Rogers functions and fluctuation theory

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Lévy Processes 7

Wrocław, July 16, 2013

...it would be worth studying Lévy processes whose jump measure has a completely monotone density, and in particular, the Wiener–Hopf factorization of such.

L.C.G. Rogers
Wiener–Hopf factorization of diffusions and Lévy processes
Proc. London Math. Soc. 47(3) (1983): 177–191

Outline

- L.C.G. Rogers's result
- Extension and further results
- 'Rogers functions' and their properties
- Wiener–Hopf factorisation
- Further research

Rogers functions and fluctuation theory
In preparation

Credits

In connection with this presentation, I thank:

- **L.C.G. Rogers**
  — for an inspiring article
- **A. Kuznetsov**
  — for letting me know about it
- **K. Kaleta**, **T. Kulczycki**, **J. Małecki**, **M. Ryznar**
  — for joint research in the symmetric case
- **P. Kim**, **Z. Vondraček**
  — for a discussion of the non-symmetric case

CM jumps

X_t is a 1-D Lévy process with Lévy measure $\nu(x)\mathrm{d}x$

Notation: **CM** = **completely monotone**

Definition

X_t has **CM jumps** $\iff$ $\nu(x)$ and $\nu(-x)$ are **CM** on $(0,\infty)$:

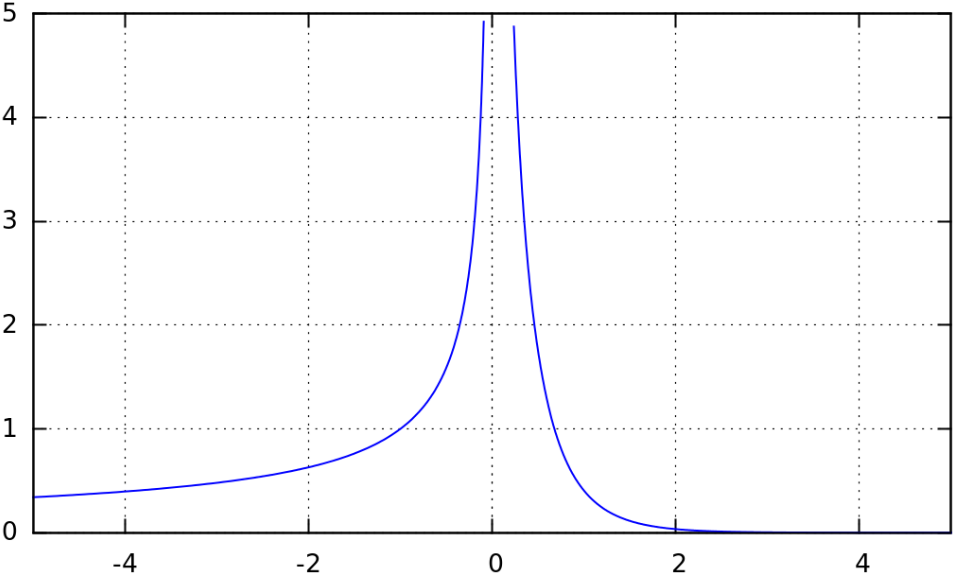
$$\nu(x) = \mathscr{L}\mu_+(x) = \int_{(0,\infty)} e^{-sx}\mu_+(\mathrm{d}s) \qquad (x > 0)$$

$$\nu(x) = \mathscr{L}\mu_-(-x) = \int_{(0,\infty)} e^{sx}\mu_-(\mathrm{d}s) \qquad (x < 0)$$

(see Bernstein's theorem)

Examples:

- Stable processes: $\nu(\pm x) = c_\pm x^{-1-\alpha}$
- Tempered stable processes: $\nu(\pm x) = c_\pm x^{-1-\alpha} e^{-m_\pm x}$
- Meromorphic processes



