Two-term asymptotics for Lévy operators in intervals

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Long-term goal

Study spectral theory of nonlocal operators on $L^2(D)$.

Short-term goal

In this talk: $D = (-1, 1)$ (and a little bit of $D = (0, \infty)$).

Joint project with **Kamil Kaleta**, **Tadeusz Kulczycki**, **Jacek Małecki**, **Michał Ryznar**, **Andrzej Stós**.

Outline:

- Overview of previous results
- Two-term asymptotics in $(-1, 1)$
- Main ideas of the proof

Part 1

Introduction:
Selected previous results

Weyl-type law

Theorem

The eigenvalues λ_n of $(-\Delta)^{\alpha/2}$ in a domain $D \subseteq \mathbf{R}^d$, with Dirichlet exterior condition, satisfy

$$\lambda_n \sim c_{d,\alpha} \left(\frac{n}{|D|}\right)^{\alpha/d}.$$

- R. M. Blumenthal, R. K. Getoor
  The asymptotic distribution of the eigenvalues...
  Pacific J. Math. 9(2) (1959)
- R. Bañuelos, T. Kulczycki
  Trace estimates for stable processes
  Probab. Theory Related Fields 142(3-4) (2009)

Two-sided bounds (1)

Theorem

For $(-\Delta)^{\alpha/2}$ in $D \subseteq \mathbf{R}^d$:

$$c\left(\lambda_n^{\Delta}\right)^{\alpha/2} \le \lambda_n \le \left(\lambda_n^{\Delta}\right)^{\alpha/2}$$

if D satisfies a version of the exterior cone condition.

If D is convex, then $c = \frac{1}{2}$.

Z.-Q. Chen, R. Song
Two sided eigenvalue estimates for subordinate...
J. Funct. Anal. 226 (2005)

R. D. DeBlassie
Higher order PDEs and symmetric stable processes
Probab. Theory Related Fields 129(4) (2004)

Two-sided bounds (2)

Theorem

For $\psi(-\Delta)$ in $D \subseteq \mathbf{R}^d$:

$$c\,\psi\left(\lambda_n^{\Delta}\right) \le \lambda_n \le \psi\left(\lambda_n^{\Delta}\right)$$

if:

- D satisfies a version of the exterior cone condition,
- ψ is complete Bernstein ($=$ operator monotone).

Again, if D is convex, then $c = \frac{1}{2}$.

Z.-Q. Chen, R. Song
Two sided eigenvalue estimates for subordinate...
J. Funct. Anal. 226 (2005)

Heat trace

Theorem

For $(-\Delta)^{\alpha/2}$ in $D \subseteq \mathbf{R}^d$:

$$t^{d/\alpha} \sum_{n=1}^{\infty} e^{-\lambda_n t} = c_{d,\alpha}|D| + c'_{d,\alpha}|\partial D| t^{1/\alpha} + o\left(t^{1/\alpha}\right)$$

as $t \to 0$, if D is a Lipschitz domain.

$\big(|\partial D|$ is the $(d-1)$-dimensional Hausdorff measure of $\partial D\big)$

R. Bañuelos, T. Kulczycki
Trace estimates for stable processes
Probab. Theory Related Fields 142(3–4) (2009)

R. Bañuelos, T. Kulczycki, B. Siudeja
On the trace of symmetric stable processes...
J. Funct. Anal 257(10) (2009)

Cesàro means

Theorem

For $(-\Delta)^{\alpha/2}$ in $D \subseteq \mathbf{R}^d$:

$$\frac{1}{N} \sum_{n=1}^{N} \lambda_n = c_{d,\alpha} \left(\frac{N}{|D|} \right)^{\alpha/d} + c'_{d,\alpha} \frac{|\partial D|}{|D|} \left(\frac{N}{|D|} \right)^{(\alpha-1)/d} + o\left(N^{(\alpha-1)/d}\right)$$

if D is a $C^{1,\varepsilon}$ set for some $\varepsilon > 0$.

