Boundary Harnack Inequality
For Jump-Type Processes

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Banff, Canada, Sep 19, 2011
Some papers of R.F. Bass

  *A probabilistic proof of the boundary Harnack principle*

  *A boundary Harnack principle for twisted Hölder domains*

  *Hölder domains and the boundary Harnack principle*

  *The Martin boundary in non-Lipschitz domains*

  *The boundary Harnack principle for non-divergence form elliptic operators*
Boundary Harnack inequality (BHI)

If $f, g > 0$ are harmonic in $D$, then:

$$\frac{f(x)}{g(x)} \approx \frac{f(x_0)}{g(x_0)}$$

(see figure)

**False** for general domains $D$!

- A. Ancona (1978)
- B. Dahlberg (1977)
BHI for probabilists

\(X_t\): Brownian motion in \(\mathbb{R}^d\)
\(\tau_U\): time of first exit from \(U\)
\(P_x\): probability for \(X_0 = x\)

\[
\frac{P_x(X_{\tau_{D\cap B}} \in E_1)}{P_x(X_{\tau_{D\cap B}} \in E_2)} \approx \frac{P_{x_0}(X_{\tau_{D\cap B}} \in E_1)}{P_{x_0}(X_{\tau_{D\cap B}} \in E_2)}
\]
(see figure)

- Laplace operator \(\leftrightarrow\) Brownian motion
- Elliptic operators \(\leftrightarrow\) diffusion processes
- Non-local operators \(\leftrightarrow\) jump-type processes
Some more papers of R.F. Bass

- R.F. Bass (1979) *Adding and subtracting jumps from Markov processes*
Yet some more papers of R.F. Bass

*The construction of Brownian motion on the Sierpinski carpet*

*Divergence form operators on fractal-like domains*

Stability of parabolic Harnack inequalities for metric measure spaces

*On the Robin problem in fractal domains*

Uniqueness for Brownian motion on Sierpinski carpets
Non-local BHI

BHI (for balls $B'$, $B$ and domain $D$)

$$\frac{P_x(X_{\tau_{D\cap B}} \in E_1)}{P_x(X_{\tau_{D\cap B}} \in E_2)} \approx \frac{P_y(X_{\tau_{D\cap B}} \in E_1)}{P_y(X_{\tau_{D\cap B}} \in E_2)}$$

for $x, y \in D \cap B'$, $E_1, E_2 \subseteq B^c$ (see figure)

(equivalently: comparability of $X_t$-harmonic functions)

Problems

- For which **processes**?
- For what **domains**?
- What about scale-invariance?
State of the art (1/3)

Isotropic stable processes in $\mathbb{R}^d$:

- K. Bogdan (1997)
  K. Bogdan, T. Byczkowski (1999): \textbf{Lipschitz} domains

  Arbitrary domains, \textbf{not} scale-invariant
  (scale-invariant for $\kappa$-fat sets)

  Arbitrary domains
State of the art (2/3)

Non-isotropic stable processes in $\mathbb{R}^d$:
  P. Sztonyk (2003):
  Arbitrary domains, **not** scale-invariant
  (scale-invariant for $\kappa$-fat sets)

More or less general subordinate Brownian motions:
- T. Grzywny, M. Ryznar (2007):
  **Lipschitz** sets
  $\kappa$-fat sets **arbitrary domains (2011)**

Sums of Brownian motion and isotropic stables:
  smooth sets
State of the art (3/3)

Censored isotropic stable processes:

- Q. Guan (arXiv): \textbf{Lipschitz} sets

Stable-like processes on Sierpiński gasket:

- K. Bogdan, A. Stós, P. Sztonyk (2003): \textit{‘smooth’} sets
- K. Kaleta, K. (2010): arbitrary sets, $\alpha < 1$

Stable-like processes on Sierpiński carpet:

- A. Stós (2005): \textit{‘smooth’} sets
Result

Theorem (K. Bogdan, T. Kumagai, K.)

