Recent progress in the study of suprema of Lévy processes

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Outline

- Introduction
- Formulae
- Asymptotic expansions
- Estimates
- A property

Joint work with **Jacek Małecki** and **Michał Ryznar**

Definitions (1/2)

- X_t is a 1-D **Lévy process** (and not a compound Poisson process)
- $\Psi(\xi)$ is the **Lévy–Khintchine exponent**:
$$\mathbf{E} e^{i\xi X_t} = e^{-t\Psi(\xi)}$$
- $M_t = \sup\{X_s : s \in [0, t]\}$ is the **supremum functional**
- $\tau_x = \inf\{t \geq 0 : X_t \geq x\}$ is the **first passage time**

Fact

$$\mathbf{P}(M_t < x) = \mathbf{P}(\tau_x > t).$$

Definitions (2/2)

- f is **completely monotone** (CM) if $(-1)^n f^{(n)} \geq 0$
- f is **regularly varying** (RV_ϱ) if $\lim\limits_{x \to \ldots} \dfrac{f(kx)}{f(x)} = k^\varrho$

We introduce three conditions:

- **CM jumps**: the Lévy measures of X_t and $-X_t$ have CM density function on $(0, \infty)$
- **symmetry**: $X_t \overset{D}{=} -X_t$
- **balance**: X_t does not become 'too asymmetric'

Reflection principle

- For the Brownian Motion:

$$\mathbf{P}(M_t \geq x) = \mathbf{P}(\tau_x \leq t) = 2\,\mathbf{P}(\tau_x \leq t; X_t \geq x) = 2\mathbf{P}(X_t \geq x)$$

- For symmetric Lévy processes:

$$\begin{aligned}\mathbf{P}(M_t \geq x) = \mathbf{P}(\tau_x \leq t) &= 2\mathbf{P}(\tau_x \leq t; X_t \geq X_{\tau_x}) \\ &\leq 2\mathbf{P}(\tau_x \leq t; X_t \geq x) = 2\mathbf{P}(X_t \geq x)\end{aligned}$$

- For general Lévy processes:

$$\begin{aligned}\mathbf{P}(M_t \geq x) = \mathbf{P}(\tau_x \leq t) &\leq \frac{\mathbf{P}(\tau_x \leq t; X_t \geq X_{\tau_x})}{\inf_{s\in[0,t]} \mathbf{P}(X_s > 0)} \\ &\leq \frac{\mathbf{P}(\tau_x \leq t; X_t \geq x)}{\inf_{s\in[0,t]} \mathbf{P}(X_s > 0)} = \frac{\mathbf{P}(X_t \geq x)}{\inf_{s\in[0,t]} \mathbf{P}(X_s > 0)}\end{aligned}$$

Regimes

Corollary

$$\mathbf{P}(X_t \geq x) \leq \mathbf{P}(M_t \geq x) \leq \frac{\mathbf{P}(X_t \geq x)}{\inf_{s\in[0,t]} \mathbf{P}(X_s > 0)}.$$

- Similar methods are used to study $x \to \infty$ or $t \to 0$
- A different approach is needed for $x \to 0$ or $t \to \infty$
- **Fluctuation theory**!
- A lot of results for processes with one-sided jumps
- We focus on symmetric (or balanced) processes

Baxter–Donsker formula

Theorem (Baxter, Donsker, 1957)

$$\int_0^\infty \int_0^\infty e^{-\xi x - \eta t} \mathbf{P}(M_t < x) dx dt$$
$$= \frac{1}{\xi} \exp\left(-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\xi \log(1 + \Psi(u)/\eta)}{u(u - i\xi)} du \right),$$

G. Baxter, M. D. Donsker
On the distribution of the supremum functional...
Trans. Amer. Math. Soc. 85 (1957)

Baxter–Donsker formula

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and for symmetric X_t:

$$\ldots = \frac{1}{\xi} \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\xi \log(1 + \Psi(u)/\eta)}{\xi^2 + u^2} du \right).$$

