Discrete Hilbert transforms on ℓ^p

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Joint work with **Rodrigo Bañuelos** (Purdue University)

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Hilbert transforms

Definition

The continuous Hilbert transform is defined by

$$Hf(x) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{f(x-z)}{z}\,dz$$

for appropriate functions $f : \mathbb{R} \to \mathbb{R}$.

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Similarly, the discrete Hilbert transform is given by

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for appropriate doubly infinite sequences $(a_n : n \in \mathbb{Z})$.

Main result #1

Theorem (Rodrigo Bañuelos, MK)

For $p \in (1, \infty)$ we have

$$\|\mathcal{H}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}.$$

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- One direction is easy: by approximation and Fatou's lemma,

$$\|\mathcal{H}\|_{\ell^p \to \ell^p} \geqslant \|H\|_{L^p \to L^p}.$$

Hilbert transforms revisited

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Theorem (Rodrigo Bañuelos, MK)

For $p = 2, 4, 6, 8, \ldots$ (or a conjugate exponent) we have

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- The other inequality remains a challenging open problem for general p.

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- They forgot to mention that some considered the problem to be rather hard.

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- The Hilbert transform

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Throughout the talk, we assume that $p \in (1,\infty)$.

Discrete Hilbert transform: prior results

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$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k}$$

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- $\|\mathcal{H}\|_{\ell^p \to \ell^p} = \|H\|_{L^p \to L^p}$ when $p = 2, 4, 8, 16, \ldots$ (or a conjugate exponent) (I.E. Verbitsky; see E. Laeng, 2007)

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Discretisations of the Hilbert transform

operator

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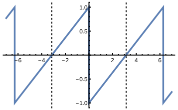
symbol



Discretisations of the Hilbert transform

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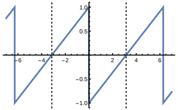

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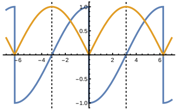

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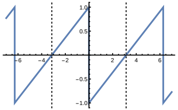

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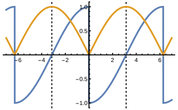
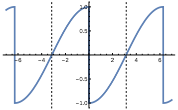


$$\mathcal{ADP}a_n = \frac{1}{\pi} \sum_{k\in\mathbb{Z}} \frac{k\, a_{n-k}}{k^2-\frac{1}{4}} \quad \leftrightsquigarrow$$



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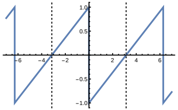

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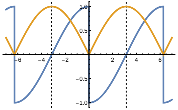

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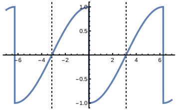
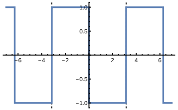


$$\mathcal{K}a_n = \frac{2}{\pi} \sum_{k \in 2\mathbb{Z}+1} \frac{a_{n-k}}{k} \quad \leftrightsquigarrow$$



Discretisations of the Hilbert transform

operator — symbol ($\xi \in (-\pi, \pi)$)

$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-k}}{k} \quad \leftrightsquigarrow \quad -i \operatorname{sign} \xi \cdot (1 - \tfrac{1}{\pi}|\xi|);$$

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Elementary inequalities

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- For second-order Riesz transforms: K. Domolevo and S. Petermichl, 2014.

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- Let $f \in L^p$. For $y > 0$ we define the Poisson integrals

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- Then u and v are conjugate harmonic functions:

$$\Delta u = \Delta v = 0, \qquad \nabla v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla u.$$

Hilbert transform and harmonic functions

- Let $f \in L^p$. For $y > 0$ we define the Poisson integrals

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x-z)\,\frac{y}{z^2+y^2}\,dz,$$
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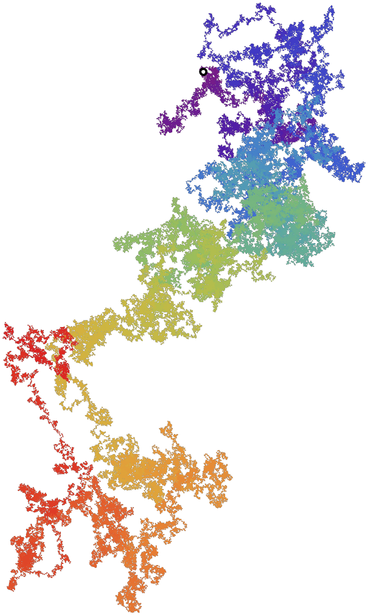
- The boundary values of u and v are given by

$$f(x) = \lim_{y \to 0^+} u(x,y), \qquad Hf(x) = \lim_{y\to 0^+} v(x,y)$$

(the limits exist in L^p and almost everywhere).

