Fractional Laplacian: explicit calculations and applications

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Outline

- **Fractional Laplace operator**
- **Explicit formulae**
- **Eigenvalues in the unit ball**
- **Spectral theory in half-line**

Based on joint work with:

- **Bartłomiej Dyda** (Wrocław)
- **Alexey Kuznetsov** (Toronto)

Question

How can one define $L = (-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2)$?

- Laplace operator: $-\widehat{\Delta f}(\xi) = |\xi|^2 \hat{f}(\xi)$.
- Use spectral theorem!

Definition 1/10 (via Fourier transform)

Write $f \in \mathscr{D}_{\mathrm{F}}$ if $f \in \mathscr{L}^p$ and there is $Lf \in \mathscr{L}^p$ such that

$$\widehat{Lf}(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

- Here $p \in [1, 2]$ in order that $\hat{f}$ is well-defined.

- Can one relax the condition $p \in [1, 2]$?
- Use distribution theory!

Definition 2/10 (weak formulation)

Write $f \in \mathscr{D}_{\mathrm{w}}$ if $f \in \mathscr{L}^p$ and there is $Lf \in \mathscr{L}^p$ such that

$$\int_{\mathbf{R}^d} Lf(x)g(x)dx = \int_{\mathbf{R}^d} f(x)Lg(x)dx$$

for $g \in \mathscr{C}_{\mathrm{c}}^{\infty}$.

- Works not only for $\mathscr{L}^p$, but also $\mathscr{C}_0$, $\mathscr{C}_{\mathrm{b}}$, $\mathscr{C}_{\mathrm{bu}}$, ...

- Is there a more explicit expression?

Definition 3/10 (as a singular integral)

Write $f \in \mathscr{D}_{\mathrm{pv}}$ if $f \in \mathscr{L}^p$ and the limit

$$-Lf(x) = c \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}^d \setminus B_\varepsilon} \frac{f(x+y) - f(x)}{|y|^{d+\alpha}} \, dy$$

exists in $\mathscr{L}^p$.

- Works in $\mathscr{C}_0$, $\mathscr{C}_{\mathrm{b}}$, $\mathscr{C}_{\mathrm{bu}}$, ...
- Allows for pointwise definition of $Lf(x)$.
- Two common variants are less general:

$$-Lf(x) = c \int_{\mathbf{R}^d} \frac{f(x+y) - f(x) - y \cdot \nabla f(x) \mathbf{1}_B(y)}{|y|^{d+\alpha}} \, dy,$$

$$-Lf(x) = c \int_{\mathbf{R}^d} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{d+\alpha}} \, dy.$$

- Yet another variant is surprisingly useful!

Definition 4/10 (as a Dynkin characteristic operator)

Write $f \in \mathscr{D}_{\mathrm{Dy}}$ if $f \in \mathscr{L}^p$ and the limit

$$-Lf(x) = c \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}^d \setminus B_\varepsilon} \frac{f(x+y) - f(x)}{|y|^d (|y|^2 - \varepsilon^2)^{\alpha/2}} \, dy$$

exists in $\mathscr{L}^p$.

- Works in $\mathscr{C}_0$, $\mathscr{C}_{\mathrm{b}}$, $\mathscr{C}_{\mathrm{bu}}$, pointwise...
- To be discussed later!

- Caffarelli–Silvestre(–Molchanov–Ostrovski) extension technique is quite similar!

Definition 5/10 (via harmonic extensions)

Write $f \in \mathscr{D}_{\mathrm{h}}$ if $f \in \mathscr{L}^p$ and the limit

$$-Lf(x) = c \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}^d} \frac{f(x+y) - f(x)}{(\varepsilon^2 + |y|^2)^{(d+\alpha)/2}} \, dy$$

exists in $\mathscr{L}^p$.