Plot of $\nu(x)$ for a sample process with **CM jumps**

Rogers's theorem

Notation: **CBF** = **complete Bernstein function**

Definition

f is a **CBF** $\iff$ $\frac{1}{f} = \mathscr{L}g$ for a **CM** g $\iff$ $\frac{1}{f} = \mathscr{L}\mathscr{L}\mu$

(there are many equivalent definitions)

Theorem [Rogers, 1983]

X_t has **CM jumps**

$\Updownarrow$

$\kappa(\tau;\xi)$ and $\hat{\kappa}(\tau;\xi)$ are **CBF**s of ξ for some/all τ

- $\kappa(\tau;\xi)$ and $\hat{\kappa}(\tau;\xi)$ are Laplace exponents of the ladder processes for X_t (describe extrema of X_t)

(more on this below)

Extension of Rogers's theorem

Theorem [K., 2013]

X_t has **CM jumps** and is **balanced**

$$\Downarrow$$

$\kappa(\tau;\xi)$ and $\hat{\kappa}(\tau;\xi)$ are **CBF**s of both τ and ξ

Furthermore, the following are **CBF**s of τ and ξ:

$$\frac{\kappa(\tau_1;\xi)}{\kappa(\tau_2;\xi)} \qquad \frac{\hat{\kappa}(\tau_1;\xi)}{\hat{\kappa}(\tau_2;\xi)} \qquad (0 \le \tau_1 \le \tau_2)$$

$$\frac{\kappa(\tau;\xi_1)}{\kappa(\tau;\xi_2)} \qquad \frac{\hat{\kappa}(\tau;\xi_1)}{\hat{\kappa}(\tau;\xi_2)} \qquad (0 \le \xi_1 \le \xi_2)$$

$$\kappa(\tau;\xi_1)\hat{\kappa}(\tau;\xi_2)$$

- The meaning of '**balanced**' is explained later

(stable are balanced; tempered stable can be made balanced)

Supremum functional

Definition

Supremum of X_t: $$M_t = \sup_{s \in [0,t]} X_s$$

Time of supremum: $$T_t \in [0,t] \; : \; M_t = X_{T_t}$$

Theorem

$$\int_0^\infty \left(\mathbf{E} e^{-\sigma T_t - \xi M_t} \right) e^{-\tau t} \mathrm{d}t \, \mathrm{d}x = \frac{1}{\tau} \, \frac{\kappa(\tau; 0)}{\kappa(\sigma + \tau; \xi)}$$

That is:

$$\mathscr{L}_{\substack{t \mapsto \tau \\ s \mapsto \sigma \\ x \mapsto \xi}} \mathbf{P}(T_t \in \mathrm{d}s, M_t \in \mathrm{d}x) = \frac{1}{\tau} \, \frac{\kappa(\tau; 0)}{\kappa(\sigma + \tau; \xi)}$$

Properties of the supremum

Corollary [Rogers, 1983]

If X_t has **CM jumps**:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty e^{-\tau t} \mathbf{P}(M_t < x) \mathrm{d}t \qquad \text{is CM in } x$$

Corollary [K.]

If X_t has **CM jumps** and is **balanced**:

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-\tau t} \mathbf{P}(T_t < s) \mathrm{d}t \qquad \text{is CM in } s$$

Corollary [K.]

If X_t has **CM jumps** and is **balanced**:

$$\mathbf{E} e^{-\xi M_t} \qquad \text{is CM in } t$$

Space-only Laplace transform

Theorem [K.]

If X_t has **CM jumps** and is **balanced**:

$$\mathbf{E}e^{-\xi M_t} = \int_0^\infty e^{-tr} \frac{\xi \operatorname{Re} \Psi^{-1}(r)}{|i\xi - \Psi^{-1}(r)|^2} \frac{\Psi_r^*(\xi)}{r} \mathrm{d}r$$

where

$$\Psi_r^*(\xi) = \exp\left(\frac{1}{\pi}\int_{\Psi_r(0)}^\infty \arg\left(1 - \frac{i\xi}{\Psi_r^{-1}(s)}\right) \frac{\mathrm{d}s}{s}\right)$$

and

$$\Psi_r(\xi) = \frac{(\xi - \Psi^{-1}(r))(\xi + \overline{\Psi^{-1}(r)})}{\Psi(\xi) - r}$$

(Ψ is the Lévy–Khintchine exponent; more on this later)

Semi-explicit formula?

