The solution of M-D problem

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1 Problem description

Let p, q be any integer numbers greater that 1. We define the *multiply-and-divide (M-D) sequence* (a_n) for the multiplier p and divisor q recursively as follows. Begin with $a_1 = 1$ and take a_{n+1} equal to either $\lfloor a_n/q \rfloor$ ($\lfloor x \rfloor$ denotes floor of x, the largest integer not exceeding x) if this number is positive and has not yet occured in the sequence or pa_n otherwise.

For instance, if $p = 3$ and $q = 2$, the corresponding M-D sequence is $1, 3, 9, 4, 2, 6, 18, 54, 27, 13, 39, 19, 57, 28, 14, 7, 21, \ldots$.

Given p, q, the M-D problem is to determine whether every positive integer occurs exactly once in the M-D sequence. In this paper we prove that the answer is positive if and only if $\log_q p$ is irrational. This condition is equivalent to $p^n \neq q^m$ unless $n = m = 0$. In particular if p and q are relatively prime, every positive integer occurs exactly once in the M-D sequence.

The M-D problem originated in *Crux Mathematicorum*, volume 26 (2000), problem 2248.

Note that the 'only if' part is clear: if $\log_q p$ is rational, then $q^n = p^m$ for some relatively prime $n, m > 0$. Hence $q^{1/m} = p^{1/n}$ is integer greater than 1; denote it by d. Clearly if (a_n) is the M-D sequence for p and q, then $a_n = d^{k_n}$ for some nonnegative integer k_n. In particular there are positive integer numbers (such as $d+1$) which do not occur in the M-D sequence. The rest of this article is dedicated to the 'if' part of the statement.

2 Notation

We denote sets of integer numbers, positive integer numbers and real numbers by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$ respectively. The quotient group of additive (topological) groups $\mathbb{R}$ and $\mathbb{Z}$ (which is isomorphic to $[0, 1)$ with addition 'modulo 1') will be denoted by $\mathbb{T}$. Elements of $\mathbb{T}$ are cosets $[a] = a + \mathbb{Z} = \{a + n : n \in \mathbb{Z}\}$, where $a \in \mathbb{R}$. Let $\kappa : \mathbb{R} \to \mathbb{T}$ be the canonical homomorphism defined by $\kappa(a) = [a]$. We recall that a subset U of $\mathbb{T}$ is open if and only if $\kappa^{-1}(U)$ is an open subset of $\mathbb{R}$ (with usual topology). Moreover, if V is any open subset of $\mathbb{R}$ then $\kappa(V)$ is open in

$\mathbb{T}$, that is κ is an open mapping. In fact, define $U = \kappa(V)$, $W = \kappa^{-1}(U)$. Then:

$$\begin{aligned} W &= \{a : [a] \in U\} = \{a : \exists_{b \in U}\ a \in [b]\} = \\ &= \{a : \exists_{b \in U} \exists_{n \in \mathbb{Z}}\ a = b + n\} = \{a : \exists_{n \in \mathbb{Z}}\ a \in V + n\} = \\ &= \bigcup_{n \in \mathbb{Z}} (V + n) \end{aligned}$$

is open since every $(V + n) = \{v + n : v \in V\}$ is open. Hence U is open.

We denote natural logarithm of x by $\ln x$ and logarithm of x in base y by $\log_y x$.

3 Formal definition of M-D sequence

Take any $p, q > 1$ such that $\log_q p$ is irrational and denote the corresponding M-D sequence (a_n). Define:

$$\begin{aligned} &A_n = \{0\} \cup \{a_m : m \leq n\}, \\ &A = \{a_m : m \in \mathbb{Z}^+\}, \\ &B = \mathbb{Z}^+ \backslash A. \end{aligned}$$

The M-D sequence (a_n) is thus defined by:

$$\begin{aligned} &a_1 = 1, \\ &a_{n+1} = p a_n && \text{if } \lfloor a_n/q \rfloor \in A_n, \\ &a_{n+1} = \lfloor a_n/q \rfloor && \text{if } \lfloor a_n/q \rfloor \notin A_n. \end{aligned} \tag{1}$$

We need to prove two statements:

$$a_n = a_m \text{ implies } n = m, \tag{2}$$

$$B = \emptyset. \tag{3}$$

4 Two properties of A_n, A and B

Fix any positive n and assume that $\lfloor a_n/q^k \rfloor \in A_n$ for some $k \in \mathbb{Z}^+$. If $\lfloor a_n/q^k \rfloor = 0$, then $\lfloor a_n/q^{k+1} \rfloor = 0 \in A_n$. If $\lfloor a_n/q^k \rfloor \neq 0$, then $\lfloor a_n/q^k \rfloor = a_m$ for some $m \leq n$; m cannot be equal to n, since $\lfloor a_n/q^k \rfloor < a_n$. Thus $m < n$. It follows that either $\lfloor a_m/q \rfloor \in A_m \subset A_n$ or $\lfloor a_m/q \rfloor = a_{m+1} \in A_n$. Hence $\lfloor a_n/q^{k+1} \rfloor = \lfloor a_m/q \rfloor \in A_n$. By an induction argument we get:

$$\text{If } \lfloor a_n/q \rfloor \in A_n, \text{ then } \lfloor a_n/q^k \rfloor \in A_n \text{ for any } k \in \mathbb{Z}^+. \tag{4}$$