G. Baxter, M. D. Donsker
On the distribution of the supremum functional...
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(Fristedt–Pecherski–)Rogozin formula

Theorem (Rogozin, 1966)

$$\int_0^\infty \int_0^\infty e^{-\xi x-\eta t}\mathbf{P}(M_t < x)dxdt$$
$$= \frac{1}{\xi\eta} \exp\left(-\int_0^\infty \int_0^\infty \frac{e^{-\eta t}(1-e^{-\xi x})}{t}\mathbf{P}(X_t \in dx)dt\right).$$

B.A. Rogozin
Distribution of certain functionals related to...
Teor. Veroyatnost. i Primenen. 11 (1966)

B.E. Fristedt
Sample functions of stochastic processes...
Adv. Probab. Rel. Top. 3, Dekker, New York, 1974

Laplace transform in space

Theorem (K., Małecki, Ryznar, 2013)

If X_t is symmetric and $\Psi(\xi)$ increases with $\xi > 0$, then:

$$\int_0^\infty e^{-\xi x} \mathbf{P}(M_t < x) dx = \int_0^\infty \frac{\Psi'(\lambda) e^{-t\Psi(\lambda)}}{\sqrt{\Psi(\lambda)}} \frac{1}{\lambda^2 + \xi^2} \times$$

$$\times \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\xi \log \frac{\lambda^2 - \zeta^2}{\Psi(\lambda) - \Psi(\zeta)}}{\xi^2 + \zeta^2} d\zeta \right) d\lambda.$$

K., J. Małecki, M. Ryznar
Suprema of Lévy processes
Ann. Probab. 41(3B) (2013)

- A similar expression can be given for balanced processes with CM jumps (work in progress)

Special cases (1/10)

Theorem (Lévy, ’37; Hunt, ’56; Dynkin, ’57)

For the Brownian motion ($\Psi(\xi) = \xi^2$):

$$\mathbf{P}(M_t < x) = 2\mathbf{P}(X_t < x) - 1.$$

P. Lévy
Théorie de l’addition des variables aléatoires
Gauthier–Villars, Paris, 1937

Special cases (2/10)

Theorem (Darling, 1956)

For the Cauchy process (symmetric 1-stable, $\Psi(\xi) = |\xi|$):

$$\mathbf{P}(M_t < x) = \int_t^\infty \frac{1}{\pi} \frac{x}{v^2 + x^2} \times$$
$$\times \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\log(v/x + u)}{1 + u^2} \, du \right) dv.$$

D.A. Darling
The maximum of sums of stable random variables
Trans. Amer. Math. Soc. 83 (1956)

Special cases (3/10)

Theorem (Baxter, Donsker, 1957)

For the two-sided Poisson process (±1 jumps, $\Psi(\xi) = 1 - \cos\xi$):

$$\mathbf{P}(M_t < n) = 1 - n\int_0^t \frac{I_n(u)}{u} e^{-u} du,$$

where I_n is the Bessel function.

G. Baxter, M. D. Donsker
On the distribution of the supremum functional...
Trans. Amer. Math. Soc. 85 (1957)

Special cases (4/10)

Theorem (Pyke, 1959; Gani, Prabhu, 1959)

For the Poisson process with drift $X_t = N_t - \gamma t$:

$$\mathbf{P}(M_t \le x) = e^{-t} \sum_{n=0}^{\lfloor \gamma t + x \rfloor} \frac{\gamma t + x - n}{\gamma^n n!} \times \sum_{j=0}^{\lfloor x \rfloor} \binom{n}{j} (j-x)^j (\gamma t + x - j)^{n-j-1}.$$

J. Gani, N.J. Prabhu
The time-dependent solution for a storage model...
J. Math. Mech. 8 (1959)

R. Pyke
The supremum and infimum of the Poisson process
Ann. Math. Stat. 30(2) (1959)

Special cases (5/10)

Theorem (Pyke, 1959)

For the reversed Poisson process with drift $X_t = \gamma t - N_t$:

$$\mathbf{P}(M_t \geq x) = x \sum_{n=0}^{\lfloor \gamma t - x \rfloor} e^{-(n+x)/\gamma} \frac{(n+x)^{n-1}}{\gamma^n n!}.$$

