Two-term asymptotics for Lévy operators in intervals

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Long-term goal
Study spectral theory of nonlocal operators on $L^2(D)$.

Short-term goal
In this talk: $D = (-1, 1)$ (and a little bit of $D = (0, \infty)$).

Joint project with Kamil Kaleta, Tadeusz Kulczycki, Jacek Małecki, Michał Ryznar, Andrzej Stós.

Outline:
- Overview of previous results
- Two-term asymptotics in $(-1, 1)$
- Main ideas of the proof
Part 1

Introduction:
Selected previous results
Weyl-type law

Theorem

The eigenvalues $\lambda_n$ of $(-\Delta)^{\alpha/2}$ in a domain $D \subseteq \mathbb{R}^d$, with Dirichlet exterior condition, satisfy

$$\lambda_n \sim c_{d,\alpha} \left( \frac{n}{|D|} \right)^{\alpha/d}.$$ 

R. M. Blumenthal, R. K. Getoor

*The asymptotic distribution of the eigenvalues...*

Pacific J. Math. 9(2) (1959)

R. Bañuelos, T. Kulczycki

*Trace estimates for stable processes*

Probab. Theory Related Fields 142(3-4) (2009)
Two-sided bounds (1)

**Theorem**

For \((-\Delta)^{\alpha/2}\) in \(D \subseteq \mathbb{R}^d\):

\[
c \left( \lambda_n^{\Delta} \right)^{\alpha/2} \leq \lambda_n \leq \left( \lambda_n^{\Delta} \right)^{\alpha/2}
\]

if \(D\) satisfies a version of the exterior cone condition.
If \(D\) is convex, then \(c = \frac{1}{2}\).

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Z.-Q. Chen, R. Song
*Two sided eigenvalue estimates for subordinate...*

R. D. DeBlassie
*Higher order PDEs and symmetric stable processes*
Two-sided bounds (2)

**Theorem**

For $\psi(-\Delta)$ in $D \subseteq \mathbb{R}^d$:

$$c \psi(\lambda_n^A) \leq \lambda_n \leq \psi(\lambda_n^A)$$

if:

- $D$ satisfies a version of the exterior cone condition,
- $\psi$ is complete Bernstein ($\equiv$ operator monotone).

Again, if $D$ is convex, then $c = \frac{1}{2}$.

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Z.-Q. Chen, R. Song

*Two sided eigenvalue estimates for subordinate...*

*J. Funct. Anal. 226 (2005)*
Heat trace

**Theorem**

For $(-\Delta)^{\alpha/2}$ in $D \subseteq \mathbb{R}^d$:

$$t^{d/\alpha} \sum_{n=1}^{\infty} e^{-\lambda_n t} = c_{d,\alpha} |D| + c'_{d,\alpha} |\partial D| t^{1/\alpha} + o\left(t^{1/\alpha}\right)$$

as $t \to 0$, if $D$ is a Lipschitz domain.

($|\partial D|$ is the $(d-1)$-dimensional Hausdorff measure of $\partial D$)

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R. Bañuelos, T. Kulczycki

*Trace estimates for stable processes*

Probab. Theory Related Fields 142(3–4) (2009)

R. Bañuelos, T. Kulczycki, B. Siudeja

*On the trace of symmetric stable processes...*

Cesàro means

**Theorem**

For \((-\Delta)^{\alpha/2}\) in \(D \subseteq \mathbb{R}^d\):

\[
\frac{1}{N} \sum_{n=1}^{N} \lambda_n = C_{d,\alpha} \left( \frac{N}{|D|} \right)^{\alpha/d} + C'_d,\alpha \frac{|\partial D|}{|D|} \left( \frac{N}{|D|} \right)^{(\alpha-1)/d} + o\left( N^{(\alpha-1)/d} \right)
\]

if \(D\) is a \(C^{1,\varepsilon}\) set for some \(\varepsilon > 0\).

R. L. Frank, L. Geisinger
*Refined semiclassical asymptotics for fractional...*
*arXiv:1105.5181*
Is there any hope for two-term asymptotics?

Asymptotic expansion of $\lambda_n$ for $(−\Delta)^{\alpha/2}$ is equivalent to semi-classical expansion of

$$\text{Tr } f\left(−(−h\Delta)^{\alpha/2}\right) \quad \text{as } h \to 0.$$ 

- Heat trace corresponds to $f(x) = e^{-x}$.
- Cesàro means correspond to $f(x) = \max(1 - |x|, 0)$.
- Ivrii-type result would require $f(x) = 1_{(-\infty,1)}(x)$.

