Spectral Theory for Subordinate Brownian Motions in Half-line

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Outline

(1) Part I: Introduction
   BM  Lévy processes  BM in interval  BM in half-line  Setting

(2) Part II: Eigenfunction expansion in half-line
   Eigenfunctions  Their properties  Eigenfunction expansion

(3) Part III: Applications
   First passage times  Fluctuation theory  Interval  Domains in $\mathbb{R}^d$

(4) Part IV: Some technical details
   Wiener-Hopf method  Heuristic derivation

K., 2010
Spectral analysis of subordinate Brownian motions in half-line

K., Jacek Małecki, Michał Ryznar, 2011
Suprema of Lévy processes
Part I

Introduction

- Brownian motion
- Lévy processes
- Warm-up: Brownian motion in an interval
- Motivation: Brownian motion in half-line
- Complete monotonicity, complete Bernstein functions and subordinate Brownian motions
Part I
Section 1
Brownian motion
Brownian motion

• $X_t$ is the position of the particle at time $t \geq 0$

Source: YouTube, http://youtube.com/watch/?v=cDcprgWiQrEY
Brownian motion: mathematical model

Definition

The **Brownian motion** (BM) is a stochastic process $X_t$ with the following properties:

- $X_0 = x$
- independent increments:
  
  \[
  0 \leq t_0 < t_1 < \ldots < t_n,
  \]
  
  \[\downarrow\]
  
  \[X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}} \text{ are independent}\]

- stationarity: law of $X_t - X_s$ depends only on $t - s$
- isotropy: law of $X_t - X_0$ is invariant under rotations
- continuity of paths: $t \mapsto X_t$ is continuous
Brownian motion: simulation

Brownian motion and PDEs (1)

Central limit theorem

Brownian motion is a Gaussian process: $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ has Gaussian distribution.

Notation

- $P_x, E_x$ correspond to process starting at $x$
- $E_x(Z; E) = \int_E Z dP_x$
- For readability, we use both $X_t$ and $X(t)$

- Components of $X_t$ are independent
- Variance of each component is $2ct$ for some $c > 0$
- Typically, $c = \frac{1}{2}$, but we take $c = 1$
The function:
\[ u(t, x) = \mathbb{E}_x f(X_t) \]
solves the **heat equation**:
\[ \frac{\partial u}{\partial t}(t, x) = c \Delta u(t, x), \quad u(0, x) = f(x) \]

- \[ \Delta = \left( \frac{\partial}{\partial x_1} \right)^2 + \ldots + \left( \frac{\partial}{\partial x_d} \right)^2 \]
Brownian motion and PDEs (3)

Definition
For $D$ open, we define the **first exit time**:

$$\tau_D = \inf \{ t \geq 0 : X_t \notin D \}$$

Theorem (Doob, Dynkin, Hunt, Feller, Kakutani, ...)
The function:

$$u(t, x) = \mathbb{E}_x(f(X_t); t < \tau_D)$$

solves **heat equation** in $D$ with **boundary condition**:

$$\frac{\partial u}{\partial t}(t, x) = c \Delta u(t, x) \quad (t \geq 0, x \in D)$$

$$u(t, x) = 0 \quad (t \geq 0, x \in \partial D)$$

$$u(0, x) = f(x) \quad (x \in D)$$
Part I
Section 2
Lévy processes
A **Lévy process** is a stochastic process $X_t$ with the following properties:

- $X_0 = x$
- independent increments
- stationarity
- no isotropy (though we will need it later)
- càdlàg paths: right-continuous with left limits (instead of continuous)

We will only study **one-dimensional** Lévy processes

In dimension one, isotropy = symmetry
Brownian motion

Symmetric 1-stable process

Lévy processes jump!
Lévy measure

Definition

The **Lévy measure** \( \nu \) describes intensity of jumps:

\[
\nu(B) = \lim_{t \to 0^+} \frac{P_x(X_t - x \in B)}{t}
\]

Theorem

\( \nu \) is a Lévy measure \( \iff \int \min(1, |y|^2) \nu(dy) < \infty \)

Theorem (Lévy-Ito decomposition)

Every Lévy process is a sum of:

- pure-jump process (described by the Lévy measure)
- Brownian motion (up to an affine map)
- uniform motion

(plus compensation)
Lévy processes and non-local PDEs (1)

Theorem (a version of the Lévy-Khintchine formula)

The function:

\[ u(t, x) = \mathbf{E}_x f(X_t) \]

solves a ‘heat’ equation:

\[ \frac{\partial u}{\partial t}(t, x) = (-A)u(t, x), \quad u(0, x) = f(x) \]

for a pseudo-differential operator:

\[ (-A)f(x) = af''(x) + bf'(x) \]

\[ + \int (f(x + y) - f(x) - f'(x)y \mathbf{1}_{|y| < 1}) \nu(dy) \]

- \( a \geq 0 \): \( d \times d \) matrix, ‘Brownian part’
- \( b \in \mathbb{R}^d \): ‘drift’
- \( \nu \): Lévy measure, ‘jump part’
The function:
\[ u(t, x) = \mathbb{E}_x(f(X_t); t < \tau_D) \]
solves the ‘heat’ equation in \( D \) with **exterior** condition:

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= (-A)u(t, x) \quad (t \geq 0, x \in D) \\
u(t, x) &= 0 \quad (t \geq 0, x \in D^c) \\
u(0, x) &= f(x) \quad (x \in D)
\end{align*}
\]

- \( A \) is **non-local**!
- Hence \( Af(x) \) requires \( f \) to be defined everywhere (not just in a neighbourhood of \( x \))
- \( X(\tau_D) \in D^c \) instead of \( X(\tau_D) \in \partial D \)
Transition operators

Definition

We define **free transition kernel**:

\[ p_t(x, A) = P_x(X_t \in A) \]

and **transition kernel on** \( D \):

\[ p^D_t(x, A) = P_x(X_t \in A; t < \tau_D) \]

Definition

We define **free transition operators**:

\[ P_tf(x) = \mathbb{E}_x f(X_t) = \int f(y)p_t(x, dy) \]

and **transition operators on** \( D \):

\[ P^D_tf(x) = \mathbb{E}_x(f(X_t); t < \tau_D) = \int_D f(y)p^D_t(x, dy) \]
Generators

- \( P_t P_s f = P_{t+s} f \)
- \( P^D_t P^D_s f = P^D_{t+s} f \)

**Definition**

- \((-A)f = \lim_{t \to 0^+} \frac{P_t f - f}{t}\) (as in the 'heat' equation)
- \((-A_D)f = \lim_{t \to 0^+} \frac{P^D_t f - f}{t}\)

- When \( f(x) = 0 \) in \( D^c \), then \( A_D f(x) = Af(x) \)
- \( A \) and \( A_D \) have different **domains**
- if \( f \in \text{Dom}(A_D) \), then \( f(x) = 0 \) in \( D^c \)
- \( A \) and \( A_D \) are positive definite
Lévy-Khintchine formula

**Definition**

We define the **Lévy-Khintchine exponent**:

\[
\Psi(\xi) = a\xi^2 + ib\xi + \int (1 - e^{-i\xi y} - i\xi y 1_{|y|<1}) \nu(dy)
\]

**Theorem (Lévy-Khintchine formula)**

\[
E_0 e^{-i\xi X_t} = e^{-t\Psi(\xi)} \\
\hat{P}_tf(\xi) = e^{-t\Psi(\xi)} \hat{f}(\xi) \\
\hat{A}f(\xi) = \Psi(\xi) \hat{f}(\xi)
\]

- \(\Psi(\xi)\) is our **initial data**:
  - all results will be given in terms of \(\Psi(\xi)\)
Kernels and densities

- Often $p_t(x, dy)$ and $p_t^D(x, dy)$ are absolutely continuous.
- Also the Lévy measure $\nu(dy)$ will typically be absolutely continuous.

**Notation**

For an absolutely continuous measure $\mu(dy)$, $\mu(y)$ denotes its density function.