R. Pyke
The supremum and infimum of the Poisson process
Ann. Math. Stat. 30(2) (1959)

Special cases (6/10)

Theorem (Skorokhod, '64; Heyde, '69; Bingham, '73)

For the spectrally negative α-stable process ($\alpha > 1$):

$$\mathbf{P}(M_t < x) = 1 - F_{1/\alpha}(t/x^{1/\alpha})\,;$$

F_β is the CDF of the β-stable distribution on $(0, \infty)$.

A.V. Skorokhod
Random Processes with Independent Increments
Izdat. Nauk. (Moscow), 1964

C.C. Heyde
On the Maximum of Sums of Random Variables...
J. Appl. Probab. 6 (1969)

N.H. Bingham
Maxima of sums of random variables and suprema...
Z. Wahrscheinlichkeit. Verw. Gebiete 26 (1973)

Note: $\mathbf{P}(M_t \geq x) = \alpha \mathbf{P}(X_t \geq x)$ by the reflection principle.

Special cases (7/10)

Theorem (Cramér; Prabhu, '61; Takács, '65; Michna, '11)

For a spectrally positive process:

$$\mathbf{P}(M_t < x)dx = \mathbf{P}(X_t < x)dx - \int_0^t \mathbf{P}(X_s \in dx)\,\frac{\mathbf{E}(-X_{t-s})_+}{t-s}\,ds.$$

N.U. Prabhu
On the ruin problem of collective risk theory
Ann. Math. Stat. 32(3) (1961)

L. Takács
On the distribution of the supremum for stochastic...
Trans. Amer. Math. Soc 119 (1965)

Z. Michna
Formula for the supremum distribution of a...
arXiv:1104.1976 (2011)

Special cases (8/10)

Theorem (Bernyk, Dalang, Peskir, 2008)

For the spectrally positive α-stable process ($\alpha > 1$):

$$\mathbf{P}(M_t < x) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(\alpha n)\Gamma(-n+1+1/\alpha)} x^{\alpha n - 1}.$$

V. Bernyk, R.C. Dalang, G. Peskir
The law of the supremum of a stable Lévy process...
Ann. Probab. 36 (2008)

R.A. Doney
On Wiener-Hopf factorization and the distribution...
Ann. Probab. 15 (1987)

P. Graczyk, T. Jakubowski
On exit time of stable processes
Stoch. Process. Appl. 122(1) (2012)

Special cases (9/10)

Theorem (Hubalek, Kuznetsov, 2011)

For ‘almost all’ α-stable processes:

$$\mathbf{P}(M_t < x) = \begin{cases} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} x^{m+\alpha n+\alpha\varrho} & (\alpha > 1) \\ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} a_{n,m} x^{-m-\alpha n} & (\alpha < 1) \end{cases}$$

where $\varrho = \mathbf{P}(X_t > 0)$,

$$a_{n,m} = \frac{\pm 1}{\Gamma(\ldots)\Gamma(\ldots)} \prod_{i=1}^{m} \frac{\sin(\ldots)}{\sin(\ldots)} \prod_{j=1}^{n} \frac{\sin(\ldots)}{\sin(\ldots)} .$$

F. Hubalek, A. Kuznetsov
A convergent series representation for the density...
Elect. Comm. Probab. 16 (2011)

Special cases (10/10)

Theorem (K., Małecki, Ryznar)

For ‘almost all’ symmetric processes with CM jumps:

$$\mathbf{P}(M_t < x) = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{\Psi'(\lambda)}{2\lambda\Psi(\lambda)}}\, e^{-t\Psi(\lambda)} F_\lambda(x) d\lambda,$$

where $F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - [\mathit{sth\ small}]$ is given explicitly.

K.
Spectral analysis of subordinate Brownian motions...
Studia Math. 206(3) (2011)

K., J. Małecki, M. Ryznar
First passage times for subordinate Brownian...
Stoch. Proc. Appl 123 (2013)

- A similar expression can be given for **some** balanced processes with CM jumps (work in progress)

Fluctuation theory (1/2)

We will need $\kappa(\eta, 0)$...