Harmonic functions and martingales

- Let B_t be the 2-D standard Brownian motion.



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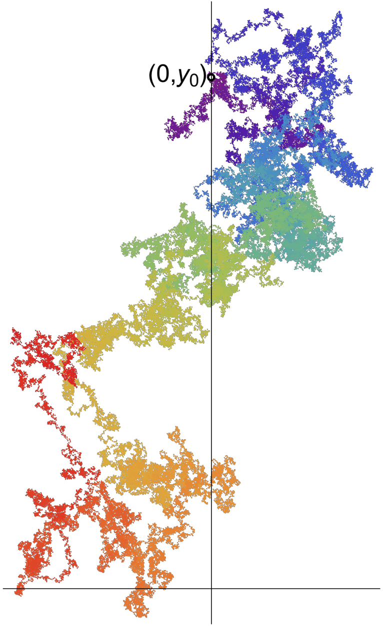

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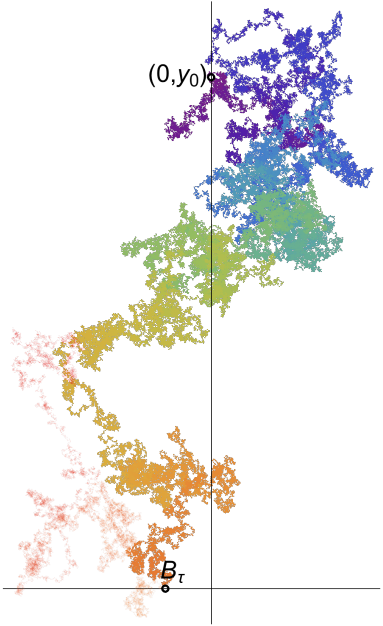

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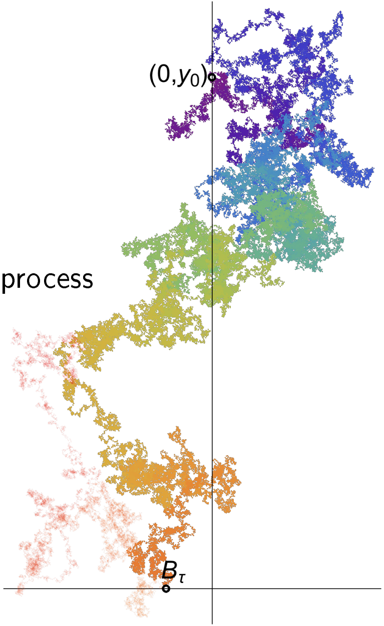

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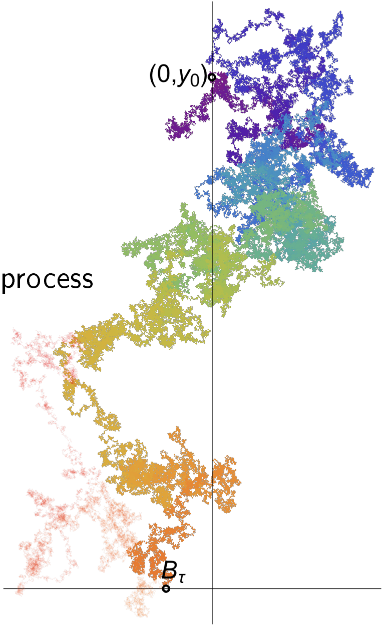

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is a martingale.
- Indeed: by the Itô formula, for $t < \tau$ we have
$$dM_t = \nabla u(B_t) \cdot dB_t,$$
$$d[M]_t = |\nabla u(B_t)|^2 dt.$$

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- We have defined two conjugate harmonic functions: $u(x, y)$ and $v(x, y)$, with boundary values $f(x)$ and $Hf(x)$, respectively.

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and

$$d[M, N]_t = \nabla u(B_t) \cdot \nabla v(B_t) dt = 0 dt$$

for $t < \tau$.