R. L. Frank, L. Geisinger
Refined semiclassical asymptotics for fractional...
arXiv:1105.5181

Is there any hope for two-term asymptotics?

Asymptotic expansion of λ_n for $(-\Delta)^{\alpha/2}$ is equivalent to semi-classical expansion of

$$\operatorname{Tr} f\left(-(-h\Delta)^{\alpha/2}\right) \qquad \text{as } h \to 0.$$

- Heat trace corresponds to $f(x) = e^{-x}$.
- Cesàro means correspond to $f(x) = \max(1 - |x|, 0)$.
- Ivrii-type result would require $f(x) = \mathbf{1}_{(-\infty,1)}(x)$.

Problems:

- Methods of semi-classical analysis typically require some smoothness.
- There is no theory of wave equation for $(-\Delta)^{\alpha/2}$.

Part 2

Our results for intervals

Cauchy process

Theorem

- For $(-\Delta)^{1/2}$ in $(-1,1)$:
$$\left|\lambda_n - \left(\frac{n\pi}{2} - \frac{\pi}{8}\right)\right| < \frac{1}{n}.$$
In particular, λ_n are simple.
- Furthermore, the corresponding eigenfunctions φ_n are uniformly bounded and
$$\varphi_n(-x) = (-1)^{n-1}\varphi_n(x).$$
- Two-sided numerical bounds:
$\lambda_1 = 1.1577738836977\ldots$ $\lambda_{10} = 15.3155549960\ldots$

Tadeusz Kulczycki, K., Jacek Małecki, Andrzej Stós
Spectral properties of the Cauchy process...
Proc. London Math. Soc. 101(2) (2010)

Stable processes

Theorem

- For $(-\Delta)^{\alpha/2}$ in $(-1,1)$:
$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\right)^{\alpha} + \mathcal{O}\left(\frac{1}{n}\right).$$
If $\alpha \geq 1$, then λ_n are simple and $\varphi_n(-x) = (-1)^{n-1}\varphi_n(x)$.
- If $\alpha \geq \frac{1}{2}$, then φ_n are uniformly bounded.
- For all α, φ_n is $\mathcal{O}\left(\frac{1}{n^{\alpha \vee 1/2}}\right)$ away in $L^2((-1,1))$ from
$$\pm\sin\left(\left(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\right)x\right) \quad \text{or} \quad \pm\cos\left(\left(\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}\right)x\right).$$

K.
Eigenvalues of the fractional Laplace operator...
J. Funct. Anal. 262(5) (2012)

Relativistic processes (1)

Theorem

- For $(-\Delta+1)^{1/2}$ in $(-a,a)$:

$$\lambda_n = \frac{n\pi}{2a} - \frac{\pi}{8a} + \mathcal{O}\left(\frac{1}{n}\right).$$

- In fact:

$$\left|\lambda_n - \sqrt{\mu_n^2+1}\right| < \frac{8}{n}\left(3+\frac{1}{a}\right)\exp\left(-\frac{a}{3}\right),$$

where:

$$a\mu_n + \vartheta(\mu_n) = \frac{n\pi}{2}$$

with:

$$\vartheta(\mu) = \frac{1}{\pi}\int_0^\infty \frac{\mu}{s^2-\mu^2} \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1+s^2}{1+\mu^2}}\right) ds.$$

- λ_n are simple.
- Furthermore, $\varphi_n(-x) = (-1)^{n-1}\varphi_n(x)$.

Relativistic processes (2)

'Physical' reformulation

For $(-\hbar^2 c^2 \Delta + (mc^2)^2)^{1/2}$ in $(-a, a)$:

$$\lambda_n = \left(\frac{n\pi}{2a} - \frac{\pi}{8a}\right)\frac{\hbar c}{a} + \mathcal{O}\left(\frac{1}{n}\right)$$

and in fact:

$$\left|\lambda_n - \sqrt{\mu_n^2 + 1}\right| < \frac{8}{n}\left(3mc^2 + \frac{\hbar c}{a}\right)\exp\left(-\frac{mca}{3\hbar}\right)$$

with μ_n as in the previous slide.