Now take any $a \in A$. Then $a = a_n$ for some n. Either $\lfloor a_n/q \rfloor = a_{n+1} \in A$ or $\lfloor a_n/q \rfloor \in A_n \subset A \cup \{0\}$. Hence $\lfloor a/q \rfloor \in A$ or $\lfloor a/q \rfloor = 0$. By induction we get $\lfloor a/q^k \rfloor \in A$ or $\lfloor a/q^k \rfloor = 0$. Writing $b = \lfloor a/q^k \rfloor$ we get:

$$\text{If } b \in B, \text{ then } q^k b + l \in B \text{ for any } k \in \mathbb{Z}^+, 0 \leq l < q^k. \tag{5}$$

5 Simple floor law

Let $pa = \lfloor px \rfloor$, where $x \in \mathbb{R}$, $a \in \mathbb{Z}$ and $p \in \mathbb{Z}^+$. Then:

$$a = \lfloor a \rfloor = \left\lfloor \frac{\lfloor px \rfloor}{p} \right\rfloor = \left\lfloor \frac{px}{p} \right\rfloor = \lfloor x \rfloor .$$

Hence:

$$pa = \lfloor px \rfloor \text{ implies } a = \lfloor x \rfloor , \tag{6}$$

$x \in \mathbb{R}$, $a \in \mathbb{Z}$, $p \in \mathbb{Z}^+$.

6 Topological dynamics lemma

The following is a simple version of a general topological dynamics result on rotations on compact topological groups:

> Let a be an irrational number, $U \subset \mathbb{T}$ a nonempty open set. Then there exists $N \in \mathbb{Z}^+$ such that for every $x \in \mathbb{R}$ there exists n, $0 \leq n \leq N$, such that $[x + na] \in U$.

Fix $x \in \mathbb{R}$ and let $A = \{[x + na] : n \geq 0\}$. Then A is dense in $\mathbb{T}$ (see any book on topological dynamics). Hence $[x + na] \in U$ for some n and so $[x] \in \bigcup_{n=0}^{\infty}(U - [na])$. Since x was arbitrary, $\bigcup_{n=0}^{\infty}(U - [na]) = \mathbb{T}$. By compactness of $\mathbb{T}$ there exist $N \in \mathbb{Z}^+$ such that $\bigcup_{n=0}^{N}(U - [na]) = \mathbb{T}$. Take any $x \in \mathbb{R}$. Then, for some n, $0 \leq n \leq N$, we have $[x + na] \in U$.

7 Proof of (2)

Assume on the contrary that $a_i = a_j$ for some $i \neq j$. Let n be the smallest number such that $a_n = a_m$ for some m, $0 < m < n$. Clearly $a_n \in A_{n-1}$. Hence $a_n = pa_{n-1}$ ($a_n = \lfloor a_{n-1}/q \rfloor$ if and only if $\lfloor a_{n-1}/q \rfloor \notin A_{n-1}$). Let $i \leq m$ be such that:

$$a_m = \lfloor a_{m-1}/q \rfloor , a_{m-1} = \lfloor a_{m-2}/q \rfloor , \ldots , a_{i+1} = \lfloor a_i/q \rfloor , a_i = pa_{i-1}.$$

(Note that such i exists, since $a_2 = pa_1$.) Take $k = m - i$. Then:

$$pa_{n-1} = a_n = a_m = \left\lfloor a_i/q^k \right\rfloor = \left\lfloor pa_{i-1}/q^k \right\rfloor .$$

Hence, by (6), $a_{n-1} = \left\lfloor a_{i-1}/q^k \right\rfloor$. Now $a_i = pa_{i-1}$ implies $\lfloor a_{i-1}/q \rfloor \in A_{i-1}$. From property (4) it follows that:

$$a_{n-1} = \left\lfloor a_{i-1}/q^k \right\rfloor \in A_{i-1}.$$

In other words, $a_{n-1} = a_j$ for some $j \leq i - 1 < n - 1$. This contradicts the choice of n. Hence our assumption was false and so we proved (2). Note that we did not use irrationality of $\log_q p$.

8 Proof of (3)

Again, assume contrary to (3) that B is nonempty. Let a be any element of B. We will show that $q^t a + s \notin B$ for some $t, s \in \mathbb{Z}^+$, $0 \le s < q^t$, which contradicts property (5). The key observation is that $b = q^t a + s$ if and only if $\log_q a - \log_q b$ is nearly an integer or, in other words, the distance between $[\log_q a]$ and $[\log_q b]$ in $\mathbb{T}$ is very small. Hence it is sufficient to show that the set of cosets $[\log_q a_n]$ may not be separated from $[\log_q a]$ for all n. It will turn out that the sequence $[\log_q a_n]$ has much in common with irrational rotations, well studied transformations on $\mathbb{T}$.