- $p_t(x, y)$ and $p_t^D(x, y)$ (if they exist) are called **transition densities** or **heat kernels**.
- $p_t(x, y)$ depends only on $y - x$:
  \[ p_t(x, y) = p_t(y - x) \]
Summary

• We are given a Lévy-Khintchine exponent $\psi(\xi)$ (think: $\psi(\xi) = |\xi|^\alpha$, $0 < \alpha \leq 2$)

• There is a corresponding pseudo-differential operator $A$ and:

$$\hat{Af}(\xi) = \psi(\xi)\hat{f}(\xi)$$

(think: $A = (-\Delta)^{\alpha/2}$)

• Given a domain $D$, $A_D$ is the operator $A$ on $D$ with ‘Dirichlet’ exterior condition

• We study eigenvalues and eigenfunctions of $A_D$

• There is a Lévy process $X_t$ corresponding to $A$

• $A_D$ corresponds to the process $X_t$ killed at $\tau_D$ (the first exit time from $D$)
Half-line

**Goal**

Study the spectral theory for $A_D$ and $P^D_t$ for the half-line:

$$D = (0, \infty) \subseteq \mathbb{R}$$

- Why half-line?
  - explicit formulae (Part II)
  - applications in fluctuation theory (Part III)
  - model case for intervals and smooth domains in $\mathbb{R}^d$ (Part III)
  - possible applications in relativistic quantum physics

- The details are very technical, but the idea is simple

- We begin with two examples for which also the details are simple
Part I
Section 3

Warm-up: Brownian motion in an interval
BM in interval: The simplest example

• Let $D = (0, \pi)$ be the interval
• Let $X_t$ be the Brownian motion with $\text{Var} X_t = 2t$
• Then:
  ▶ $\Psi(\xi) = \xi^2$
  ▶ $(-A)f = \Delta f = f''$
  ▶ $(-A_D) = \Delta_D$ is the Dirichlet Laplacian
• Goal: eigenvalues and eigenfunctions of $A_D$ and $P^D_t$
BM in interval: eigenvalues and eigenfunctions

- Eigenfunctions of $\mathcal{A}$ are sines and cosines
- Eigenfunctions and eigenvalues of $\mathcal{A}_D$ are
  \[ \begin{cases} 
  f_n(x) = \sin(nx)1_{x \in D} \\
  \mu_n = n^2 
  \end{cases} \quad n = 1, 2, \ldots 
\]

Indeed:
- $f_n''(x) = -n^2f_n(x)$ in $D$
- $f_n(x) = 0$ in $D^c$
- $f_n$ is continuous
BM in interval: solution

Solution

For:

\[ f_n(x) = \sin(nx) \mathbf{1}_{x \in D}, \quad \mu_n = n^2 \]

we have:

\[ P_t^D f_n = e^{-\mu_nt} f_n, \quad A_D f_n = \mu_n f_n \]

- Similar explicit solutions exist for balls, cubes etc.
- \( f_n \) form a complete orthogonal set in \( L_2(D) \)
- \( \| f_n \|_2 = \sqrt{\frac{\pi}{2}} \)
- \( \frac{2}{\pi} \langle f, f_n \rangle \) is the Fourier series coefficient of \( f \)
BM in interval: eigenfunction expansion

Corollary

\[ P^D_t f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\mu_n t} \langle f, f_n \rangle f_n(x) \]
\[ p^D_t (x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\mu_n t} f_n(x)f_n(y) \]

- There are better formulae for \( p^D_t (x, y) \) for small \( t \)
- Results extend to:
  - more general processes
  - in general bounded domains

But in general, there are no explicit formulae for \( \mu_n \) and \( f_n \)!
Part I

Section 4

Motivation: Brownian motion in half-line
BM in half-line: The unbounded example

- Let $D = (0, \infty)$ be the **half-line**
- Let $X_t$ be again the **Brownian motion**, $\text{Var} X_t = 2t$:
  - $\psi(\xi) = \xi^2$
  - $(-\mathcal{A})f = \Delta f = f''$
  - $(-\mathcal{A}_D) = \Delta_D$ is the Dirichlet Laplacian
- Goal: eigenvalues and eigenfunctions of $\mathcal{A}_D$ and $P^D_t$
- Main change: $P^D_t$ are no longer **compact** operators
BM in half-line: eigenvalues and eigenfunctions

- Again, eigenfunctions and eigenvalues of $A_D$ are:
  \[
  \begin{cases}
  F_\lambda(x) = \sin(\lambda x)1_{x>0} \\
  \mu_\lambda = \lambda^2
  \end{cases}
  \quad \lambda \in (0, \infty)
  
  Indeed:
  - $F_\lambda''(x) = -\lambda^2 F_\lambda(x)$ in $D$
  - $F_\lambda(x) = 0$ in $D^c$
  - $F_\lambda$ is continuous
- This time $F_\lambda \notin L^2(D)$!
- Note that $\mu_\lambda = \psi(\lambda)
BM in half-line: solution

Solution

For:

\[ F_\lambda(x) = \sin(\lambda x) \mathbf{1}_{x>0}, \quad \mu_\lambda = \lambda^2 = \psi(\lambda) \]

we have:

\[ P_t^D F_\lambda = e^{-t\psi(\lambda)} F_\lambda, \quad A_D F_\lambda = \psi(\lambda) F_\lambda \]

- \( F_\lambda \) are not in \( L^2(D) \)
- There are uncountably many eigenfunctions
- The Fourier sine transform of \( f \) is given by:

\[ \langle f, F_\lambda \rangle = \int f(y) F_\lambda(y) dy \]
BM in half-line: eigenfunction expansion

**Corollary**

\[
P^D_t f(x) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda)} \langle f, F_\lambda \rangle F_\lambda(x) d\lambda
\]

\[
p^D_t(x, y) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda)} F_\lambda(x)F_\lambda(y) d\lambda
\]

- Here \( f \in L^1(D) \)
  Can be extended to \( f \in L^2(D) \)

- By **reflection principle**, there is a better formula:
  \[
p^D_t(x, y) = p_t(y - x) - p_t(y + x) \quad (x, y \in D)
\]
  No reflection principle for jump-type processes

- No similar results for other Lévy processes! (until very recently)
Problems:

(1) Let $X_t$ be the Brownian motion and $D = (0, \infty)$ . Prove, by a direct calculation, that $F_{\lambda}(x) = \sin(\lambda x)$ is the eigenfunction of $P^D_t$.
   (Hint: use $p^D_t(x, y) = p_t(y - x) - p_t(y + x)$)

(2) Let $X_t$ be the Brownian motion and $D = (0, \pi)$. Prove that for $x, y \in D$:

$$p^D_t(x, y) = \sum_{n=-\infty}^{\infty} p_t(y - x + 2n\pi) - \sum_{n=-\infty}^{\infty} p_t(y + x + 2n\pi)$$

(3) Show that for $D = (0, \infty)$, $A_D$ and $P^D_t$ may fail to be normal operators when $X_t$ is not symmetric.

(4) Let $X_t$ be the Brownian motion with drift $2b \in \mathbb{R}$ (that is, $\text{Var} X_t = 2t$, $\mathbb{E} X_t = 2bt$) and $D = (0, \infty)$. Prove that $F_{\lambda}(x) = e^{-bx} \sin(\lambda x) 1_{x>0}$ $(\lambda > 0)$ satisfies $A_D F_{\lambda}(x) = (\lambda^2 + b^2) F_{\lambda}$ and $P^D_t F_{\lambda} = e^{-t(\lambda^2+b^2)} F_{\lambda}$. Give eigenfunction expansion for $P^D_t$ on the weighted $L^2(D; e^{bx} dx)$ space. Find a (well-known) closed-form formula for $p^D_t(x, y)$. 
Part I

Section 5

Complete monotonicity, complete Bernstein functions and subordinate Brownian motions
Assumptions

Goal
For a class of Lévy processes in half-line $D = (0, \infty)$:

- find a formula for eigenfunctions of $A_D$ and $P_t^D$
- prove eigenfunctions expansion

- $X_t$ should be symmetric ( = isotropic) (otherwise, $A_D, P_t^D$ are not self-adjoint)
- some regularity is needed

Assumption (†)
- There is no drift
- The Lévy measure $\nu$ is:
  - symmetric
  - has completely monotone density on $(0, \infty)$
Complete monotonicity

**Definition**

\( g(y) \) is **completely monotone** if:

\[
(-1)^n g^{(n)}(y) \geq 0 \quad (n = 0, 1, 2, \ldots, \ y > 0)
\]

**Theorem (Sergei Natanovich Bernstein, 1929)**

Equivalently: \( g \) is the **Laplace transform** of a measure:

\[
g(y) = \mathcal{L}m(y) = \int_0^\infty e^{-sy} m(ds)
\]

**Proof**

- (\( \Leftarrow \)) direct differentiation (easy)
- (\( \Rightarrow \)) inversion of Laplace transform (hard)
Complete Bernstein functions

**Definition**

\( \Phi(\xi) \) is a **complete Bernstein function (CBF)** if:

- \( \Phi : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C} \setminus (-\infty, 0] \) is **holomorphic**
- \( \text{Im} \Phi(\xi) \geq 0 \) when \( \text{Im} \xi \geq 0 \)
- \( \text{Im} \Phi(\xi) \leq 0 \) when \( \text{Im} \xi \leq 0 \)

- Equivalent definitions and properties of CBFs will be discussed later
- CBF \( \equiv \) operator monotone function \( \cong \) Pick function

**Theorem**

Assumption (\( \triangleright \)) \( \iff \) \( \Psi(\xi) = \Phi(\xi^2) \) for a **CBF** \( \Phi(\xi) \)

- \( \Psi(\xi) \) is the Lévy-Khintchine exponent of \( X_t \)
- Proof will be given later in this part
Subordinators

Definition

A **subordinator** is a nonnegative Lévy process (starting at 0).