- $m \to 0$ — zero mass case (Cauchy process)
- $c \to \infty$ — non-relativistic limit (Brownian motion)
- $\hbar \to 0$ — semi-classical limit (surprising one!)

Kamil Kaleta, K., Jacek Małecki
One-dimensional quasi-relativistic particle in the box
arXiv:1110.5887

Extensions

Natural **feasible** directions for further research:

- More general processes
  (work in progress)
- One or two more terms, as in the **sloshing problem**
  (too technical for me)

Many other natural questions seem to be very difficult.

Part 3

Proof of two-term asymptotic formula (main ideas)

Setting

Assumption

$$-\mathbf{A}f(x) = bf''(x) + \text{pv}\int_{-\infty}^{\infty}(f(y) - f(x))\nu(y - x)dy$$

with:

- $b \geq 0$;
- $\nu(z) \geq 0$, $\nu(z) = \nu(-z)$, $\int_{-\infty}^{\infty} \min(1, z^2)\nu(z)dz < \infty$;
- ν is completely monotone on $(0, \infty)$
  (i.e. $(-1)^n \nu^{(n)}(z) \geq 0 \quad \text{for} \quad z > 0, n = 0, 1, 2, \ldots$).

Examples:

$$(-\Delta)^{\alpha/2} \qquad (-\Delta + 1)^{1/2} \qquad \log(-\Delta + 1) \qquad ((-\Delta)^{-1} + 1)^{-1}$$

Complete Bernstein functions

$$\mathbf{A}f(x) = bf''(x) + \mathrm{pv}\int_{-\infty}^{\infty}(f(y) - f(x))\nu(y - x)dy$$

Proposition

$$\begin{cases} b \geq 0,\ \nu(z) = \nu(-z), \\ \nu \text{ completely monotone on } (0, \infty) \end{cases}$$

is equivalent to $\mathbf{A} = -\psi(-\Delta)$ for a **complete Bernstein function** ψ:

$$\psi(s) = bs + \int_0^{\infty} \frac{s}{u + s}\,\mu(du)$$

- $\mathbf{A}$ is the generator of a symmetric Lévy process
- More precisely: of a subordinate Brownian motion
- Yet more precisely: for a subordinator with completely monotone density of the Lévy measure

Theory for half-line

Theorem

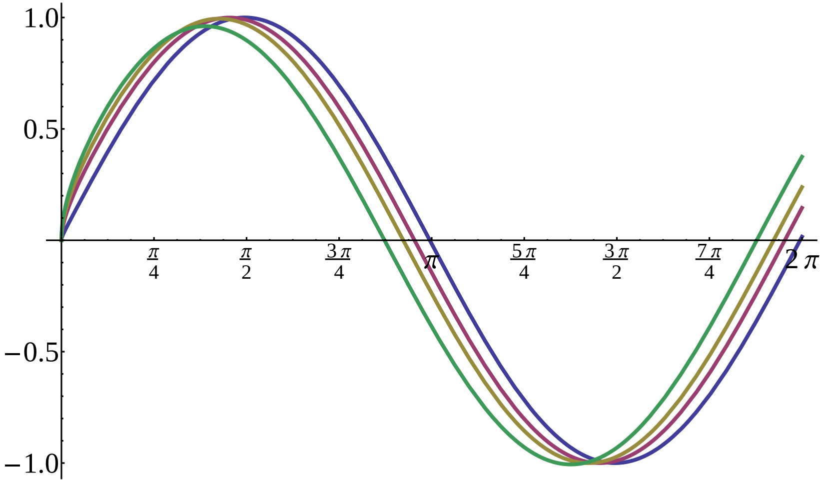
$\mathbf{A}$ in $(0,\infty)$ has explicit (generalized) eigenfunctions:

$$F_s(x) = \sin(sx + \vartheta_s) - \int_{(0,\infty)} e^{-xu} g_s(du)$$

for $\vartheta_s \in [0, \frac{\pi}{2})$ and g_s positive, integrable.