If one can justify the use of Fubini:

Theorem?

If X_t has **CM jumps** and is **balanced**:

$$\mathbf{P}(M_t < x) = \int_0^\infty e^{-tr} F_r(x) \mathrm{d}r$$

where

$$F_r(x) = c_r e^{\alpha_r x} \sin(\beta_r x + \vartheta_r) - \{\textbf{CM} \text{ correction}\}$$

$$\alpha_r = \mathrm{Im}(\Psi^{-1}(r))$$

$$\beta_r = \mathrm{Re}(\Psi^{-1}(r))$$

c_r, ϑ_r and the **CM** correction are given semi-explicitly

Potential applications

- Semi-explicit expression for the distribution of M_t
- Asymptotic expansions and estimates of the above
- Eigenfunction expansion for X_t in half-line

For the symmetric case, see:

K.
Spectral analysis of subordinate Brownian motions...
Studia Math. 206(3) (2011)

K., J. Małecki, M. Ryznar
Suprema of Lévy processes
Ann. Probab. 41(3B) (2013)

K., J. Małecki, M. Ryznar
First passage times for subordinate Brownian...
Stoch. Proc. Appl 123 (2013)

Lévy–Khintchine exponent

Definition

$$\mathbf{E} e^{-i\xi X_t} = e^{-t\Psi(\xi)}$$

Lévy–Khintchine formula

$$\Psi(\xi) = a\xi^2 - ib\xi + \int_{\mathbf{R}} (1 - e^{i\xi x} + i\xi x \mathbf{1}_{|x|<1}) \nu(x) \mathrm{d}x$$

- $\operatorname{Re} \Psi(\xi) \geq 0$

CM jumps revisited

Observation

If X_t has **CM jumps**:

$$\nu(x) = \mathscr{L}\mu_+(x) \qquad (x > 0)$$
$$\nu(x) = \mathscr{L}\mu_-(-x) \qquad (x < 0)$$

then

$$\Psi(\xi) = a\xi^2 - ib\xi + \int_{\mathbf{R}\setminus\{0\}} \left(\frac{\xi}{\xi + is} + \frac{i\xi s}{1 + s^2} \right) \frac{\mu(ds)}{|s|}$$

(b is different here and in the previous slide)

with

$$\mu(E) = \mu_+(E \cap (0, \infty)) + \mu_-((-E) \cap (-\infty, 0))$$

Rogers functions

Definition

f is a **Rogers function** if

$$f(\xi) = a\xi^2 - ib\xi + c + \int_{\mathbf{R}\setminus\{0\}} \left(\frac{\xi}{\xi + is} + \frac{i\xi s}{1 + s^2} \right) \frac{\mu(ds)}{|s|}$$

for $a \ge 0$, $b \in \mathbf{R}$, $c \ge 0$, $\mu \ge 0$

- f extends to $\mathbf{C} \setminus i\mathbf{R}$
- $f(-\overline{\xi}) = \overline{f(\xi)}$
- It suffices to consider f on $\{\xi : \operatorname{Re} \xi > 0\}$

Equivalent definitions

Proposition

The following are equivalent:

(a) for $a \ge 0$, $b \in \mathbf{R}$, $c \ge 0$, $\mu \ge 0$:

$$f(\xi) = a\xi^2 - ib\xi + c + \int_{\mathbf{R}\setminus\{0\}} \left(\frac{\xi}{\xi + is} + \frac{i\xi s}{1 + s^2} \right) \frac{\mu(ds)}{|s|}$$