Definition

We define the **Laplace exponent**:

\[
\Phi(\xi) = b\xi + \int_0^\infty (1 - e^{-\xi y}) \nu(dy)
\]

**(b ≥ 0)**

\(\nu((−\infty, 0)) = 0\)

\(\int_0^\infty \min(1, y) \nu(dy) < \infty\)

Theorem (Lévy-Khintchine formula for subordinators)

\[
E_0 e^{-\xi X_t} = e^{-t\Phi(\xi)}
\]

\[
\mathcal{L}(P_t f)(\xi) = e^{-t\Phi(\xi)} \mathcal{L}f(\xi)
\]

\[
\mathcal{L}(Af)(\xi) = \Phi(\xi) \mathcal{L}f(\xi)
\]
Subordinators and CBFs (1)

Theorem

(of a subordinator)

Laplace exponent $\Phi(\xi)$ is a CBF

$\updownarrow$

(of a subordinator)

Lévy measure $\nu$ has completely monotone density

Proof

• Suppose that $\nu(y) = \mathcal{L}m(y)$ (that is, $\nu(dy) = (\mathcal{L}m(y))dy$)

• $\Phi(\xi) = b\xi + \int_0^\infty (1 - e^{-\xi y})\nu(dy)$

$$= b\xi + \int_0^\infty \int_0^\infty (1 - e^{-\xi y})e^{-sy}m(ds)dy$$

$$= b\xi + \int_0^\infty \frac{\xi}{s + \xi} \frac{m(ds)}{s}$$
Subordinators and CBFs (2)

Proof (cont.)

- \( \Phi(\xi) = b\xi + \int_0^\infty \frac{\xi}{s+\xi} \frac{m(ds)}{s} \) (more precisely: extends to a CBF)
- By checking \( \text{Im } \Phi(\xi) \), \( \Phi(\xi) \) is a CBF
- Reasoning can be reversed by the next result.

Theorem

\[ \Phi(\xi) \text{ is a CBF } \iff \Phi(\xi) = a + b\xi + \int_0^\infty \frac{\xi}{s+\xi} \frac{m(ds)}{s} \]  
\((a, b \geq 0)\)  
\((\int_0^\infty \min(1, y)m(dy) < \infty)\)

Proof

- \((\iff)\) direct calculation (easy)
- \((\implies)\) representation of positive harmonic functions by the Poisson kernel (harder)
Subordination

Definition

- If $Y_t$ is a stochastic process, $Z_t$ is a subordinator, and $Y_t, Z_t$ are independent processes, then $X_t = Y(Z_t)$ is a **subordinate process**
- If $Y_t$ is Brownian motion, then $X_t$ is **subordinate Brownian motion**

Theorem

Suppose that:
- $Y_t$ is Brownian motion ($\text{Var} Y_t = 2t, \psi_Y(\xi) = \xi^2$)
- $Z_t$ is a subordinator
- $\Phi_Z(\xi)$ is the Laplace exponent of $Z_t$

Then $\psi_X(\xi) = \Phi_Z(\xi^2)$ is the Lévy-Khintchine exp. of $X_t$

Proof

- Direct calculation: a nice exercise
Summary

Theorem (equivalent forms of Assumption (†))

- \( X_t \) has no drift
- The Lévy measure \( \nu \) of \( X_t \) is:
  - symmetric
  - has completely monotone density on \((0, \infty)\)
- \( \Psi(\xi) = \Phi(\xi^2) \) for a CBF \( \Phi(\xi) \)
- \( X_t = Y(Z_t) \) is a subordinate Brownian motion
- The Lévy measure \( \nu_Z \) of \( Z_t \) has completely monotone density on \((0, \infty)\)
# Examples

<table>
<thead>
<tr>
<th>Process:</th>
<th>Stable</th>
<th>Relativistic</th>
<th>Var. gamma</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Parameter:</strong></td>
<td>(\alpha \in (0, 2))</td>
<td>(m \in (0, \infty))</td>
<td></td>
</tr>
<tr>
<td>(\Psi(\xi))</td>
<td>(</td>
<td>\xi</td>
<td>^{\alpha})</td>
</tr>
<tr>
<td>(\nu_x(y))</td>
<td>(c_{\alpha} \frac{</td>
<td>y</td>
<td>^{1+\alpha}}{</td>
</tr>
<tr>
<td>(\Phi(\xi))</td>
<td>(\xi^{\alpha/2})</td>
<td>(\sqrt{\xi + m^2 - m})</td>
<td>(\log(\xi + 1))</td>
</tr>
<tr>
<td>(\nu_Z(s))</td>
<td>(c_{\alpha} \frac{s^{1+\alpha/2}}{s^{1+\alpha/2}})</td>
<td>(\frac{e^{-ms}}{2\sqrt{\pi} s^{3/2}})</td>
<td>(e^{-s})</td>
</tr>
</tbody>
</table>

\(K_1\) is a Bessel function.
Proof ((3) ⇒ (1))

- Suppose that \( X_t = Y(Z_t) \), \( Y_t \) is BM, \( Z_t \) is a subordinator and \( \nu_Z(s) = \mathcal{L}m(s) \) \((X_t \text{ symmetric } ⇒ \text{no drift})\)

- \( p_{X,t}(y) = \int_0^\infty p_{Y,s}(y)p_{Z,t}(s)ds \) (subordination formula)

- \( \nu_X(y) = \lim_{t \to 0^+} \frac{p_{X,t}(y)}{t} = \int_0^\infty p_{Y,s}(y)\nu_Z(s)ds \)

- By \( p_{Y,s}(y) = \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{y^2}{4s}\right) \), \( \nu_Z(s) = \int_0^\infty \exp(-st)m(dt) \):

\[
\nu_X(y) = \int_0^\infty \left( \int_0^\infty \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{y^2}{4s}\right)e^{-st}ds \right) m(dt)
\]

\[
= \int_0^\infty \frac{1}{2\sqrt{t}} e^{-\sqrt{t}|y|} m(dt) = \mathcal{L}\tilde{m}(|y|)
\]
Proof ((1) $\Rightarrow$ (3))

- Reverse the reasoning

Proof ((2) $\iff$ (3))

- $\Psi_X(\xi) = \Phi_Z(\xi^2) \iff X_t = Y(Z_t)$
  (a theorem above)

- $\Phi_Z(\xi)$ is a CBF $\iff \nu_Z(s)$ is completely monotone
  (another theorem above)

Rene Schilling, Renming Song, Zoran Vonraček

*Bernstein Functions: Theory and Applications*

De Gruyter, 2010
Problems:

(1) Let $\Psi_Y(\xi)$ be the Lévy-Khintchine exponent of a Lévy process $Y_t$, and $\Phi_Z(\xi)$ be the Laplace exponent of a subordinator $Z_t$. Suppose that $Y_t$ and $Z_t$ are independent processes. Show that the Lévy-Khintchine exponent of $X_t = Y(Z_t)$ is $\Psi_X(\xi) = \Phi_Z(\Psi_Y(\xi))$.

(Note: Re $\Psi_Y(\xi) \geq 0$ and $\Phi_Z(\xi)$ is well-defined if Re $\xi \geq 0$)

(2) Prove that $\Phi(\xi)$ is a Laplace exponent (a.k.a. **Bernstein function**) if and only if $\Phi(0) \geq 0$ and $\Phi'(\xi)$ is completely monotone.