Problems:
- Methods of semi-classical analysis typically require some smoothness.
- There is no theory of wave equation for $(−\Delta)^{\alpha/2}$. 
Part 2

Our results for intervals
Cauchy process

Theorem

- For $(-\Delta)^{1/2}$ in $(-1, 1)$:
  \[ |\lambda_n - \left(\frac{n\pi}{2} - \frac{\pi}{8}\right)| < \frac{1}{n}. \]
  In particular, $\lambda_n$ are simple.
- Furthermore, the corresponding eigenfunctions $\varphi_n$ are uniformly bounded and
  \[ \varphi_n(-x) = (-1)^{n-1} \varphi_n(x). \]
- Two-sided numerical bounds:
  $\lambda_1 = 1.1577738836977... \quad \lambda_{10} = 15.3155549960...$

Tadeusz Kulczycki, K., Jacek Małecki, Andrzej Stós

*Spectral properties of the Cauchy process...*

Stable processes

Theorem

- For \((-\Delta)^{\alpha/2}\) in \((-1, 1)\):

\[
\lambda_n = \left( \frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8} \right)^\alpha + O\left(\frac{1}{n}\right).
\]

If \(\alpha \geq 1\), then \(\lambda_n\) are simple and
\(\varphi_n(-x) = (-1)^{n-1}\varphi_n(x)\).

- If \(\alpha \geq \frac{1}{2}\), then \(\varphi_n\) are uniformly bounded.

- For all \(\alpha\), \(\varphi_n\) is \(O\left(\frac{1}{n^{\alpha \vee 1/2}}\right)\) away in \(L^2((-1, 1))\) from
\[\pm \sin\left(\left( \frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8} \right)x\right) \quad \text{or} \quad \pm \cos\left(\left( \frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8} \right)x\right).
\]

K.

*Eigenvalues of the fractional Laplace operator...*

*J. Funct. Anal. 262(5) (2012)*
Relativistic processes (1)

**Theorem**

- For $(-\Delta + 1)^{1/2}$ in $(-a, a)$:
  \[
  \lambda_n = \frac{n\pi}{2a} - \frac{\pi}{8a} + \mathcal{O}\left(\frac{1}{n}\right).
  \]

- In fact:
  \[
  \left| \lambda_n - \sqrt{\mu_n^2 + 1} \right| < \frac{8}{n} \left(3 + \frac{1}{a}\right) \exp\left(-\frac{a}{3}\right),
  \]
  where:
  \[
  a\mu_n + \vartheta(\mu_n) = \frac{n\pi}{2}
  \]
  with:
  \[
  \vartheta(\mu) = \frac{1}{\pi} \int_0^{\infty} \frac{\mu}{s^2 - \mu^2} \log\left(\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1+s^2}{1+\mu^2}}\right) ds.
  \]

- $\lambda_n$ are simple.

- Furthermore, $\varphi_n(-x) = (-1)^{n-1} \varphi_n(x)$. 
Relativistic processes (2)

‘Physical’ reformulation

For \((-\hbar^2 c^2 \Delta + (mc^2)^2)^{1/2}\) in \((-a, a)\):

\[
\lambda_n = \left( \frac{n\pi}{2a} - \frac{\pi}{8a} \right) \frac{\hbar c}{a} + \mathcal{O}\left(\frac{1}{n}\right)
\]

and in fact:

\[
\left| \lambda_n - \sqrt{\mu_n^2 + 1} \right| < \frac{8}{n} \left( 3mc^2 + \frac{\hbar c}{a} \right) \exp\left(-\frac{mca}{3\hbar}\right)
\]

with \(\mu_n\) as in the previous slide.

- \(m \to 0\) — zero mass case (Cauchy process)
- \(c \to \infty\) — non-relativistic limit (Brownian motion)
- \(\hbar \to 0\) — semi-classical limit (surprising one!)

Kamil Kaleta, K., Jacek Małecki

*One-dimensional quasi-relativistic particle in the box*

arXiv:1110.5887
Extensions

Natural **feasible** directions for further research:

- More general processes
  (work in progress)

- One or two more terms, as in the **sloshing problem**
  (too technical for me)

Many other natural questions seem to be very difficult.
Part 3

Proof of two-term asymptotic formula (main ideas)
Setting

Assumption

\[-Af(x) = bf''(x) + pv \int_{-\infty}^{\infty} (f(y) - f(x))\nu(y - x)dy\]

with:

- \(b \geq 0;\)
- \(\nu(z) \geq 0, \nu(z) = \nu(-z), \int_{-\infty}^{\infty} \min(1, z^2)\nu(z)dz < \infty;\)
- \(\nu\) is completely monotone on \((0, \infty)\)
  (i.e. \((-1)^n\nu^{(n)}(z) \geq 0\) for \(z > 0, n = 0, 1, 2, \ldots\)).

Examples:

\((-\Delta)^{\alpha/2} \quad (-\Delta + 1)^{1/2} \quad \log(-\Delta + 1) \quad ((-\Delta)^{-1} + 1)^{-1}\)
Complete Bernstein functions

\[ A f(x) = b f''(x) + \text{pv} \int_{-\infty}^{\infty} (f(y) - f(x)) \nu(y - x) dy \]

Proposition

\[ \begin{cases} b \geq 0, \ \nu(z) = \nu(-z), \\ \nu \text{ completely monotone on } (0, \infty) \end{cases} \]

is equivalent to \( A = -\psi(-\Delta) \) for a \textbf{complete Bernstein function} \( \psi \):

\[ \psi(s) = bs + \int_{0}^{\infty} \frac{s}{u + s} \mu(du) \]

- \( A \) is the generator of a symmetric Lévy process
- More precisely: of a subordinate Brownian motion
- Yet more precisely: for a subordinator with completely monotone density of the Lévy measure
Theory for half-line

**Theorem**

\( A \) in \((0, \infty)\) has explicit (generalized) eigenfunctions:

\[
F_s(x) = \sin(sx + \vartheta_s) - \int_{(0,\infty)} e^{-xu} g_s(du)
\]

for \( \vartheta_s \in [0, \frac{\pi}{2}) \) and \( g_s \) positive, integrable.