(b) for $k \ge 0$, $\varphi \in [0, \pi]$:

$$f(\xi) = k \exp\left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\xi}{\xi + is} - \frac{1}{1 + |s|} \right) \frac{\varphi(s) ds}{|s|} \right)$$

(c) f is holomorphic in $\{\xi : \operatorname{Re} \xi > 0\}$ and:

$$\operatorname{Re} \frac{f(\xi)}{\xi} \ge 0 \qquad \text{if } \operatorname{Re} \xi > 0$$

(that is, $f(\xi)/\xi$ is a Nevanlinna–Pick function)

Real values

Theorem [K.]

If f is a Rogers function, then:

(a) For $r > 0$ there is at most one solution of

$$f(\xi) = r \qquad (\mathrm{Re}\,\xi > 0)$$

Write $\xi = f^{-1}(r)$

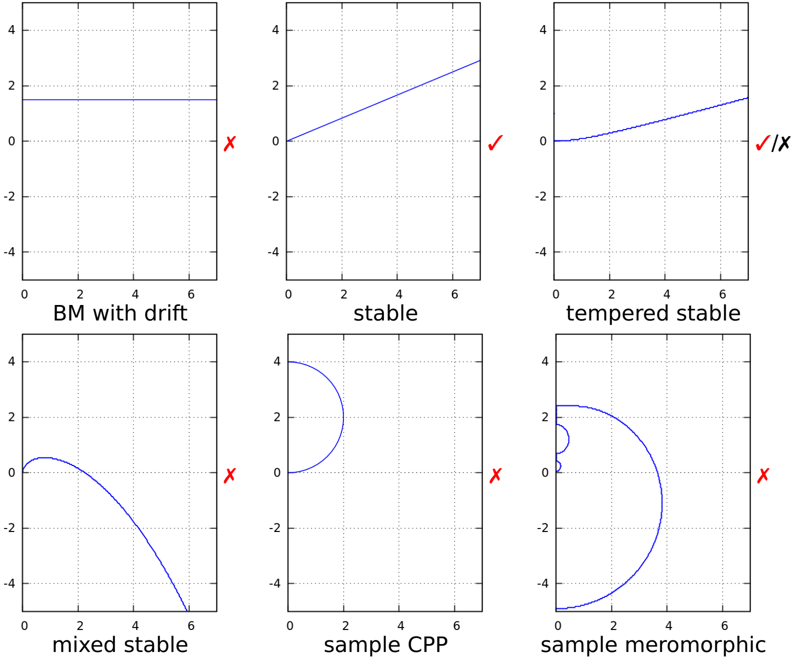
(b) $|f^{-1}(r)|$ is increasing

Definition

A Rogers function f is **balanced** if

$$-\tfrac{\pi}{2} + \varepsilon \le \arg(f^{-1}(r)) \le \tfrac{\pi}{2} - \varepsilon$$

X_t is **balanced** if Ψ is **balanced**



Real lines $\{\xi : f(\xi) \in (0, \infty)\}$ for some Rogers functions

Extension

Definition

A Rogers function f is **nearly balanced** if

$$f \circ \Phi$$

is **balanced** for some Möbius transformation Φ which preserves $\{\xi : \operatorname{Re} \xi > 0\}$ (e.g. vertical translation)