(3) Suppose that $\Phi(\xi)$, $\Phi_1(\xi)$, $\Phi_2(\xi)$ are non-zero CBFs and $c > 0$, $0 < \alpha < 1$. Prove that:

(a) $c\Phi(\xi)$, $\Phi_1(\xi) + \Phi_2(\xi)$, $\Phi_1(\Phi_2(\xi))$, $\frac{\xi}{\Phi(\xi)}$, $(\Phi_1(\xi))^{\alpha}(\Phi_2(\xi))^{1-\alpha}$ are CBFs;

(b) $\xi^{1-\alpha}\Phi(\xi^\alpha)$ is a CBF;

(Hint: use only ‘$\Phi(\xi)$ is CBF $\Rightarrow$ $\Phi(\xi^\alpha)$ is CBF’ and ‘$\Phi(\xi)$ is CBF $\Rightarrow$ $\xi/\Phi(\xi)$ is CBF’)

(c) $(\Phi(\xi^\alpha))^{1/\alpha}$ is a CBF;

(d) $\Phi$ maps \{\xi \in \mathbb{C} : \text{Arg } \xi \in (0, \alpha\pi)\} into itself;

(e) $(\Phi_1(\xi^\alpha) + \Phi_2(\xi^\alpha))^{1/\alpha}$, $((\Phi_1(\xi))^{\alpha} + (\Phi_2(\xi))^\alpha)^{1/\alpha}$, $\Phi_1(\xi^\alpha)\Phi_2(\xi^{1-\alpha})$ are CBFs.
Part II

Eigenfunction expansion in half-line

- Formula for eigenfunctions
- Properties of eigenfunctions
- Eigenfunction expansion

Note: There are a lot of ugly formulae in this part!
Part II
Section 1

Formula for eigenfunctions
### Setting

#### Assumptions

Throughout this part we assume that:

- $X_t$ is a (symmetric) Lévy process in $\mathbb{R}$
- $\psi(\xi)$ is the Lévy-Khintchine exponent of $X_t$
- Assumption ($\dagger$):
  \[ \psi(\xi) = \Phi(\xi^2) \text{ for a CBF } \Phi(\xi) \]
- $P_t$ are free transition operators of $X_t$
- $A$ is the generator of $P_t$
- $D = (0, \infty)$
- $P^D_t$ are transition operators of $X_t$ on $D$
- $A_D$ is the generator of $P^D_t$
Eigenfunctions: intuition

- For \( F(x) = \sin(\lambda x + \theta) \):
  \[
  \mathcal{A}F = \psi(\lambda)F, \quad P_tF = e^{-t\psi(\lambda)}F
  \]
  (Lévy-Khintchine formula, \( \psi(\lambda) = \psi(-\lambda) \))

- For \( g(x) = f(x)1_{x>0} \) and \( x \in D \) large:
  \[
  \mathcal{A}_D g(x) = \mathcal{A}g(x) \approx \mathcal{A}f(x)
  \]

**Guess**

For each \( \lambda > 0 \) there is \( F_\lambda(x) \) such that:

\[
\mathcal{A}_D F_\lambda = \psi(\lambda)F_\lambda, \quad P^D_t F_\lambda = e^{-t\psi(\lambda)}F_\lambda
\]

and \( F_\lambda(x) \approx \sin(\lambda x + \theta_\lambda) \) as \( x \to \infty \). That is:

\[
F_\lambda(x) = \sin(\lambda x + \theta_\lambda)1_{x>0} - G_\lambda(x)
\]

where \( G_\lambda \) is small.

- In this part, always \( x > 0 \)
- For simplicity, we drop \( 1_{x>0} \) from the notation
Theorem [K, 2010]

For each \( \lambda > 0 \) there are:

- \( \theta_\lambda \in [0, \pi/2) \)
- completely monotone \( G_\lambda(x) \)

such that \( F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - G_\lambda(x) \) satisfies:

\[
\mathcal{A}_D F_\lambda = \psi(\lambda) F_\lambda, \quad P_t^D F_\lambda = e^{-t\psi(\lambda)} F_\lambda
\]
Eigenfunctions: formula (2)

- \( F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - G_\lambda(x) \)
- \( G_\lambda \) is completely monotone: \( G_\lambda(x) = \mathcal{L} \gamma_\lambda(x) \)

**Theorem [K, 2010]**

\[
\theta_\lambda = \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 - u^2} \log \frac{2\lambda(\psi(\lambda) - \psi(u))}{\psi'(\lambda)(\lambda^2 - u^2)} \, du
\]

\[
\gamma_\lambda(ds) = \frac{1}{2\pi} \left( \text{Im} \frac{\psi'(\lambda)}{\psi(\lambda) - \Phi^+(-s^2)} \right)
\]
\[
\times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + u^2} \log \frac{2\lambda(\psi(\lambda) - \psi(u))}{\psi'(\lambda)(\lambda^2 - u^2)} \, du \right) \, ds
\]

- \( \gamma_\lambda \) may fail to have density!
- \( \Phi^+(-s^2) = \lim_{\varepsilon \to 0^+} \Phi(-s^2 + \varepsilon i) \) in the distributional sense
Example

(symmetric)

For the $\alpha$-stable process, $\psi(\xi) = |\xi|^\alpha$, $\alpha \in (0, 2)$:

$$F_\lambda(x) = \sin \left( \lambda x + \frac{(2 - \alpha)\pi}{8} \right) - \int_0^\infty \gamma(s)e^{-\lambda sx} \, ds$$

$$\gamma(s) = \frac{\sqrt{2\alpha} \sin(\alpha \pi/2)}{2\pi} \frac{s^\alpha}{1 + s^{2\alpha} - 2s^\alpha \cos(\alpha \pi/2)}$$

$$\times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{1}{1 + u^2} \log \frac{1 - s^2u^2}{1 - s^\alpha u^\alpha} \, du \right)$$

- Scaling: $F_\lambda(x) = F_1(\lambda x)$
- $\frac{1}{\sqrt{\alpha/2} \Gamma(\alpha/2)} (\lambda x)^{\alpha/2}$ as $x \to 0^+$
Eigenfunctions: relativistic processes

Example

For the relativistic process, \( \psi(\xi) = \sqrt{\xi^2 + m^2} - m \):

- \( \theta_\lambda \) increases from 0 to \( \pi/8 \)
- \( F_\lambda(x) \sim \sqrt{\frac{2\lambda x}{\pi}} \) as \( x \to 0^+ \)
### Example

For the **variance gamma process**, $\psi(\xi) = \log(\xi^2 + 1)$:

- $\theta_\lambda$ increases from 0 to $\pi/4$
- $F_\lambda(x) \sim \frac{\lambda}{\sqrt{2}} \frac{1}{\sqrt{\lambda^2 + 1}} \frac{1}{\sqrt{|\log x|}}$ as $x \to 0^+$

### Example

For the **mixture of stables**, $\psi(\xi) = \xi^\alpha + \xi^\beta$, (sum of two independent stables, $0 < \alpha \leq \beta \leq 2$):

- $\theta_\lambda$ decreases from $\frac{(2 - \alpha)\pi}{8}$ to $\frac{(2 - \beta)\pi}{8}$
- $F_\lambda(x) \sim \frac{1}{\sqrt{\beta/2}} \frac{1}{\Gamma(\beta/2)} (\lambda x)^{\beta/2}$ as $x \to 0^+$
Eigenfunctions: yet another two examples

Example

For the Brownian motion, $\psi(\xi) = \xi^2$:

- $2\lambda(\psi(\lambda) - \psi(u))$
- $\log \frac{\psi'(\lambda)(\lambda^2 - u^2)}{\psi(\lambda)} = 0$
- $\theta_\lambda = 0$, $\gamma_\lambda = 0$ and $F_\lambda(x) = \sin(\lambda x)$, as expected

Example

For $\psi(\xi) = \frac{\xi}{1 + \xi}$, $\nu(y) = \frac{e^{-|y|}}{2}$ (compound Poisson with Laplace distributed jumps):

- $\theta_\lambda = \arctan \lambda$ increases from 0 to $\pi/2$
- $\gamma_\lambda$ vanishes!
- $F_\lambda(x) = \sin(\lambda x + \arctan \lambda) 1_{x>0}$
- $F_\lambda$ is discontinuous at 0!
Part II
Section 2

Properties of eigenfunctions
Laplace transform of eigenfunctions

- Derivation of the formula for $F_\lambda$ will be sketched in Part IV
- **Bounds** and **asymptotics** of $F_\lambda$ can be proved in a fairly general setting
- Most of them follow from the formula for $\mathcal{L}F_\lambda$
- In most applications, exact formula is not needed