BHI holds for a Hunt process $X_t$ on a metric measure space $X_t$ under 4 assumptions:

- jumps of $X_t$ are regular enough
- potential kernel of $X_t$ is bounded away from the diagonal
- generator has rich domain
- $X_t$ is doubly Feller and there is a dual process $X_t^*$

for arbitrary open sets

A sufficient criterion for scale-invariance is provided
Features

- **uniformity**: constants do not depend on geometry!
- **generality**: relatively mild assumptions
- **examples**:
  - stable-like processes in (subsets of) $\mathbb{R}^d$
  - subordinate diffusions on fractals
  - processed killed by multiplicative functionals
  - processes with diffusion component (no scale-invariance here!)
- **explicitness**: good control of constants
- **stability** under certain perturbations
Let $B_s$ be the Brownian motion on the Sierpiński carpet
Let $Z_t$ be the $\alpha/d_w$-stable subordinator $\quad (0 < \alpha < d_w)$
Let $X_t = B_{Z_t}$ (jumping kernel: $\nu(x, y) \approx |x - y|^{-d-\alpha}$)

Then scale-invariant BHI holds for arbitrary domains.
A priori supremum estimate

An intermediate result:

Theorem (K. Bogdan, T. Kumagai, K.)

If \( f \geq 0 \) is \( X_t \)-subharmonic in \( B \), then
\[
\sup_{B'} f \leq C \int_{(B')^c} \nu(x_0, y)f(y)dy
\]
where \( \nu \) is the jumping kernel of \( X_t \)

(For example: \( f(x) = P_x(X_{\tau_{dB}} \in E) \))

Note 1: Integral does not charge \( B' \)!

Note 2: This sup-estimate implies BHI under even milder assumptions!
**Assumption**

The jumping kernel \( \nu(x, y) \) is **everywhere positive** and \( \nu(x_1, y) \approx \nu(x_2, y) \) when \( x_1 \approx x_2 \) are away from \( y \)

**Why?** If \( \nu(x_1, y) \gg \nu(x_2, y) \), then
\[
P_{x_1}(X_{\tau_{DnB}} \in E_1) \gg P_{x_2}(X_{\tau_{DnB}} \in E_1)
\]
but for \( E_2 = B^c \),
\[
P_{x_1}(X_{\tau_{DnB}} \in E_2) \approx P_{x_2}(X_{\tau_{DnB}} \in E_2)
\]
This contradicts (BHI):
\[
\frac{P_{x_1}(X_{\tau_{DnB}} \in E_1)}{P_{x_1}(X_{\tau_{DnB}} \in E_2)} \gg \frac{P_{x_2}(X_{\tau_{DnB}} \in E_1)}{P_{x_2}(X_{\tau_{DnB}} \in E_2)}
\]
Potential kernel

**Assumption**

Green functions of balls are bounded off diagonal

**Why?** To eliminate partially degenerate processes, for example: sum of *degenerate* Brownian motion and a compound Poisson process.

Note: Brownian motion plus a compound Poisson is perfectly fine! (But (BHI) will not be scale-invariant)
Generators

Assumption

There are **bump functions** (for example, $C^\infty_c(\mathbb{R}^d)$) in the domain of the Feller generator of $X_t$

**Why?** Required to use Dynkin’s results

Note: We don’t like this assumption!
Can one do better using Dirichlet forms?
Potential theory

Assumption

- $X_t$ is Feller and strong Feller
- There is a dual ($\approx$ time-reversed) process $X^*_t$
- $X^*_t$ satisfies all conditions imposed on $X_t$
- Hunt's hypothesis (H) holds true (polar = semi-polar)

Why? Required for potential-theoretic developments
Supremum estimate: idea

\[
\sup_{B'} f \leq C \int_{(B')^c} \nu(x_0, y)f(y)dy
\]

- Consider potential \( V(x) \) such that
  - \( V(x) = \infty \) for \( x \notin B \)
  - \( V(x) = 0 \) for \( x \in B' \)
  - \( 1/(V(x) + 1) \) is smooth
- Stop \( X_t \) according to \( V(x) \):
  - \( A_t = \int_0^t V(X_s)ds \)
  - \( T \) — exponential variable (independent of \( X_t \))
  - \( \tau = \inf\{t : A_t \geq T\} \)
- If \( f \) is \( X_t \)-subharmonic, then
  \[
  f(x) \leq \mathbb{E}_xf(X_\tau)
  \]