Theorem

Main results extend to **nearly balanced** processes

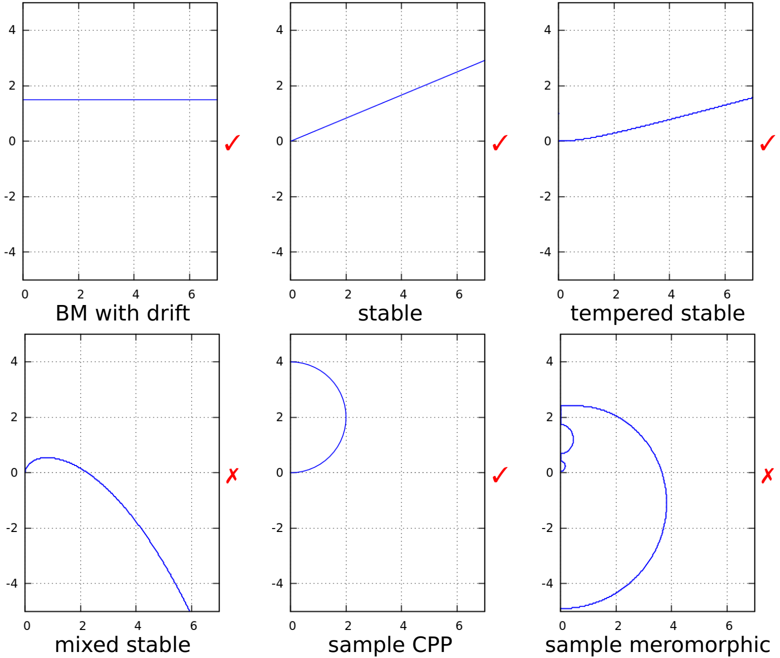
Examples of **nearly balanced** processes:

- Non-monotone strictly stable and their mixtures
- Tempered strictly stable:

$$\nu(\pm x) = c_{\pm} x^{-1-\alpha} e^{-m_{\pm} x}$$

- (Completely) subordinate to above

(that is, with a subordinator corresponding to a CBF)



Real lines $\{\xi : f(\xi) \in (0, \infty)\}$ for some Rogers functions

Analytical approach

Wiener–Hopf method

For $A \in \mathscr{S}'(\mathbf{R})$ write

$$A = A^+ * A^- \qquad (\text{or } \mathscr{F}A = \mathscr{F}A^+ \cdot \mathscr{F}A^-)$$

where $\operatorname{supp} A^+ \subseteq [0, \infty)$, $\operatorname{supp} A^- \subseteq (-\infty, 0]$

- Fourier transform of A^+ extends to $\{\xi : \operatorname{Im} \xi > 0\}$:

$$\log \mathscr{F}A^+(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\xi - z} \log \mathscr{F}A(z) dz$$

- Fourier transform of A^- extends to $\{\xi : \operatorname{Im} \xi < 0\}$:

$$\log \mathscr{F}A^-(\xi) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\xi - z} \log \mathscr{F}A(z) dz$$

(these are principal value intergals at $\pm\infty$)

- Developed to solve integral equations and PDEs with mixed boundary conditions on $(-\infty, 0)$ and $(0, \infty)$

Wiener–Hopf in fluctuation theory

Wiener–Hopf factorization

$$\frac{1}{\Psi(\xi)+\tau} = \frac{1}{\kappa(\tau;-i\xi)} \cdot \frac{1}{\hat{\kappa}(\tau;i\xi)}$$

- $\frac{1}{\Psi(\xi)+\tau} = \mathscr{F}U^\tau(\xi)$ with $U^\tau(E) = \int_0^\infty e^{-\tau t}\mathbf{P}(X_t \in E)\mathrm{d}t$

  $\left(\text{or } \frac{\tau}{\Psi(\xi)+\tau} \text{ is the Fourier transform of } X_{\mathbf{e}(\tau)}\right)$

- $\frac{1}{\kappa(\tau;-i\xi)} = \mathscr{F}V^\tau(\xi)$ and $\frac{1}{\kappa(\tau;\xi)} = \mathscr{L}V^\tau(\xi)$

  ($V^\tau(\mathrm{d}x)$ is the renewal measure of the ascending ladder height process for X_t killed at rate τ)

  (and a dual version with $\hat{\kappa}$ and $\hat{V}^\tau$)

