Boundary Harnack Inequality For Jump-Type Processes

Mateusz Kwaśnicki

Polish Academy of Sciences
Wrocław University of Technology

mateusz.kwasnicki@pwr.wroc.pl

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Some papers of R.F. Bass

- R.F. Bass, K. Burdzy (1989)
  A probabilistic proof of the ***boundary Harnack principle***
- R.F. Bass, K. Burdzy (1991)
  A ***boundary Harnack principle*** *for twisted Hölder domains*
- R. Bañuelos, R.F. Bass, K. Burdzy (1991)
  Hölder domains and the ***boundary Harnack principle***
- R.F. Bass, K. Burdzy (1993)
  The ***Martin boundary*** *in non-Lipschitz domains*
- R.F. Bass, K. Burdzy (1994)
  The ***boundary Harnack principle*** *for non-divergence form elliptic operators*

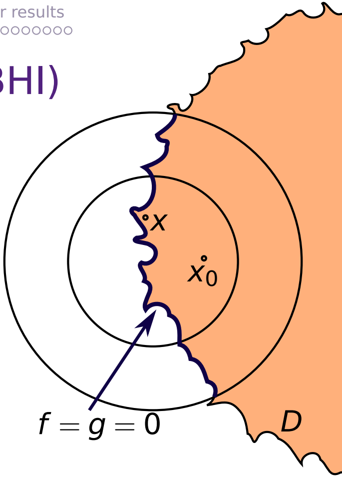
Boundary Harnack inequality (BHI)

BHI

If $f, g > 0$ are harmonic in D, then:

$$\frac{f(x)}{g(x)} \approx \frac{f(x_0)}{g(x_0)}$$

(see figure)



False for general domains D!

- A. Ancona (1978)
  B. Dahlberg (1977)
  J.-M. Wu (1978): Lipschitz domains
- D.S. Jerison, C.E. Kenig (1982): NTA sets
- R.F. Bass, K. Burdzy (1991): twisted Hölder sets

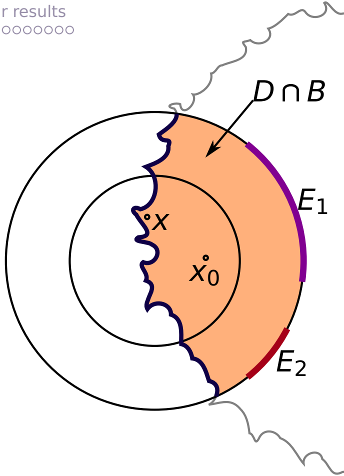
BHI for probabilists

X_t: Brownian motion in $\mathbf{R}^d$
τ_U: time of first exit from U
$\mathbf{P}_x$: probability for $X_0 = x$

BHI

$$\frac{\mathbf{P}_x(X_{\tau_{D\cap B}} \in E_1)}{\mathbf{P}_x(X_{\tau_{D\cap B}} \in E_2)} \approx \frac{\mathbf{P}_{x_0}(X_{\tau_{D\cap B}} \in E_1)}{\mathbf{P}_{x_0}(X_{\tau_{D\cap B}} \in E_2)}$$

(see figure)



- Laplace operator $\longleftrightarrow$ Brownian motion
- elliptic operators $\longleftrightarrow$ diffusion processes
- non-local operators $\longleftrightarrow$ jump-type processes

Some more papers of R.F. Bass

R.F. Bass (1979)
Adding and subtracting ***jumps*** *from Markov processes*

R.F. Bass, M. Cranston (1983)
Exit times for symmetric ***stable*** *processes in* $\mathbf{R}^n$

R.F. Bass (1988)
Uniqueness in law for ***pure jump*** *Markov processes*

R.F. Bass, D.A. Levin (2002)
Harnack inequalities for ***jump*** *processes*

M.T. Barlow, R.F. Bass, Z.-Q. Chen, M. Kassmann
Non-local Dirichlet forms and symmetric ***jump*** *processes* (2009)

Yet some more papers of R.F. Bass

- M.T. Barlow, R.F. Bass (1989)
  The construction of Brownian motion on the ***Sierpinski carpet***
- M.T. Barlow, R.F. Bass (2000)
  Divergence form operators on ***fractal****-like domains*
- M.T. Barlow, R.F. Bass, T. Kumagai (2006)
  Stability of parabolic Harnack inequalities for **metric measure spaces**
- R.F. Bass, K. Burdzy, Z.-Q. Chen (2008)
  On the Robin problem in ***fractal*** *domains*
- M.T. Barlow, R.F. Bass, T. Kumagai, A. Teplyaev
  Uniqueness for Brownian motion on **Sierpinski carpets** (2010)

Non-local BHI

BHI (for balls B', B and domain D)