**Theorem [K, 2010]**

$$\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2 + s^2} \times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + u^2} \log \frac{\psi'(\lambda)(\lambda^2 - u^2)}{2\lambda(\psi(\lambda) - \psi(u))} \, du \right)$$
Common elements (the worst slide ever!)

\[
\theta_\lambda = \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 - u^2} \log \frac{2\lambda(\psi(\lambda) - \psi(u))}{\psi'(\lambda)(\lambda^2 - u^2)} \, du
\]

\[
\gamma_\lambda(ds) = \frac{1}{2\pi} \left( \text{Im} \frac{\psi'(\lambda)}{\psi(\lambda) - \phi^+(-s^2)} \right)
\times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + u^2} \log \frac{2\lambda(\psi(\lambda) - \psi(u))}{\psi'(\lambda)(\lambda^2 - u^2)} \, du \right) \, ds
\]

\[
\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2 + s^2} \exp \left( \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + u^2} \log \frac{\psi'(\lambda)(\lambda^2 - u^2)}{2\lambda(\psi(\lambda) - \psi(u))} \, du \right)
\]

**Definition**

\[
\Phi_\lambda^+(\xi) = \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + u^2} \log \frac{\psi'(\lambda)(\lambda^2 - u^2)}{2\lambda(\psi(\lambda) - \psi(u))} \, du \right)
\]
Simplification

- \( \psi(\xi) = \phi(\xi^2) \)
- We will now use mostly \( \phi(\xi) \), not \( \psi(\xi) \)

**Definition**

- \( \phi_\lambda(\xi^2) = \frac{\phi'(\lambda^2)(\lambda^2 - \xi^2)}{\phi(\lambda^2) - \phi(\xi^2)} = \frac{\psi'(\lambda)(\lambda^2 - \xi^2)}{2\lambda(\psi(\lambda) - \psi(\xi))} \)
- \( \phi^\dagger(\xi) = \exp\left( \frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + u^2} \log \phi(u^2) du \right) \)
- \( \phi^\dagger_\lambda = (\phi_\lambda)^\dagger \)
- \( \text{Arg} \phi^\dagger(i\xi) = -\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 - u^2} \log \phi(u^2) du \)
### Theorem

We have:

\[ F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - \mathcal{L}\gamma_\lambda(x) \]

with:

\[ \theta_\lambda = \text{Arg}(\Phi^\dagger_\lambda(i\lambda)) \]

\[ \gamma_\lambda(ds) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + s^2} \frac{\text{Im}(\Phi_\lambda)^{-}(s)}{\Phi^\dagger_\lambda(s)} \, ds \]

\[ \mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2 + s^2} \Phi^\dagger_\lambda(s) \]

- Technical details are now moved to definitions
Divide and rule

**Strategy** [K, 2010], [K-Małecki-Ryznar, 2011]

1. study $\Phi_\lambda$
2. study $\Phi^\dagger$
3. use (1) and (2) to $\theta_\lambda$
4. apply (1) and (2) to $\mathcal{L}F_\lambda$
5. use tauberian theory (Jovan Karamata et al.)
   (properties of $\mathcal{L}F_\lambda \Rightarrow$ properties of $F_\lambda$)

- Alternatively, for a specific $\Psi$, one may try:
  4’. apply (1) and (2) to $\gamma_\lambda$
  5’. use abelian theory
   (properties of $\gamma_\lambda \Rightarrow$ properties of $\mathcal{L}\gamma_\lambda$)
Properties of $\Phi_\lambda$

$$\Phi_\lambda(\xi) = \frac{\Phi'(\lambda^2)(\lambda^2 - \xi)}{\Phi(\lambda^2) - \Phi(\xi)}$$

Lemma

$\Phi(\xi)$ is a CBF $\Rightarrow \Phi_\lambda(\xi)$ is a CBF

- Proof: nice exercise
- estimates of $\Phi_\lambda$ depend on bounds on $\frac{-\xi \Phi''(\xi)}{\Phi'(\xi)}$
- $\Phi_\lambda(\xi^2) \sim \Phi'(\lambda^2) \frac{\xi^2}{\Phi(\xi^2)}$ as $\xi \to \infty$
- $\Phi_\lambda(\xi^2) \to \Phi'(\lambda^2) \frac{\lambda^2}{\Phi(\lambda^2)}$ as $\xi \to 0^+$
Properties of $\Phi^\dagger (1)$

$$\Phi^\dagger(\xi) = \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + u^2} \log \Phi(u^2) \, du \right)$$

Lemma [K, 2010], [Kim-Song-Vondraček, 2010], [Rogers, 1983]
(and discoverers of the Wiener-Hopf factorization for Lévy processes)

- $\Phi(\xi)$ is a CBF $\Rightarrow$ $\Phi^\dagger(\xi)$ is a CBF
- $\Phi^\dagger(\xi)\Phi^\dagger(-\xi) = \Phi(-\xi^2)$

- Proof: smart contour integration
- A fundamental lemma!
- It enables inversion of the Laplace transform in

$$\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2 + s^2} \Phi^\dagger_\lambda(s)$$

- Residues at $\pm i\lambda \mapsto \sin(\lambda x + \theta_\lambda)$
- Jump along $(-\infty, 0] \mapsto \mathcal{L}\gamma_\lambda(x)$
Properties of $\Phi^\dagger$ (2)

$$\Phi^\dagger(\xi) = \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + u^2} \log \Phi(u^2) du \right)$$

Bounds [K-Małecki-Ryznar, 2011], [Kim-Song-Vondraček, 2010]

$$\frac{1}{2} \sqrt{\Phi(\xi^2)} \leq \Phi^\dagger(\xi) \leq 2 \sqrt{\Phi(\xi^2)}$$

Proof

- $\min \left( 1, \frac{u^2}{\xi^2} \right) \leq \frac{\Phi(u^2)}{\Phi(\xi^2)} \leq \max \left( 1, \frac{u^2}{\xi^2} \right)$
- $\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + u^2} \log \left( \max \left( 1, \frac{u^2}{\xi^2} \right) \right) du \leq \log 2$
- Similar estimate for the lower bound
Properties of $\Phi^\dagger$ (3)

**Definition**

$\Phi(\xi)$ is **regularly varying** of order $\alpha$ (\textit{\textbf{\alpha-RV}}) at $0^+$ if

\[
\lim_{\xi \to 0^+} \frac{\Phi(c\xi)}{\Phi(\xi)} = c^\alpha \quad \text{for all } c > 0
\]

$\Phi(\xi)$ is **regularly varying** of order $\alpha$ (\textit{\textbf{\alpha-RV}}) at $\infty$ if

\[
\lim_{\xi \to \infty} \frac{\Phi(c\xi)}{\Phi(\xi)} = c^\alpha \quad \text{for all } c > 0
\]

**Asymptotics** [K-Małecki-Ryznar, 2011], [Kim-Song-Vondraček, 2010]

- $\Phi^\dagger(\xi) \sim \sqrt{\Phi(\xi^2)}$ as $\xi \to \infty$ if $\Phi(\xi)$ is RV at $\infty$
- $\Phi^\dagger(\xi) \sim \sqrt{\Phi(\xi^2)}$ as $\xi \to 0^+$ if $\Phi(\xi)$ is RV at $0^+$

- Proof: explicit estimates
Properties of $\theta_\lambda$

$$\theta_\lambda = \text{Arg} \Phi^\dagger_\lambda(i\lambda)$$

- $\theta_\lambda \in [0, \pi/2)$ \quad ($\Phi^\dagger_\lambda(\xi) \not\equiv c \xi$, so $\theta_\lambda \not\equiv \pi/2$)
- $\theta_\lambda$ close to $\pi/2$ generates problems
- $\theta_\lambda \leq \arctan \sqrt{\frac{\Phi(\lambda^2)}{\lambda^2 \Phi'(\lambda^2)}} - 1$
  
  (good for $\Phi(\xi)$ with power-type growth)

**Bounds** [K-Małecki-Ryznar, 2011]

$$\left( \inf_{\xi > 0} \frac{-\xi \Phi''(\xi)}{\Phi'(\xi)} \right) \frac{\pi}{4} \leq \theta_\lambda \leq \left( \sup_{\xi > 0} \frac{-\xi \Phi''(\xi)}{\Phi'(\xi)} \right) \frac{\pi}{4}$$