- $U^\tau(E) = \int_{\mathbf{R}} \hat{V}^\tau(x-E)V^\tau(\mathrm{d}x)$

Wiener–Hopf in fluctuation theory

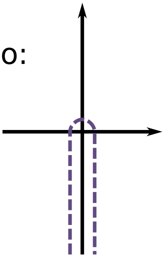
Wiener–Hopf factorization

$$\frac{1}{\Psi(\xi) + \tau} = \frac{1}{\kappa(\tau; -i\xi)} \cdot \frac{1}{\hat{\kappa}(\tau; i\xi)}$$

- Baxter–Donsker-type formula:

$$\log \frac{\kappa(\tau;\xi)}{\kappa(\tau;1)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{i\xi - z} - \frac{1}{i - z} \right) \log(\Psi(z) + \tau) dz$$

- Deform the contour of integration from **R** to:



- Exponential **CBF** representation of $\kappa(\tau;\xi)$ in ξ follows (proving Rogers's result)

Wiener–Hopf in fluctuation theory

Wiener–Hopf factorization

$$\frac{1}{\Psi(\xi) + \tau} = \frac{1}{\kappa(\tau; -i\xi)} \cdot \frac{1}{\hat{\kappa}(\tau; i\xi)}$$

- Baxter–Donsker-type formula:

$$\log \frac{\kappa(\tau; \xi_1)}{\kappa(\tau; \xi_2)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{i\xi_1 - z} - \frac{1}{i\xi_2 - z} \right) \log(\Psi(z) + \tau) dz$$

- Deform the contour of integration from **R** to

$$\{\xi \in \mathbf{C} : \Psi(\xi) \in (0, \infty)\}$$

- Then $\log(\Psi(z) + \tau)$ is holomorphic in $\tau \in \mathbf{C} \setminus (-\infty, 0]$ (a major step towards the extension)

Non-balanced processes

Problem

Show that if X_t has **CM jumps**, then:

(that is, drop the assumption that X_t is balanced)

$$\kappa(\tau;\xi) \qquad \hat\kappa(\tau;\xi)$$

$$\frac{\kappa(\tau_1;\xi)}{\kappa(\tau_2;\xi)} \qquad \frac{\hat\kappa(\tau_1;\xi)}{\hat\kappa(\tau_2;\xi)} \qquad (0 \le \tau_1 \le \tau_2)$$

$$\frac{\kappa(\tau;\xi_1)}{\kappa(\tau;\xi_2)} \qquad \frac{\hat\kappa(\tau;\xi_1)}{\hat\kappa(\tau;\xi_2)} \qquad (0 \le \xi_1 \le \xi_2)$$

$$\kappa(\tau;\xi_1)\hat\kappa(\tau;\xi_2)$$

are **CBF**s of τ and ξ

Problem

When the above are **CBF**s of τ only?

Bivariate CBFs

Problem

Describe functions $f(\xi, \eta)$ such that

$$f(\xi, \eta), \quad \frac{f(\xi, \eta_1)}{f(\xi, \eta_2)} \quad \text{and} \quad \frac{f(\xi_1, \eta)}{f(\xi_2, \eta)} \qquad \begin{matrix} (0 \le \xi_1 \le \xi_2) \\ (0 \le \eta_1 \le \eta_2) \end{matrix}$$

are **CBF**s of ξ, η

Distribution of the supremum functional

Problem

Justify the use of Fubini for the formula for $\mathbf{P}(M_t < x)$

Problem

Prove generalised eigenfunction expansion for X_t killed upon leaving half-line

- Work in progress

Hitting time of a point

$$\mathbf{P}(M_t < x) = \mathbf{P}(\tau_x > t) \qquad \text{with} \qquad \tau_x = \inf\{t : X_t \geq x\}$$

Problem

Find a formula, estimates and asymptotic expansion of $\mathbf{P}(\sigma_x > t)$ for

$$\sigma_x = \inf\{t : X_t = x\}$$

For the symmetric case, see:

K.
Spectral theory for one-dimensional symmetric...
Electron. J. Probab. 17 (2012)

T. Juszczyszyn, K.
Hitting times of points for symmetric Lévy...
In preparation