- Pecherski–Rogozin formula:
$$\kappa(\eta, 0) = \exp\left(\int_0^\infty \frac{e^{-t} - e^{-\eta t}}{t} \mathbf{P}(X_t > 0) dt\right)$$
- $\kappa(\eta, 0)$ is the Laplace exponent of the ascending ladder time process (the 'easier' part of $\kappa(\eta, \xi)$)
- for symmetric processes: $\kappa(\eta, 0) = \sqrt{\eta}$
- if $\mathbf{P}(X_t > 0) = \varrho$, then $\kappa(\eta, 0) = \eta^\varrho$

Fluctuation theory (2/2)

...and $V(x)$

- $$\int_0^\infty e^{-\xi x} V(x)dx = \frac{1}{\xi\kappa(0,\xi)}$$
- Pecherski–Rogozin formula:
$$\kappa(\eta,\xi) = \exp\left(\int_0^\infty \frac{e^{-t} - e^{-\eta t - \xi x}}{t} \mathbf{P}(X_t \in dx)dt\right)$$
- $V(x)$ is the renewal function of the ascending ladder height process
- $V(x)$ is nondecreasing and subadditive on $(0,\infty)$

Large t (1/2)

Theorem (Greenwood, Novikov, 1986)

If $\varrho = \lim_{t\to\infty} \mathbf{P}(X_t > 0) \in (0,1)$, then:

$$\lim_{t\to\infty} \frac{\mathbf{P}(M_t < x)}{\kappa(1/t, 0)} = \frac{V(x)}{\sqrt{\pi}}$$

and $\kappa(\eta, 0)$ is RV_ϱ as $\eta \to 0^+$.

P.E. Greenwood, A.A. Novikov
One-side boundary crossing for processes...
Teor. Veroyatnost. i Primenen. 31(2) (1986)

Large t (2/2)

Theorem (K., Małecki, Ryznar)

For ‘almost all’ symmetric processes with CM jumps:

$$\lim_{t \to \infty} \left(t^{n+1/2} \frac{d^n}{dt^n} \mathbf{P}(M_t < x) \right) = \frac{(-1)^n \Gamma(n+1/2)}{\pi} V(x).$$

K., J. Małecki, M. Ryznar
First passage times for subordinate Brownian...
Stoch. Proc. Appl 123 (2013)

Large t (2/2)

Theorem (K., Małecki, Ryznar)

For 'almost all' symmetric processes with CM jumps:

$$\lim_{t\to\infty}\left(t^{n+1/2}\,\frac{d^n}{dt^n}\overbrace{\mathbf{P}(M_t<x)}^{\mathbf{P}(\tau_x>t)}\right)=\frac{(-1)^n\Gamma(n+1/2)}{\pi}V(x).$$

Therefore, τ_x has 'eventually CM' density.

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First passage times for subordinate Brownian...
Stoch. Proc. Appl 123 (2013)

Small x (1/2)

Theorem (Bingham, 1973)

For α-stable processes:

$$\lim_{x \to 0^+} \frac{\mathbf{P}(M_t < x)}{x^{\alpha\varrho}} = \frac{c}{t^{\varrho}} .$$

N.H. Bingham
Maxima of sums of random variables and suprema...
Z. Wahrscheinlichkeit. Verw. Gebiete 26 (1973)

Small x (2/2)

Theorem (K., Małecki, Ryznar)

For ‘almost all’ symmetric processes with CM jumps:

$$\lim_{x \to 0^+} \left(\sqrt{\Psi(1/x)} \, \frac{d^n}{dt^n} \mathbf{P}(M_t < x) \right) = \frac{(-1)^n \Gamma(n + 1/2)}{\pi \Gamma(1 + \frac{\alpha}{2})} \, \frac{1}{t^{n+1/2}}$$

if $\Psi(\xi)$ is RV_α for $\alpha \in [0, 2]$ at ∞.