- Proof: bounds for $\Phi_\lambda$, explicit formula for $\Phi(\xi) = \xi^{\alpha/2}$
Properties of $F_{\lambda}$ (1)

$$\mathcal{L}F_{\lambda}(s) = \frac{\lambda}{\lambda^2 + s^2} \Phi_{\lambda}^+(s)$$

$$F_{\lambda}(x) = \sin(\lambda x + \theta_{\lambda}) - \mathcal{L}\gamma_{\lambda}(x)$$

**Bounds** [K-Małęcki-Ryznar, 2011]

When $\lambda x \leq \frac{1}{2}(\frac{\pi}{2} - \theta_{\lambda})$, then:

$$\frac{1}{5} \lambda x \sqrt{\Phi_{\lambda}(1/x^2)} \leq F_{\lambda}(x) \leq 30(\frac{\pi}{2} - \theta_{\lambda}) \lambda x \sqrt{\Phi_{\lambda}(1/x^2)}$$

- Proof: concavity of $F_{\lambda}(x)$ for small $x$, comparison of Laplace transforms
- Kind of uniform continuity of $F_{\lambda}(x/\lambda)$
Properties of \( F_\lambda \) (2)

\[
\mathcal{L}F_\lambda (s) = \frac{\lambda}{\lambda^2 + s^2} \Phi_\lambda^\dagger(s)
\]

\[
F_\lambda (x) = \sin(\lambda x + \theta_\lambda) - \mathcal{L} \gamma_\lambda (x)
\]

Asymptotics [K, 2010], [K-Małecki-Ryznar, 2011]

- \( F_\lambda (x) \sim \sqrt{\lambda^2 \Phi'(\lambda^2)} \frac{1}{\Gamma(1 + \alpha)} \frac{1}{\sqrt{\Phi(1/x^2)}} \) as \( x \to 0^+ \)

\[
\text{if } \Phi(\xi) \text{ is } \alpha\text{-RV at } \infty
\]

- \( F_\lambda (x) \sim V(x) \sqrt{\lambda^2 \Phi'(\lambda^2)} \) as \( \lambda \to 0^+ \)

\[
\text{if } \limsup_{\lambda \to 0^+} \theta_\lambda < \pi/2
\]

- Proof: technical, nothing interesting

- \( V(x) \) comes from fluctuation theory, \( \mathcal{L}V(\xi) = \frac{1}{\xi \Phi^\dagger(\xi)} \)
Part II
Section 3
Eigenfunction expansion
Some delicate problems with integrability arise when $e^{-t\psi(\lambda)}$ is not integrable.

Solution: use continuous $L^2(D)$ extension.
Eigenfunction expansion (2)

**Definition**

\[ \Pi f(\lambda) = \int_0^\infty f(x)F_\lambda(x)dx = \langle f, F_\lambda \rangle \]

\[ \Pi^* g(x) = \int_0^\infty g(\lambda)F_\lambda(x)d\lambda \]

**Theorem [K, 2010], [K-Małecki-Ryznar, 2011]**

- \( \sqrt{\frac{2}{\pi}} \Pi, \sqrt{\frac{2}{\pi}} \Pi^* \) extend to **unitary** operators on \( L^2(D) \)
- \( f \in \text{Dom}_{L^2(D)}(A_D) \iff \psi(\lambda)\Pi f(\lambda) \in L^2(D) \)
- \( \Pi(A_D f)(\lambda) = \psi(\lambda)\Pi f(\lambda) \) for \( f \in \text{Dom}_{L^2(D)}(A_D) \)
- \( \Pi(P^D_t f)(\lambda) = e^{-t\psi(\lambda)}\Pi f(\lambda) \) for \( f \in L^2(D) \)
Eigenfunction expansion (3)

**Corollary**

If \( f \in L^1(D) \) and \( e^{-t\psi(\lambda)} \) is integrable, then:

\[
P^D_t f(x) = \frac{2}{\pi} \Pi^* (\psi \cdot (\Pi f))(x) \\
= \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda)} \langle f, F_\lambda \rangle F_\lambda(x) d\lambda
\]

If \( e^{-t\psi(\lambda)} \) is integrable, then:

\[
\rho^D_t(x, y) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda)} F_\lambda(x)F_\lambda(y) d\lambda
\]

- Proofs will be sketched in Part IV
- Most difficult part: completeness of \( F_\lambda \)
Problems:

(1) Show that if \( \Phi(\xi) \) is a CBF, then \( \Phi_\lambda(\xi) \) is a CBF.

(2) Prove, by a direct calculation, that for \( \Psi(\xi^2 + 1) \) (that is, \( \nu(y) = e^{-|y|/2} \)), \( F_\xi(x) = \sin(\xi x + \arctan \xi) \mathbf{1}_{\xi > 0} \) is an eigenfunction in \((0, \infty)\).

(3) Prove that \( P^D_t \) has no \( L^2(D) \) eigenfunctions when \( X_t \) is the symmetric \( \alpha \)-stable process, \( \Psi(\xi) = |\xi|^\alpha \).

   Note: this is true in the general case under Assumption (\( \ddagger \)), but the proof is much more difficult.

(4) Show that \( P^D_t \) may have \( L^2(D) \) eigenfunctions when \( X_t \) is not symmetric.

Open problems:

(1) Are there any other eigenfunctions \( F \) of \( P^D_t \)?

(2) Formula for \( \mathcal{L}F_\lambda \) makes sense for a much more general class of Lévy-Khintchine exponents \( \Psi(\xi) \). For which exponents does this formula indeed define the Laplace transform of a function?

(3) Is it true that \( p^D_t(x, y) = \lim_{\varepsilon \to 0^+} \frac{2}{\pi} \int_0^\infty e^{-\varepsilon \lambda - t\Psi(\lambda)} F_\lambda(x)F_\lambda(y) d\lambda \) when \( e^{-t\Psi(\lambda)} \) is not integrable?
Part III

Applications

- Supremum functional and first passage times
- Connections with fluctuation theory
- Eigenvalues for intervals
- Higher-dimensional domains
Part III

Section 1

Supremum functional and first passage times
Supremum functional and FPT (1)

- In this section we often write $P$ for $P_0$ (that is, $X_t$ starts at 0)

**Definition**

We define the the **first passage time (FPT)**:

$$\tau_x = \inf \{s \geq 0 : X_s \geq x\}$$

and the **supremum functional (or sup. process)**:

$$M_t = \sup_{s\in[0,t]} X_s$$

- Important in many areas of applied probability
- Distribution is rather difficult to compute
Supremum functional and FPT (2)

Proposition

\( P(M_t < x) = P(\tau_x > t) \)

Proof

- \( \{M_t < x\} \) is almost equal to \( \{\tau_x > t\} \)
- Use càdlàg paths and quasi left continuity

- Let \( D = (0, \infty) \)
- When \( X_t \) is symmetric, then \( P(\tau_x > t) = P_x(\tau_D > t) \)
- (In the general case, \( P(\tau_x > t) = P_{-x}(\tau_{(-\infty,0)} > t) \))

- \( P_x(\tau_D > t) = \int_0^{\infty} p_t^D(x, y)dy \)
- We have a formula for \( p_t^D(x, y) \)
Theorem [K, 2010], [K-Małecki-Ryznar, 2011]

Under Assumption (\(\star\)), and if:

- \(\sup_{\lambda>0} \theta_\lambda < \frac{\pi}{2}\)
- \(\sqrt{\frac{\psi'(\lambda)}{2\lambda \psi(\lambda)}} \ e^{-t\psi(\lambda)}\) is integrable at \(\infty\) (note that \(\frac{\psi'(\lambda)}{2\lambda \psi(\lambda)} = \frac{\phi'(\lambda^2)}{\phi(\lambda^2)} \leq \frac{1}{\lambda^2}\))

we have:

\[
P(\tau_x > t) = 2 \frac{\pi}{\int_0^\infty \sqrt{\frac{\psi'(\lambda)}{2\lambda \psi(\lambda)}} \ e^{-t\psi(\lambda)} F_\lambda(x)d\lambda}
\]

- Integrability near 0 is automatic!
Examples

- Assumptions are relatively mild, examples include:
  - Symmetric stable processes
  - Relativistic processes
  - Variance gamma process
  - Mixtures of stables
  - $\psi(\xi) = \log(\log(\xi^2 + 1) + 1)$ when $t \geq 1/2$