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- **Tadeusz Kulczycki, K., Jacek Małecki, Andrzej Stós**
  *Spectral properties of the Cauchy process...*

- **K.**
  *Spectral analysis of subordinate Brownian motions...*
  Studia Math. 206(3) (2011)

- **K., Jacek Małecki, Michał Ryznar**
  *First passage times for subordinate Brownian motions*
  arXiv:1110.0401
Eigenfunctions in half-line

Plot of $F_s(x/s)$ for $s = \frac{1}{20}, \frac{1}{2}, 1, 10$
(all plots for the relativistic process in $(-1, 1)$)
Approximation to first eigenfunction

Approximation to $\varphi_1$
Approximation to second eigenfunction

Approximation to $\varphi_2$
Approximation to third eigenfunction

Approximation to $\varphi_3$
Approximation to fourth eigenfunction

Approximation to $\varphi_4$
Sketch of the proof (1)

Key technical lemma

\[ A_{(-a,a)}\tilde{\varphi}_n = \tilde{\lambda}_n \tilde{\varphi}_n + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \quad \text{in } L^2((-a, a)), \]

where:

- \( \tilde{\varphi}_n \) is the approximation of \( \varphi_n \),
- \( \tilde{\lambda}_n \) is the ‘best guess’ for \( \lambda_n \).

Remarks:

- Recall that \( F_s(x) = \sin(sx + \vartheta_s) - G_s(x) \) with \( G_s \) small
- \( A_{(0,\infty)}F_s = \psi(s^2)F_s \).
- \( \tilde{\varphi}_n \) is constructed with \( F_s \) for \( s = \frac{n\pi}{2a} - \frac{1}{a} \vartheta_s \).
- \( \tilde{\lambda}_n = \psi(s^2) = \psi\left(\left(\frac{n\pi}{2a} - \frac{1}{a} \vartheta_s\right)^2\right) \).
Sketch of the proof (2)

\[ \mathbf{A}_{(-a,a)} \tilde{\varphi}_n = \tilde{\lambda}_n \tilde{\varphi}_n + O\left(\frac{1}{\sqrt{n}}\right) \quad \text{in } L^2((−a, a)), \]

Corollary

\[ |\tilde{\lambda}_n - \lambda_{k(n)}| = O\left(\frac{1}{n}\right) \]

for some \( k(n) \).

Proof:

- Expand \( \tilde{\varphi}_n \) in the basis of \( \varphi_k \).
- Apply this to the estimate of \( \| \mathbf{A}_{(-a,a)} \tilde{\varphi}_n - \tilde{\lambda}_n \tilde{\varphi}_n \| \).
Bounds for eigenvalues $a\lambda_k(n)$ ($n = 3, 4, \ldots$) plotted versus $a$.
Sketch of the proof (3)

**Corollary**
The numbers $k(3)$, $k(4)$, $k(5)$, ... are distinct.

**Lemma**
There are no other eigenvalues than:
- $\lambda_1$, $\lambda_2$ (estimates in Z.-Q. Chen, R. Song, JFA 2005)
- $\lambda_{k(3)}$, $\lambda_{k(4)}$, $\lambda_{k(5)}$, ...

**Proof:**
- $\int p_t^{(-a,a)}(x, x) dx = \sum_{k=1}^{\infty} e^{-t\lambda_k}$,
- Use $p_t^{(-a,a)}(x, x) \leq p_t(0)$ for the upper bound of LHS.
- Use upper bounds for $\lambda_{k(n)}$ to estimate RHS.
- In the limit $t \to 0$, $\left| \int p_t^{(-a,a)}(x, x) dx - \sum_{n=3}^{\infty} e^{-t\lambda_{k(n)}} \right| < 3$. 
Eigenfunctions in half-line revisited

\[ \psi_s(u^2) = \frac{\psi'(s^2)(s^2 - u^2)}{\psi(s^2) - \psi(u^2)} \]

\[ \psi^*_s(u) = \exp \left( \frac{1}{\pi} \int_0^\infty \frac{u}{u^2 + v^2} \log \psi_s(v^2) dv \right) \]

\[ \mathcal{L}F_s(u) = \frac{s}{s^2 + u^2} \psi^*_s(u) \]

\[ F_s(x) = \sin(sx + \vartheta_s) - \int_{(0,\infty)} e^{-xu} g_s(du) \]

\[ \vartheta_s = \text{Arg}(\psi^*_s(is)) \]

\[ g_s(du) = \frac{1}{\pi} \frac{s}{s^2 + u^2} \frac{\text{Im}(\psi_s)^+(-u^2)}{\psi^*_s(u)} du \]