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Stable processes

- For α-stable processes:

$$\mathbf{P}(M_t < x) = \mathbf{P}(M_1 < x/t^{1/\alpha})$$

- By the asymptotic expansion:

$$c_1 \min\left(1, \frac{x^{\alpha\varrho}}{t^{\varrho}}\right) \leq \mathbf{P}(M_t < x) \leq c_2 \min\left(1, \frac{x^{\alpha\varrho}}{t^{\varrho}}\right)$$

General Lévy processes

Theorem (K., Małecki, Ryznar)

(a) $$\mathbf{P}(M_t < x) \le \min\left(1, 2\kappa(1/t, 0)V(x)\right).$$

(b) If $\varepsilon \le \mathbf{P}(X_s > 0) \le 1 - \varepsilon$ for $s > 0$, then:

$$\mathbf{P}(M_t < x) \ge \min\left(\tfrac{1}{100}, c(\varepsilon)\kappa(1/t, 0)V(x)\right).$$

K., J. Małecki, M. Ryznar
Suprema of Lévy processes
Ann. Probab. 41(3B) (2013)

- In fact, the result is slightly more general

Symmetric Lévy processes (1/2)

- For symmetric processes, $\kappa(\eta, 0) = \sqrt{\eta}$

Corollary

For symmetric processes:

$$\frac{1}{2000} \leq \frac{\mathbf{P}(M_t < x)}{\min\left(1, \frac{V(x)}{\sqrt{t}}\right)} \leq 2.$$

Lemma

If in addition $\Psi(\xi)$ i $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$, then:

$$\frac{1}{5\sqrt{\Psi(1/x)}} \leq V(x) \leq \frac{5}{\sqrt{\Psi(1/x)}}.$$

Symmetric Lévy processes (2/2)

Theorem (K., Małecki, Ryznar)

For symmetric X_t, if $\Psi(\xi)$, $\xi^2/\Psi(\xi)$ are increasing in $\xi > 0$:

$$\frac{1}{10000} \le \frac{\mathbf{P}(M_t < x)}{\min\left(1, \dfrac{1}{\sqrt{t\Psi(1/x)}}\right)} \le 10$$

- The result includes all subordinate Brownian motions

K., J. Małecki, M. Ryznar
Suprema of Lévy processes
Ann. Probab. 41(3B) (2013)

- More general estimates for V:

K. Bogdan, T. Grzywny, M. Ryznar
Density and tails of unimodal convolution...
arXiv:1305.0976

Symmetric Lévy processes with CM jumps

Theorem (K., Małecki, Ryznar)

For symmetric processes with CM jumps:

$$\frac{c_1(n)}{t^{n+1/2}\sqrt{\Psi(1/x)}} \le (-1)^n \frac{d^n}{dt^n} \mathbf{P}(\tau_x > t) \le \frac{c_2(n)}{t^{n+1/2}\sqrt{\Psi(1/x)}}$$

for $t \ge \dfrac{c_0(n, \alpha)}{\Psi(1/x)}$, if Ψ is RV at 0 and ∞ of positive indices.

- Regular variation can be slightly relaxed
- Weaker results hold if Ψ is slowly varying

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First passage times for subordinate Brownian...
Stoch. Proc. Appl 123 (2013)

Balanced Lévy processes with CM jumps

Theorem (K.)

For balanced processes with CM jumps:

$$\kappa(\tau, \xi), \quad \frac{\kappa(\tau, \xi_1)}{\kappa(\tau, \xi_2)}, \quad \frac{\kappa(\tau_1, \xi)}{\kappa(\tau_2, \xi)}, \quad \kappa(\tau, \xi_1)\hat{\kappa}(\tau, \xi_2)$$

are **complete Bernstein functions** of τ and ξ if $0 < \xi_1 < \xi_2$, $0 < \tau_1 < \tau_2$.

- κ and $\hat{\kappa}$ are Laplace exponents of ladder processes
- New even for symmetric processes

K.
Rogers functions and fluctuation theory
(in preparation)