Relation (2) implies that $\lim_{n\to\infty} a_n = \infty$. It follows that there are infinitely many n such that $a_{n+1} = pa_n$; let $k_1, k_2, \ldots$ be increasing sequence of all those n. Hence:

$$a_{k_{n+1}} = \left\lfloor \frac{a_{k_n+1}}{q^{k_{n+1}-k_n-1}} \right\rfloor, \quad a_{k_n+1} = pa_{k_n}. \tag{7}$$

Let $b_n = \log_q a_n$, $\alpha = \log_q p$; α is irrational. Define:

$$\epsilon_n = \log_q \frac{a_{k_n+1}}{q^{k_{n+1}-k_n-1}} - \log_q a_{k_{n+1}} = b_{k_n+1} - b_{k_{n+1}} - k_{n+1} + k_n + 1.$$

Relations (7) and $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$ for $x \in \mathbb{R}$ imply that:

$$\begin{aligned} 0 \le \epsilon_n &< \log_q(a_{k_{n+1}} + 1) - \log_q a_{k_{n+1}} = \\ &= \log_q \left(\frac{a_{k_{n+1}} + 1}{a_{k_{n+1}}} \right) < \frac{1}{a_{k_{n+1}} \ln q}. \end{aligned} \tag{8}$$

By definition of ϵ_n and (7):

$$\begin{aligned} \left[b_{k_{n+1}}\right] &= \left[b_{k_n+1} - \epsilon_n\right], \\ \left[b_{k_n+1}\right] &= \left[\log_q(pa_{k_n})\right] = \left[b_{k_n} + \alpha\right]. \end{aligned} \tag{9}$$

This is exactly what we needed: $[b_{k_{n+1}}]$ is very close to $[b_{k_n+1}]$, which is an irrational rotation of b_{k_n}. Hence we can apply topological dynamics methods. Before we proceed with some technical details, let us recall that we are looking for a_m such that $a_m = q^t a + s$ for some $t, s \in \mathbb{Z}^+$, $0 \le s < q^t$, which is equivalent to $\log_q a_m \in (\log_q a + t, \log_q(a+1) + t)$.

Define:

$$\begin{aligned} \delta &= \log_q(a + \tfrac{1}{2}) - \log_q a, \\ V &= \left(\log_q(a + \tfrac{1}{2}), \log_q(a+1)\right), \\ U &= \kappa(V). \end{aligned}$$

Since U is a nonempty open subset of $\mathbb{T}$ and α is irrational, a topogical dynamics lemma implies that there exists $L \in \mathbb{Z}^+$ such that for every $x \in \mathbb{R}$ we have $[x + l\alpha] \in U$ for some l, $0 \le l \le L$.

Let M be large enough so that $L < M\delta \ln q$. Since $\lim_{n\to\infty} a_n = \infty$, there exists N such that $a_n > M$ for $n \geq N$. Fix any n such that $k_n \geq N$. For some l, $0 \leq l \leq L$, we have $[b_{k_n} + l\alpha] \in U$. Equivalently, for some $i \in \mathbb{Z}$:

$$b_{k_n} + l\alpha + i \in V.$$

By definition of V:

$$\log_q(a + \tfrac{1}{2}) < b_{k_n} + l\alpha + i < \log_q(a+1) \tag{10}$$

Let $m = n + l$. Using (9) and simple induction argument we get:

$$[b_{k_m}] = [b_{k_n} + l\alpha - \epsilon_n - \epsilon_{n+1} - \cdots - \epsilon_{m-1}]\,.$$

Again this is equivalent to:

$$b_{k_m} = b_{k_n} + l\alpha - \epsilon_n - \epsilon_{n+1} - \cdots - \epsilon_{m-1} + j$$

for some $j \in \mathbb{Z}$. Recall that if $n \leq \nu < m$, then $k_{\nu+1} > k_n \geq N$ and so $a_{k_{\nu+1}} > M$. According to (8) we get $0 \leq \epsilon_\nu < (a_{k_{\nu+1}} \ln q)^{-1} < (M \ln q)^{-1}$, so that:

$$b_{k_n} + l\alpha + j - \frac{l}{M \ln q} < b_{k_m} \leq b_{k_n} + l\alpha + j.$$

But M was defined so that $l/(M \ln q) \leq L/(M \ln q) < \delta$. Hence:

$$b_{k_n} + l\alpha + j - \delta < b_{k_m} \leq b_{k_n} + l\alpha + j.$$

Together with (10) and the definition of δ this leads us to:

$$\log_q(a + \tfrac{1}{2}) + j - i - (\log_q(a + \tfrac{1}{2}) - \log_q a) < b_{k_m} < \log_q(a+1) + j - i.$$

Finally we get:

$$\log_q a + j - i < b_{k_m} < \log_q(a+1) + j - i.$$

In terms of a_ν this means:

$$q^{j-i} a < a_{k_m} < q^{j-i}(a+1).$$

It follows that $j - i > 0$ and $a_{k_m} = q^t a + s$ for $t = i - j$ and some s such that $0 < s < q^t$. Contradiction with (5).

This proves that B is empty and completes our solution.