- Works in $\mathscr{C}_0$, $\mathscr{C}_{\mathrm{b}}$, $\mathscr{C}_{\mathrm{bu}}$, pointwise...
- Originates as the Dirichlet-to-Neumann operator

$$\begin{cases} \Delta_x u(x,y) + c\, y^{2-2/\alpha} \frac{\partial^2 u}{\partial y^2}(x,y) = 0 & \text{for } y > 0 \\ u(x,0) = f(x) \\ \partial_y u(x,0) = Lf(x) \end{cases}$$

- Semigroup definition is of the same kind!
- Let $\hat{p}_t(\xi) = \exp(-t|\xi|^\alpha)$.

Definition 6/10 (as a generator of a $\mathscr{C}_0$-semigroup)

Write $f \in \mathscr{D}_s$ if $f \in \mathscr{L}^p$ and the limit

$$-Lf(x) = c \lim_{t \to 0^+} \int_{\mathbf{R}^d} (f(x+y) - f(x))\, p_t(y) dy$$

exists in $\mathscr{L}^p$.

- Works in $\mathscr{C}_0$, $\mathscr{C}_b$, $\mathscr{C}_{bu}$, pointwise...
- General theory of $\mathscr{C}_0$-semigroups is a powerful tool!

- The inverse of the generator is called potential
- The inverse of L is the Riesz potential

Definition 7/10 (as the inverse of a Riesz potential)

Write $f \in \mathscr{D}_{\mathrm{R}}$ if $f \in \mathscr{L}^p$ and there is $Lf \in \mathscr{L}^p$ such that

$$f(x) = c \int_{\mathbf{R}^d} \frac{Lf(x+y)}{|y|^{d-\alpha}}\, dy.$$

- Requires $\alpha < d$ (when $d = 1$).
- The convolution is well-defined if $p \in [1, \frac{d}{\alpha})$.

- The semigroup $\exp(-tL)$ is subordinate (in the sense of Bochner) to the semigroup $\exp(-t\Delta)$.
- $\lambda^{\alpha/2} = \dfrac{1}{|\Gamma(-\frac{\alpha}{2})|} \displaystyle\int_0^\infty (1 - e^{-t\lambda})\, t^{-1-\alpha/2} dt.$

Definition 8/10 (via Bochner's subordination)

Write $f \in \mathscr{D}_{\mathrm{Bo}}$ if $f \in \mathscr{L}^p$ and the integral

$$Lf = \frac{1}{|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty (f - e^{t\Delta} f)\, t^{-1-\alpha/2} dt$$

exists in $\mathscr{L}^p$.

- Works in $\mathscr{C}_0$, $\mathscr{C}_{\mathrm{b}}$, $\mathscr{C}_{\mathrm{bu}}$, pointwise...
- $e^{t\Delta}$ is the convolution with Gauss–Weierstrass kernel.

- A closely related idea is due to Balakrishnan.
- $\lambda^{\alpha/2} = \dfrac{\sin\frac{\alpha\pi}{2}}{\pi}\displaystyle\int_0^\infty \frac{\lambda}{s+\lambda}\, s^{\alpha/2-1} ds.$

Definition 9/10 (Balakrishnan's definition)

Write $f \in \mathscr{D}_{\mathrm{Ba}}$ if $f \in \mathscr{L}^p$ and the integral

$$Lf = \frac{\sin\frac{\alpha\pi}{2}}{\pi}\int_0^\infty \Delta(s-\Delta)^{-1} f\, s^{\alpha/2-1} ds$$

exists in $\mathscr{L}^p$.

- Works in $\mathscr{C}_0$, $\mathscr{C}_{\mathrm{b}}$, $\mathscr{C}_{\mathrm{bu}}$, pointwise...
- $(s-\Delta)^{-1}$ is the s-resolvent for Δ; not very useful.

- Quadratic form is a natural approach in $\mathscr{L}^2$.
- Define

$$\mathscr{E}(f,g) = c\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}}\,dx\,dy$$
$$= \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} |\xi|^\alpha \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.$$

Definition 10/10 (via quadratic forms)

Write $f \in \mathscr{D}_{\mathrm{q}}$ if $f \in \mathscr{L}^2$ and there is $Lf \in \mathscr{L}^2$ such that

$$\mathscr{E}(f,g) = -\int_{\mathbf{R}^d} Lf(x)g(x)dx.$$

Theorem (*many authors*)