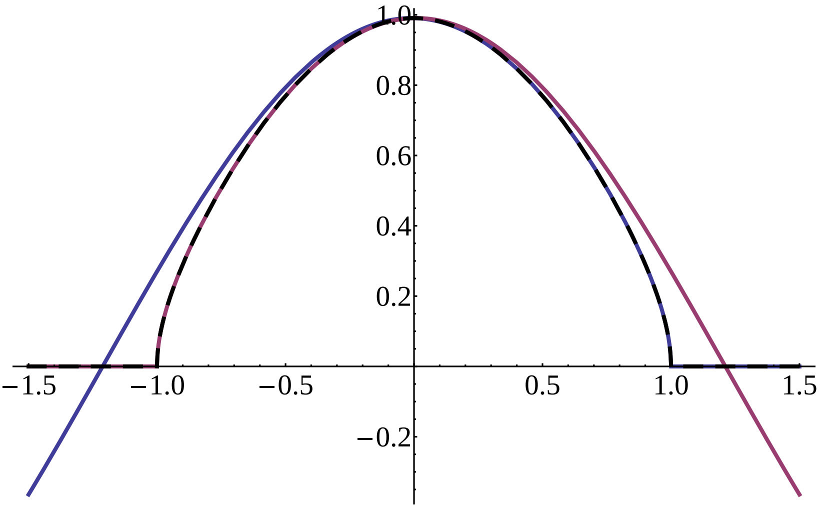
- Tadeusz Kulczycki, K., Jacek Małecki, Andrzej Stós
  Spectral properties of the Cauchy process...
  Proc. London Math. Soc. 101(2) (2010)
- K.
  Spectral analysis of subordinate Brownian motions...
  Studia Math. 206(3) (2011)
- K., Jacek Małecki, Michał Ryznar
  First passage times for subordinate Brownian motions
  arXiv:1110.0401

Eigenfunctions in half-line



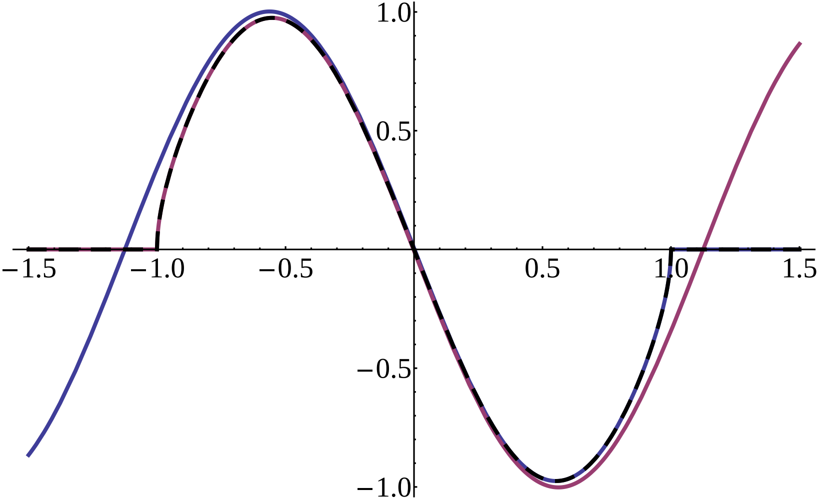
Plot of $F_s(x/s)$ for $s = \frac{1}{20}, \frac{1}{2}, 1, 10$
(all plots for the relativistic process in $(-1, 1)$)

