Ikeda-Watanabe formula / Dynkin formula
  - rough estimate of Green function for $\alpha < 1$