Burkholder's inequality

Theorem (R. Bañuelos, G. Wang, 1995)

If M_t and N_t are martingales and

- N_t is differentially subordinate to M_t:
$$d[N]_t \leqslant d[M]_t;$$
- M_t and N_t are orthogonal:
$$d[M, N]_t = 0 dt,$$

then

$$\mathbb{E}|N_\infty - N_0|^p \leqslant (C_p)^p \, \mathbb{E}|M_\infty - M_0|^p,$$

with $C_p = \max\{\tan(\frac{\pi}{2p}), \cot(\frac{\pi}{2p})\}$.

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Pichorides estimate

- Since B_τ has a Cauchy distribution on $\mathbb{R} \times \{0\}$, we have

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- We multiply both sides by y_0 and pass to the limit as $y_0 \to \infty$ to get

$$\|Hf\|_{L^p}^p \leqslant (C_p)^p \|f\|_{L^p}^p$$

(the Pichorides–Cole bound).

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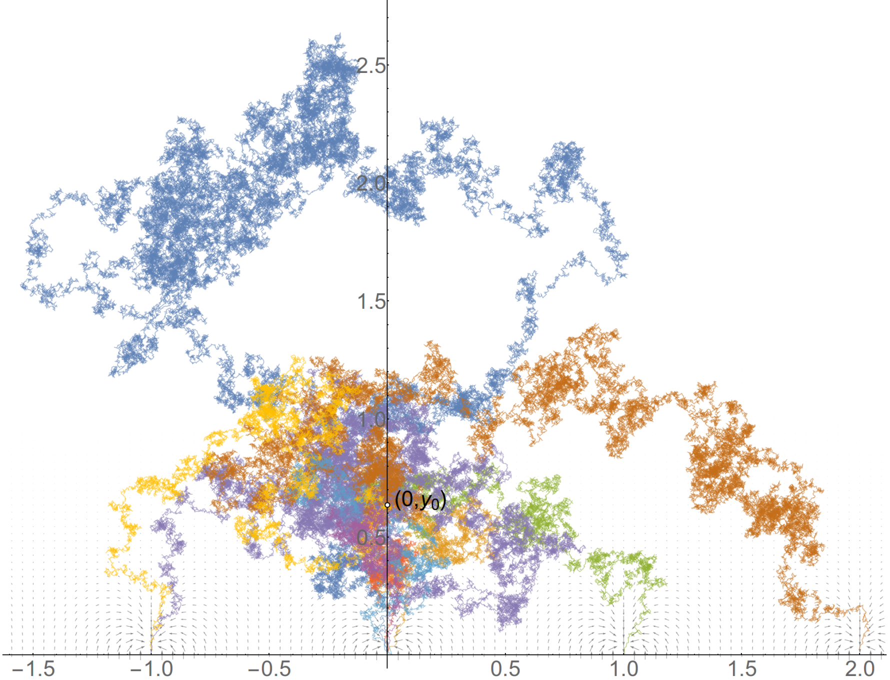
- To study discrete transform, we replace B_t by a diffusion X_t which only hits a discrete subset of the boundary: $\mathbb{Z} \times \{0\}$.
- The process X_t is obtained by conditioning the Brownian motion so that

$$B_\tau \in \left(\bigcup_{k \in \mathbb{Z}} (k - \varepsilon, k + \varepsilon) \right) \times \{0\},$$

and passing to the limit as $\varepsilon \to 0^+$.



2.5
2.0
1.5
1.0
0.5
(0,y0)
−1.5
−1.0
−0.5
0.5
1.0
1.5
2.0

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- (Initially I made a mistake and dropped a minus sign, and I got $\tilde{\mathcal{H}} = \mathcal{H}$.)

Convolution trick

- To complete the proof, we show that

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- (Had I not send an enthusiastic email to Rodrigo before noticing the error, I would have never found enough motivation to do that.)

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- Therefore, no similar argument can be given for the unitary operator $\mathcal{R}$.