Approximation to first eigenfunction



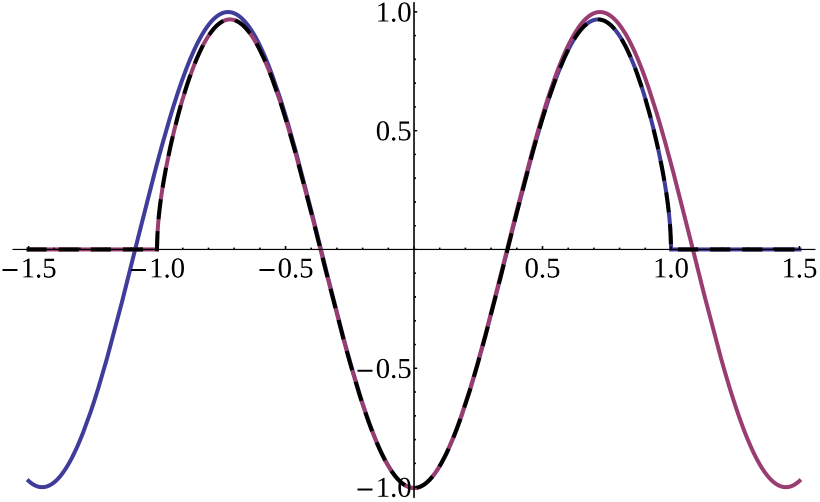
Approximation to φ_1

Approximation to second eigenfunction



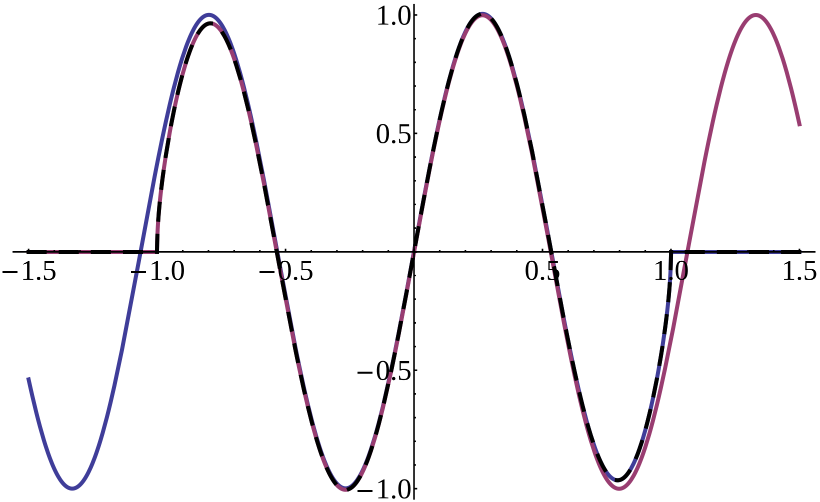
Approximation to φ_2

Approximation to third eigenfunction



Approximation to φ_3

Approximation to fourth eigenfunction



Approximation to φ_4

Sketch of the proof (1)

Key technical lemma

$$\mathbf{A}_{(-a,a)}\tilde{\varphi}_n = \tilde{\lambda}_n\tilde{\varphi}_n + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \qquad \text{in } L^2((-a,a)),$$

where:

- $\tilde{\varphi}_n$ is the approximation of φ_n,
- $\tilde{\lambda}_n$ is the 'best guess' for λ_n.

Remarks:

- Recall that $F_s(x) = \sin(sx + \vartheta_s) - G_s(x)$ with G_s small
- $\mathbf{A}_{(0,\infty)}F_s = \psi(s^2)F_s$.
- $\tilde{\varphi}_n$ is constructed with F_s for $s = \frac{n\pi}{2a} - \frac{1}{a}\vartheta_s$.
- $\tilde{\lambda}_n = \psi(s^2) = \psi\left(\left(\frac{n\pi}{2a} - \frac{1}{a}\vartheta_s\right)^2\right)$.