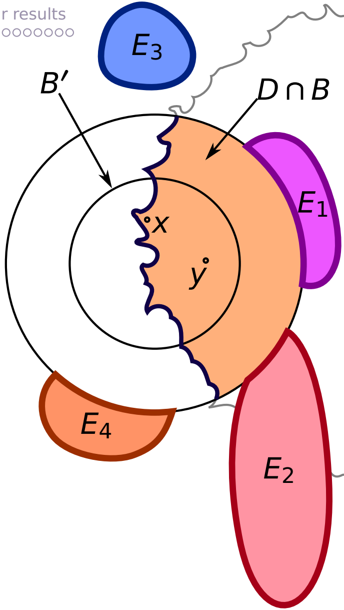
$$\frac{\mathbf{P}_x(X_{\tau_{D\cap B}} \in E_1)}{\mathbf{P}_x(X_{\tau_{D\cap B}} \in E_2)} \approx \frac{\mathbf{P}_y(X_{\tau_{D\cap B}} \in E_1)}{\mathbf{P}_y(X_{\tau_{D\cap B}} \in E_2)}$$

for $x, y \in D \cap B'$, $E_1, E_2 \subseteq B^c$
(see figure)

(equivalently: comparability of X_t-harmonic functions)

Problems

- For which **processes**?
- For what **domains**?
- What about scale-invariance?



State of the art (1/3)

Isotropic stable processes in $\mathbf{R}^d$:

- K. Bogdan (1997)
  K. Bogdan, T. Byczkowski (1999):
  Lipschitz domains
- R. Song, J.-M. Wu (1999):
  Arbitrary domains, **not** scale-invariant
  (scale-invariant for κ**-fat** sets)
- K. Bogdan, T. Kulczycki, K. (2008):
  Arbitrary domains

State of the art (2/3)

Non-isotropic stable processes in $\mathbf{R}^d$:

- K. Bogdan, A. Stós, P. Sztonyk (2002)
  P. Sztonyk (2003):
  Arbitrary domains, **not** scale-invariant
  (scale-invariant for κ**-fat** sets)

More or less general subordinate Brownian motions:

- T. Grzywny, M. Ryznar (2007):
  Lipschitz sets
- P. Kim, R. Song, Z. Vondraček (2007, 2008, 2009):
  ~~κ-**fat** sets~~ **arbitrary domains (2011)**

Sums of Brownian motion and isotropic stables:

- Z.-Q. Chen, P. Kim, R. Song, Z. Vondraček (in press):
  smooth sets

State of the art (3/3)

Censored isotropic stable processes:

- K. Bogdan, K. Burdzy, Z.-Q. Chen (2003):
  smooth sets
- Q. Guan (arXiv):
  Lipschitz sets

Stable-like processes on Sierpiński gasket:

- K. Bogdan, A. Stós, P. Sztonyk (2003):
  ‘smooth’ sets
- K. Kaleta, K. (2010):
  arbitrary sets, $\alpha < 1$

Stable-like processes on Sierpiński carpet:

- A. Stós (2005):
  ‘smooth’ sets

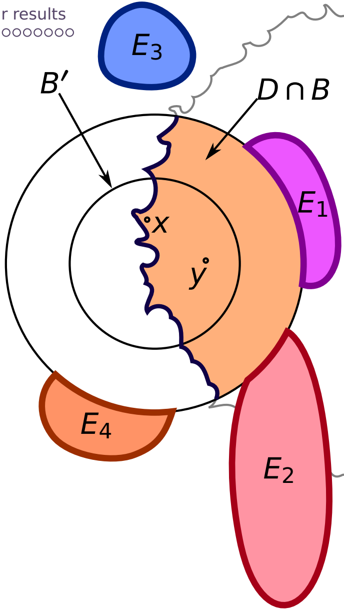
Result

Theorem (K. Bogdan, T. Kumagai, K.)

BHI holds for a **Hunt process** X_t on a **metric measure space** X_t under 4 assumptions:

- jumps of X_t are regular enough
- potential kernel of X_t is bounded away from the diagonal
- generator has rich domain
- X_t is doubly Feller and there is a dual process X_t^*

for **arbitrary** open sets

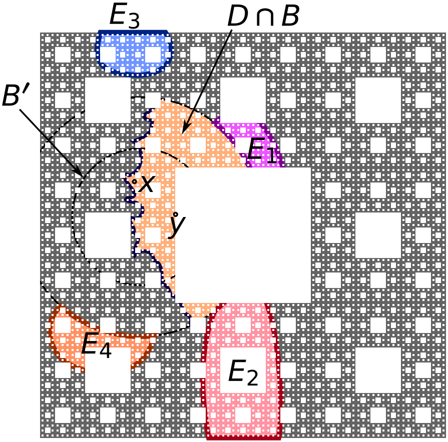


A sufficient criterion for scale-ivariance is provided

Features

- **uniformity**: constants do not depend on geometry!
- **generality**: relatively mild assumptions
- **examples**:
  - stable-like processes in (subsets of) $\mathbf{R}^d$
  - subordinate diffusions on fractals
  - processed killed by multiplicative functionals
  - processes with diffusion component (no scale-invariance here!)
- **explicitness**: good control of constants
- **stability** under certain perturbations

Example



Let B_s be the Brownian motion on the Sierpiński carpet
Let Z_t be the α/d_w-stable subordinator $(0 < \alpha < d_w)$
Let $X_t = B_{Z_t}$ (jumping kernel: $\nu(x,y) \approx |x-y|^{-d-\alpha}$)

Then scale-invariant BHI holds for arbitrary domains.