- Problems when $\psi(\xi)$ grows very slowly, for example, for compound Poisson processes

- Formula is applicable for numerical computations (although there are essential problems with numerical stability)
Plots of $P(\tau_x > t)$ for $x = 0.5, 1, 1.5$ and $2$
Formula for FPT (2)

Proof no. 1

- \( P(\tau_x > t) = \int_0^\infty p_t^D(x, y) dy = \lim_{\varepsilon \to 0^+} \int_0^\infty e^{-\varepsilon y} p_t^D(x, y) dy \)

- \( p_t^D(x, y) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda)} F_\lambda(x) F_\lambda(y) d\lambda \)

- \( \int_0^\infty e^{-\varepsilon y} F_\lambda(y) = \mathcal{L} F_\lambda(y) \)

- \( P(\tau_x > t) = \frac{2}{\pi} \lim_{\varepsilon \to 0^+} \int_0^\infty e^{-t\psi(\lambda)} F_\lambda(x) \mathcal{L} F_\lambda(\varepsilon) d\lambda \)

- \( \mathcal{L} F_\lambda(\varepsilon) \to \sqrt{\frac{\psi'(\lambda)}{2\lambda \psi(\lambda)}} \) as \( \varepsilon \to 0^+ \)

- Show uniform integrability as \( \varepsilon \to 0^+ \)

(very technical)
Formula for FPT (3)

Proof no. 2

\[ \int_0^\infty \int_0^\infty e^{-\xi x} e^{-zt} P(\tau_x > t) \, dx \, dt \]

is known

(Baxter-Donsker formula, discussed later)

\[ \int_0^\infty \int_0^\infty e^{-\xi x} e^{-zt} \left( \frac{2}{\pi} \int_0^\infty \frac{\psi'(\lambda)}{2\lambda \psi(\lambda)} e^{-t\psi(\lambda)} F_\lambda(x) \, d\lambda \right) \, dx \, dt \]

can be computed (slightly less technical)

Both turn out to be equal

Use uniqueness argument for Laplace transform
Properties of FPT (1)

Corollary

The formula can be differentiated under the integral when

\[ \sqrt{\frac{\psi'(\lambda)}{2\lambda \psi(\lambda)}} \ e^{-t\psi(\lambda)(\psi(\lambda))^n} \text{ is integrable at } \infty: \]

\[ (-1)^n \frac{d^n}{dt^n} \ P(\tau_x > t) \]

\[ = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{\psi'(\lambda)}{2\lambda \psi(\lambda)}} e^{-t\psi(\lambda)(\psi(\lambda))^n} F_\lambda(x) d\lambda \]

- For large \( t \), the integral over \([0, C/x]\) dominates
- We have good estimates for \( F_\lambda(x) \) when \( \lambda x \in [0, C] \)
Properties of FPT (2)

Corollary

\[ \tau_x \text{ has } \textbf{ultimately completely monotone} \text{ distribution:} \]

\[ (-1)^n \frac{d^n}{dt^n} P(\tau_x > t) > 0 \quad \text{for } t \text{ large enough} \]

\( (t > C(n, x, \Psi)) \)

Asymptotics

\[ (-1)^n \frac{d^n}{dt^n} P(\tau_x > t) \sim \frac{\Gamma(n + 1/2)}{\pi t^{n+1/2}} \frac{V(x)}{\left(1 + \alpha\right)} \]

\[ \text{as } t \to \infty \]

\[ (-1)^n \frac{d^n}{dt^n} P(\tau_x > t) \sim \frac{\Gamma(n + 1/2)}{\pi \Gamma(1 + \alpha)} \frac{1}{t^{n+1/2} \sqrt{\psi(1/x)}} \]

\[ \text{as } x \to 0^+, \text{ if } \psi(\xi) \text{ is } \alpha\text{-RV at } \infty \]

• Bounds with explicit constants are also available
• For \( n = 0 \), some of the above has been known before
• More information on \( V(x) \) in the next section
Part III
Section 2
Connections with fluctuation theory
Baxter-Donsker formula

**Theorem** [Baxter-Donsker, 1957]

When $X_t$ is a symmetric Lévy process:

$$
\int_0^\infty \int_0^\infty e^{-\xi x - z t} P(\tau_x > t) \, dx \, dt = \frac{1}{\xi \sqrt{z}} \frac{1}{(z + \Phi) \dagger(\xi)}
$$

$$
= \frac{1}{\xi \sqrt{z}} \exp \left( -\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + u^2} \log(z + \psi(u)) \, du \right)
$$

- There is a variant for asymmetric processes

**Glen Baxter, Monroe David Donsker, 1957**

*On the distribution of the supremum functional for processes with stationary independent increments*

*Trans. Amer. Math. Soc.* 85
Inversion of the Laplace transform (1)

- If $\Psi(\xi) = \Phi(\xi^2)$ for a CBF $\Phi(\xi)$ (Assumption (\#)), then our formula for $P(\tau_x > t)$ inverts the double Laplace transform in Baxter-Donsker formula.

- In general, partial inverse in space is known:
  $$\int_0^\infty e^{-zt} P(\tau_x > t)dt = \frac{V^z(x)}{\sqrt{Z}}$$

But $V^z(x)$ is not explicit:
  $$V^z(x) = E \left( \int_0^\infty e^{-zt} 1_{M_t < x} dL_t \right)$$

where $L_t$ is the local time of $M_t - X_t$ at 0
Inversion of the Laplace transform (2)

**Theorem [K-Małecki-Ryznar, 2011]**

If $X_t$ is a symmetric Lévy process and $\Psi(\xi)$ is increasing on $(0, \infty)$, then:

$$
\int_0^\infty e^{-\xi x} P(\tau_x > t) dx = \frac{1}{\pi} \int_0^\infty \frac{\xi}{\lambda^2 + \xi^2} \sqrt{\frac{\Psi'(\lambda)}{\Psi(\lambda)}} e^{-t\Psi(\lambda)}
$$

$$
\times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + u^2} \log \frac{\Psi'(\lambda)(\lambda^2 - u^2)}{2\lambda(\Psi(\lambda) - \Psi(u))} du \right) d\lambda
$$

- Proof: analytic continuation, contour integration and smart substitution, rather standard
- It remained undiscovered for more than 50 years!
- This theorem is used in the ‘Proof no. 2’ of the formula for $P(\tau_x > t)$
Increasing harmonic function (1)

- \( V^z(x) = E\left(\int_0^\infty e^{-zt} \mathbf{1}_{t<\tau_x} dL_t\right) \)
- Let \( V(x) = V^0(x) = E L(\tau_x) \)
- As usual, \( V(x) = 0 \) for \( x \leq 0 \)
- Then \( V(x) \) is **harmonic** in \((0, \infty)\):
  \[ \mathcal{A}V(x) = 0 \quad \text{for } x > 0 \]
- It is the unique increasing harmonic function
- It already appeared twice in the slides
Increasing harmonic function (2)

- Suppose that Assumption $(\ddagger)$ is satisfied
- $V(x) = \lim_{\lambda \to 0^+} \frac{F_\lambda(x)}{\lambda \sqrt{\psi(\lambda)}}$
- $V(x) = \lim_{t \to \infty} \left( \sqrt{\pi} t P(\tau_x > t) \right)$
- $\mathcal{L}V(\xi) = \frac{1}{\xi \Phi^+(\xi)}$ (this holds in greater generality)

Bounds [K-Małecki-Ryznar, 2011], [Kim-Song-Vondraček, 2010]

If $X_t$ is a symmetric Lévy process, and $\psi(\xi), \xi^2/\psi(\xi)$ are increasing, then: (more general than Assumption $(\ddagger)$)

$$\frac{2}{5} \frac{1}{\sqrt{\psi(1/x)}} \leq V(x) \leq 5 \frac{1}{\sqrt{\psi(1/x)}}$$
Bounds for FPT

Theorem [K-Małecki-Ryznar, 2011]

If $X_t$ is a symmetric Lévy process, and $\psi(\xi), \xi^2/\psi(\xi)$ are increasing, then:

$$\frac{1}{100} \min\left(1, \frac{1}{200\sqrt{t\psi(1/x)}}\right)$$

$$\leq P(\tau_x > t) \leq \min\left(1, \frac{10}{\sqrt{t\psi(1/x)}}\right)$$

- Theorem applies for all subordinate BM! That is, $\psi(\xi) = \Phi(\xi^2)$ satisfies the assumptions for any Laplace exponent $\Phi(\xi)$ (not just for CBFs)
- There is a (less explicit) version for asymmetric processes
- Note: $P(\tau_x \leq t) = 1 - P(\tau_x > t)$ is much easier
Increasing harmonic function (3)