Sketch of the proof (2)

$$\mathbf{A}_{(-a,a)}\tilde{\varphi}_n = \tilde{\lambda}_n\tilde{\varphi}_n + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \qquad \text{in } L^2((-a,a)),$$

Corollary

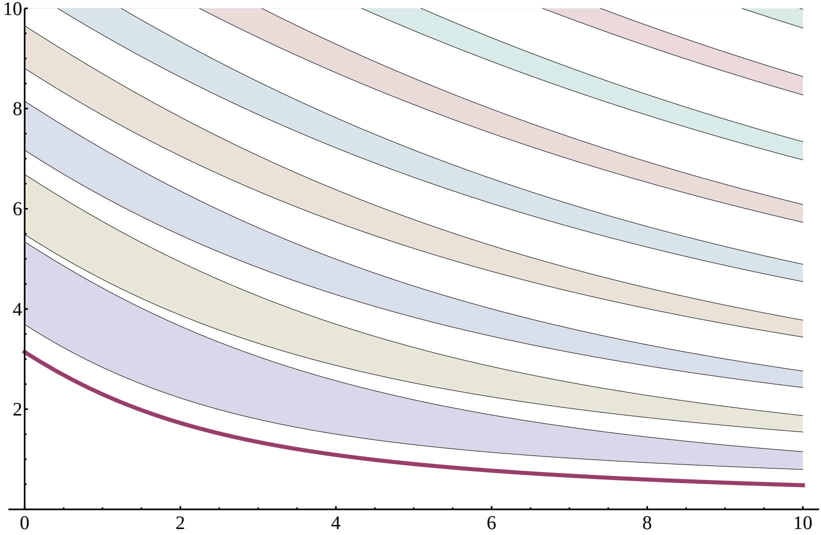
$$\left|\tilde{\lambda}_n - \lambda_{k(n)}\right| = \mathcal{O}\left(\frac{1}{n}\right)$$

for some $k(n)$.

Proof:

- Expand $\tilde{\varphi}_n$ in the basis of φ_k.
- Apply this to the estimate of $\left\|\mathbf{A}_{(-a,a)}\tilde{\varphi}_n - \tilde{\lambda}_n\tilde{\varphi}_n\right\|$.

Bounds for eigenvalues



Bounds for $a\lambda_{k(n)}$ $(n = 3, 4, \ldots)$ plotted versus a

Sketch of the proof (3)

Corollary

The numbers $k(3)$, $k(4)$, $k(5)$, ... are distinct.

Lemma

There are no other eigenvalues than:

- λ_1, λ_2 (estimates in Z.-Q. Chen, R. Song, JFA 2005)
- $\lambda_{k(3)}$, $\lambda_{k(4)}$, $\lambda_{k(5)}$, ...

Proof:

- $$\int p_t^{(-a,a)}(x,x)dx = \sum_{k=1}^{\infty} e^{-t\lambda_k},$$
- Use $p_t^{(-a,a)}(x,x) \leq p_t(0)$ for the upper bound of LHS.
- Use upper bounds for $\lambda_{k(n)}$ to estimate RHS.
- In the limit $t \to 0$, $\left| \int p_t^{(-a,a)}(x,x)dx - \sum_{n=3}^{\infty} e^{-t\lambda_{k(n)}} \right| < 3$.

Eigenfunctions in half-line revisited

$$\psi_s(u^2) = \frac{\psi'(s^2)(s^2 - u^2)}{\psi(s^2) - \psi(u^2)}$$

$$\psi_s^*(u) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{u}{u^2+v^2} \log \psi_s(v^2) dv\right)$$

$$\mathcal{L}F_s(u) = \frac{s}{s^2+u^2}\,\psi_s^*(u)$$

$$F_s(x) = \sin(sx + \vartheta_s) - \int_{(0,\infty)} e^{-xu} g_s(du)$$

$$\vartheta_s = \operatorname{Arg}(\psi_s^*(is))$$

$$g_s(du) = \frac{1}{\pi}\,\frac{s}{s^2+u^2}\,\frac{\operatorname{Im}(\psi_s)^+(-u^2)}{\psi_s^*(u)}\,du$$