A priori supremum estimate

An intermediate result:

Theorem (K. Bogdan, T. Kumagai, K.)

If $f \geq 0$ is X_t-subharmonic in B, then

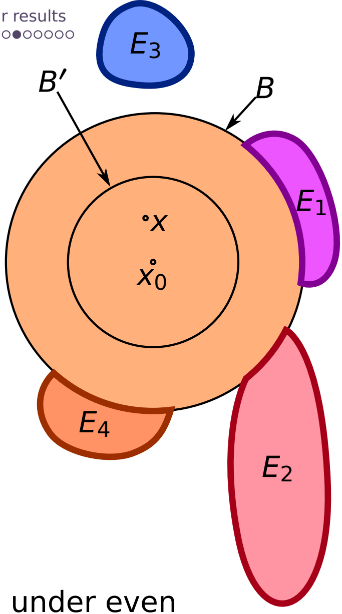
$$\sup_{B'} f \leq C \int_{(B')^c} \nu(x_0, y) f(y) dy$$

where ν is the jumping kernel of X_t

(For example: $f(x) = \mathbf{P}_x(X_{\tau_{D \cap B}} \in E)$)

Note 1: Integral does not charge B'!

Note 2: This sup-estimate implies BHI under even milder assumptions!



Jumps

Assumption

The jumping kernel $\nu(x,y)$ is **everywhere positive** and

$$\nu(x_1,y) \approx \nu(x_2,y) \quad \text{when } x_1 \approx x_2 \text{ are away from } y$$

Why? If $\nu(x_1,y) \gg \nu(x_2,y)$, then

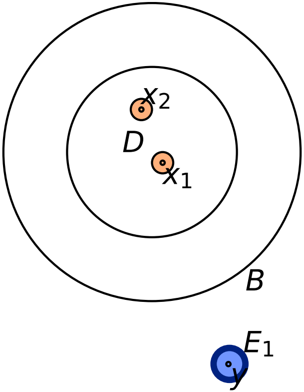
$$\mathbf{P}_{x_1}(X_{\tau_{D\cap B}} \in E_1) \gg \mathbf{P}_{x_2}(X_{\tau_{D\cap B}} \in E_1)$$

but for $E_2 = B^c$,

$$\mathbf{P}_{x_1}(X_{\tau_{D\cap B}} \in E_2) \approx \mathbf{P}_{x_2}(X_{\tau_{D\cap B}} \in E_2)$$

This contradicts (BHI):

$$\frac{\mathbf{P}_{x_1}(X_{\tau_{D\cap B}} \in E_1)}{\mathbf{P}_{x_1}(X_{\tau_{D\cap B}} \in E_2)} \gg \frac{\mathbf{P}_{x_2}(X_{\tau_{D\cap B}} \in E_1)}{\mathbf{P}_{x_2}(X_{\tau_{D\cap B}} \in E_2)}$$



Potential kernel

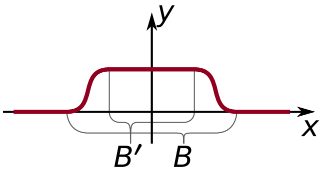
Assumption

Green functions of balls are bounded off diagonal

Why? To eliminate partially degenerate processes, for example: sum of **degenerate** Brownian motion and a compound Poisson process.

Note: Brownian motion plus a compound Poisson is perfectly fine! (But (BHI) will not be scale-invariant)

Generators



Assumption

There are **bump functions** (for example, $C_c^\infty(\mathbf{R}^d)$) in the domain of the Feller generator of X_t

Why? Required to use Dynkin's results

Note: We don't like this assumption!
Can one do better using Dirichlet forms?

Potential theory

Assumption

- X_t is Feller and strong Feller
- There is a dual (≈ time-reversed) process X_t^*
- X_t^* satisfies all conditions imposed on X_t
- Hunt's hypothesis (H) holds true (polar = semi-polar)

Why? Required for potential-theoretic developments

Supremum estimate: idea

$$\sup_{B'} f \le C \int_{(B')^c} \nu(x_0, y) f(y) dy$$

- Consider potential $V(x)$ such that
  - $V(x) = \infty$ for $x \notin B$
  - $V(x) = 0$ for $x \in B'$
  - $1/(V(x)+1)$ is **smooth**
- Stop X_t according to $V(x)$:
  - $A_t = \int_0^t V(X_s) ds$
  - T — exponential variable (independent of X_t)
  - $\tau = \inf\{t : A_t \ge T\}$
- If f is X_t-subharmonic, then

$$f(x) \le \mathbf{E}_x f(X_\tau)$$