Asymptotics [K-Małecki-Ryznar, 2011], [Kim-Song-Vondraček, 2010]

If $X_t$ is a symmetric Lévy process, and $\Psi(\xi)$, $\xi^2/\Psi(\xi)$ are increasing, then:

- $V(x) \sim \frac{1}{\Gamma(1+\alpha) \sqrt{V(1/x)}}$ as $x \to 0$
  
  if $\Psi(\xi)$ is $\alpha$-RV at $\infty$

- $V(x) \sim \frac{1}{\Gamma(1+\alpha) \sqrt{V(1/x)}}$ as $x \to \infty$
  
  if $\Psi(\xi)$ is $\alpha$-RV at $0^+$

- Some special cases have been known before
- Under Assumption ($\heartsuit$):
  - $V(x)$ is a Bernstein function
  - explicit formula for $V(x)$ can be given
- One can obtain similar results for $V^z(x)$
- $V(x)$ is much simpler than $F_\lambda(x)$ and $\mathbb{P}(\tau_x < t)$
Part III
Section 3

Eigenvalues for intervals
Interval: idea

- Let $D = (a, b)$
- As for the BM, there are $f_n$ and $\mu_n$ such that
  \[ P^D_t f_n = e^{-\mu_n t} f_n, \quad \mathcal{A}_D f_n = \mu_n f_n \]
- $f_n$ form a complete orthogonal set in $L^2(D)$

**Guess**

We should have:
\[
\begin{align*}
  f_n(x) &\approx c_1 F_{\lambda_n}(x - a) & \text{for } x \approx a \\
  f_n(x) &\approx c_2 F_{\lambda_n}(b - x) & \text{for } x \approx b
\end{align*}
\]

for $\lambda_n$ such that:
\[ \psi(\lambda_n) \approx \mu_n \]
Interval: sketch of the proof

- Define ‘approximate eigenfunction’ $\tilde{f}_n$ so that:
  \[
  \tilde{f}_n(x) = F_{\lambda_n}(x - a) \quad \text{for } x \in (a, \frac{2}{3}a + \frac{1}{3}b)
  \]
  \[
  f_n(x) = (-1)^{n-1}F_{\lambda_n}(b - x) \quad \text{for } x \in (\frac{1}{3}a + \frac{2}{3}b, b)
  \]
  and $\tilde{f}_n$ changes ‘smoothly’ on $(\frac{2}{3}a + \frac{1}{3}b, \frac{1}{3}a + \frac{2}{3}b)$

- This is possible only when:
  \[
  \lambda_n = \frac{n\pi}{2} - \theta_{\lambda_n}
  \]
  (then sine parts of $F_{\lambda_n}(x - a)$ and $F_{\lambda_n}(b - x)$ coincide)

- Show that $A_D\tilde{f}_n \approx \psi(\lambda_n)\tilde{f}_n$

- Deduce that $\tilde{\mu}_n \approx \psi(\lambda_n)$
Interval: construction of the approximations

$\tilde{f}_n(x)$ (solid line), $F_{\lambda_n}(x-a)$ (dashed line) and $\pm F_{\lambda_n}(b-x)$ (dotted line) for $\Psi(\xi) = |\xi|^{1/10}$, $D = (a, b) = (-1, 1)$, $n = 1, 2, 3, 4$. 
Interval: results

Theorem [Kulczycki-K-Małecki-Stós, 2010]
For the symmetric 1-stable process, \( \psi(\xi) = |\xi| \):
\[
| (b - a)\mu_n - \left( n\pi - \frac{\pi}{4} \right) | < \frac{2}{n}
\]

Theorem [K, 2010]
For the symmetric \( \alpha \)-stable process, \( \psi(\xi) = |\xi|^\alpha \):
\[
(b - a)^\alpha \mu_n = \left( n\pi - \frac{2 - \alpha}{4}\pi \right)^\alpha + O\left( \frac{1}{n} \right)
\]

Theorem [Kaleta-K-Małecki]
For the relativistic process, \( \psi(\xi) = \sqrt{\xi^2 + m^2} - m \):
\[
(b - a)\mu_n = \left( n\pi - \frac{\pi}{4} \right) + O\left( \frac{1}{n} \right)
\]
Part III
Section 4

Higher-dimensional domains
Multidimensional domains: introduction

- Let $X_t$ be the isotropic $\alpha$-stable process in $\mathbb{R}^d$
- Let $D \subseteq \mathbb{R}^d$ be a bounded domain
- There are $f_n$ and $\mu_n$ such that
  \[ P_t^D f_n = e^{-\mu_n t} f_n, \quad A_D f_n = \mu_n f_n \]
- $f_n$ form a complete orthogonal set in $L^2(D)$
- $N(\lambda) = \# \{ n : \mu_n \leq \lambda \}$ is the partition function

Theorem (Robert M. Blumenthal, Ronald K. Getoor, 1959)

For $C_1 = \frac{1}{2^d \pi^{d/2} \Gamma(d/2 + 1)}$:

\[ \frac{N(\lambda)}{\lambda^{d/\alpha}} = C_1 |D| + o(1) \quad \text{as} \ \lambda \rightarrow \infty \]
Multidimensional domains: second term

Theorem (Rodrigo Bañuelos, Tadeusz Kulczycki, 2008)

(Abel means) As $t \to 0^+$:

$$
t \mathcal{L} N(t) \frac{\Gamma(\frac{d}{\alpha} + 1)t^{d/\alpha}}{C_1|D| - C_2^{(1)}|\partial D|t^{1/\alpha} + o(t^{1/\alpha})}
$$

• $C_2^{(1)}$ given only implicitly

Theorem (Rupert L. Frank, Leander Geisinger, 2011)

(Cesaro means) As $\lambda \to \infty$:

$$
\frac{\int_0^\lambda N(u)du}{(\frac{d}{\alpha} + 1)\lambda^{(d+1)/\alpha}} = C_1|D| - C_2^{(2)}|\partial D|\lambda^{-1/\alpha} + o(t^{1/\alpha})
$$

• $C_2^{(2)}$ given explicitly in terms of the eigenfunctions $F_\lambda(x)$ for $\psi(\xi) = (\xi^2 + 1)^{\alpha/2} - 1$!
Conjecture

As $\lambda \to \infty$:

$$\frac{N(\lambda)}{\lambda^{d/\alpha}} = C_1 |D| - C_2^{(3)} |\partial D| t^{1/\alpha} + o(t^{1/\alpha})$$

Or even:

$$\frac{\mu_n}{\lambda^{d/\alpha}} = C_1 |D| - C_2^{(4)} |\partial D| t^{1/\alpha} + o(t^{1/\alpha})$$

- This conjecture seems to be extremely difficult
- Constants $C_2^{(n)} (n = 1, 2, 3, 4)$ are related to each other through simple formulae
Problems:

(1) Using the strong Markov property, prove that \((X(\tau_x + t) - X(\tau_x))\) is independent from the \(\sigma\)-algebra \(\mathcal{F}_{\tau_x}\) and has the same law as the process \(X_t\).

(2) Prove the reflection principle: if \(X_t\) is a symmetric Lévy process and \(P(X_t = 0) = 0\) for all \(t > 0\), then:
\[
P(M_t \geq x) = P(\tau_x \leq t) = 2P(\tau_x \leq t, X_t \geq X_{\tau_x})
\]
Prove similar inequalities when \(P(X_t = 0) > 0\) for \(t > 0\).

(3) Show that for the Brownian motion:
\[
P(M_t \geq x) = 2P(X_t \geq x)
\]

(4) Show the Lévy inequality: for symmetric Lévy processes \(X_t\):
\[
P(X_t \geq x) \leq P(M_t \geq x) \leq 2P(X_t \geq x)
\]

(5) Prove that if \(X_t\) is a symmetric Lévy process and \(e^{-t\psi(\xi)}\) is integrable in \(\xi \in \mathbb{R}\), then:
\[
P(X_t \geq x) = \frac{1}{\pi} \int_0^\infty \frac{\sin(\xi)}{\xi} (1 - e^{-t\psi(\xi/x)}) d\xi
\]
Part IV

Some technical details

- Wiener-Hopf method
- Heuristic derivation of the formula for eigenfunctions

This part will be available soon
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