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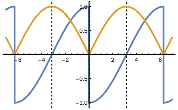

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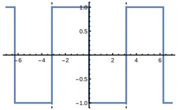

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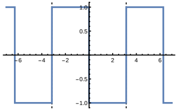

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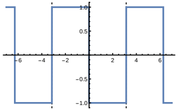

- The operators $\mathcal{R}$ and $\mathcal{K}$ are equivalent:

$$\mathcal{K}a_n = b_n \iff \begin{cases} \mathcal{R}[a_{2n}] = [b_{2n+1}], \\ \mathcal{R}[a_{2n-1}] = [b_{2n}]. \end{cases}$$

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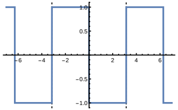

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- In particular, $\|\mathcal{R}\|_{\ell^p\to\ell^p} = \|\mathcal{K}\|_{\ell^p\to\ell^p}$.

Scaled discrete Hilbert transform

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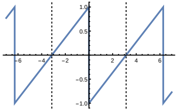

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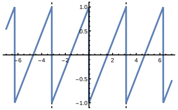

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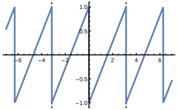

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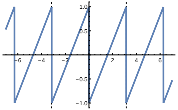

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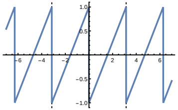

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- In particular, $\|\mathcal{H}^{\text{original}}\|_{\ell^p \to \ell^p} = \|\mathcal{H}^{\text{scaled}}\|_{\ell^p \to \ell^p}$.

Factorization

- Write $\mathcal{H} = \mathcal{H}^{\text{scaled}}$.

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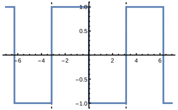

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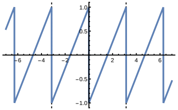

Factorization

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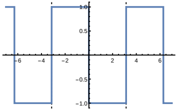

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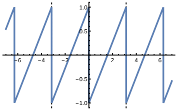

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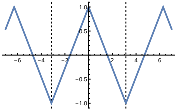

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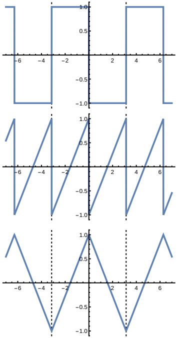

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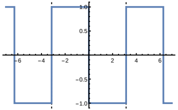

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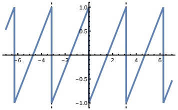

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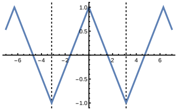

- $\mathcal{J}a_n$ is the convolution of a_n with a probability kernel.
- We have $\mathcal{H} = \mathcal{J}\mathcal{K}$.

Product rule

Lemma (see Titchmarsh, 1926)

We have

$$\mathcal{K}a_n \cdot \mathcal{K}b_n = \mathcal{K}[\mathcal{H}a_n \cdot b_n] + \mathcal{K}[a_n \cdot \mathcal{H}b_n] + \mathcal{I}[a_n \cdot b_n].$$

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- Compare with the cotangent of sum formula

$$\cot\alpha \cot\beta = \cot(\alpha+\beta)\cot\alpha + \cot(\alpha+\beta)\cot\beta + 1.$$

$p \rightsquigarrow 2p$

- By the product rule:

$$(\mathcal{K}a_n)^2 = 2\mathcal{K}[\mathcal{H}a_n \cdot a_n] + \mathcal{I}[a_n^2].$$

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- Note: we can replace $\|\mathcal{H}\|_{\ell^p\to\ell^p} = \cot\frac{\pi}{2p}$ by $\|\mathcal{H}\|_{\ell^p\to\ell^p} \leqslant \|\mathcal{K}\|_{\ell^p\to\ell^p}$.

$p \rightsquigarrow 3p$

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$$\begin{aligned}(\mathcal{K}a_n)^3 &= 2\mathcal{K}a_n \cdot \mathcal{K}[\mathcal{H}a_n \cdot a_n] + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2] \\ &= 2\mathcal{K}[(\mathcal{H}a_n)^2 \cdot a_n] + 2\mathcal{K}[a_n \cdot \mathcal{H}[\mathcal{H}a_n \cdot a_n]] \\ &\qquad + 2\mathcal{I}[\mathcal{H}a_n \cdot a_n^2] + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2].\end{aligned}$$

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- Note: we use $\|\mathcal{H}\|_{\ell^{p/2}\to\ell^{p/2}} = \cot\frac{\pi}{p}$ in an essential way.

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- Enumeration of all intermediate terms is a non-obvious task.
- To get things under control, we introduce frames, skeletons and buildings.