The above ten definitions are all equivalent:

$$\mathscr{D}_{\mathsf{F}} = \mathscr{D}_{\mathsf{w}} = \mathscr{D}_{\mathsf{pv}} = \mathscr{D}_{\mathsf{Dy}} = \mathscr{D}_{\mathsf{h}} = \mathscr{D}_{\mathsf{s}} = \mathscr{D}_{\mathsf{R}} = \mathscr{D}_{\mathsf{Bo}} = \mathscr{D}_{\mathsf{Ba}} = \mathscr{D}_{\mathsf{q}}$$

in $\mathscr{L}^p$, $p \in [1, \infty)$ (whenever meaningful).

Norm convergence implies a.e. convergence in four definitions:

pv, Dy, h, s.

- Very well-known for smooth functions.
- Some parts are very general (e.g. $\mathscr{D}_{\mathsf{s}} = \mathscr{D}_{\mathsf{Bo}} = \mathscr{D}_{\mathsf{Ba}}$).
- Some pieces were apparently missing.

Theorem (*many authors*)

Seven out of ten definitions are equivalent:

$$\mathscr{D}_{\mathsf{w}} = \mathscr{D}_{\mathsf{pv}} = \mathscr{D}_{\mathsf{Dy}} = \mathscr{D}_{\mathsf{h}} = \mathscr{D}_{\mathsf{s}} = \mathscr{D}_{\mathsf{Bo}} = \mathscr{D}_{\mathsf{Ba}}$$

in $\mathscr{C}_0$ and $\mathscr{C}_{\mathsf{bu}}$.

Uniform convergence is *equivalent* to pointwise convergence with a limit in $\mathscr{C}_0$ or $\mathscr{C}_{\mathsf{bu}}$ in five definitions:

pv, Dy, h, s, Bo.

- The remaining three definitions (F, R, q) are meaningless for $\mathscr{C}_0$ or $\mathscr{C}_{\mathsf{bu}}$.

M. Kwaśnicki
Ten equivalent definitions of the fractional Laplace operator
arXiv:1507.07356

- The study of the fractional Laplacian was initiated by **Marcel Riesz** in 1938.
- His seminal article contains a lot of results!
- Some of them are often attributed to other authors.

M. Riesz
Intégrales de Riemann–Liouville et potentiels
Acta Sci. Math. Szeged 9 (1938): 1–42

M. Riesz
Rectification au travail “Intégrales de Riemann–Liouville et potentiels”
Acta Sci. Math. Szeged 9 (1938): 116–118.

Theorem (M. Riesz)

$$L\Big[(1-|x|^2)_+^{\alpha/2}\Big] = c \qquad \text{for } x \in B.$$

Theorem (M. Riesz; Kac)

The Poisson kernel for L in B is given by

$$P_B(x,z) = c\left(\frac{1-|x|^2}{|z|^2-1}\right)^{\alpha/2} \frac{1}{|x-z|^d},$$

where $x \in B$, $z \in \mathbf{R}^d \setminus B$.

Theorem (M. Riesz; Kac; Blumenthal–Getoor–Ray)

The Green function for L in B is given by

$$G_B(x,y) = \frac{c}{|y-z|^{d-\alpha}} \int_0^{\frac{(r^2-|y|^2)(r^2-|z|^2)}{r^2|y-z|^2}} \frac{s^{\alpha/2-1}}{(1+s)^{d/2}}\, ds,$$

where $x, y \in B$.

Theorem (Hmissi, Bogdan)

If $z \in \partial B$, then

$$L\left[\frac{(1-|x|^2)_+^{\alpha/2}}{|x-z|^d}\right] = 0 \qquad \text{for } x \in B;$$

that is:

$$\Delta f = 0 \text{ in } B \quad \Longleftrightarrow \quad L\Big[(1-|x|^2)_+^{\alpha/2} f(x)\Big] = 0 \text{ in } B.$$

In particular,

$$L\Big[(1-|x|^2)_+^{\alpha/2-1}\Big] = 0 \qquad \text{for } x \in B.$$

Theorem (Biler–Imbert–Karch; Dyda)

$$L\Big[(1-|x|^2)_+^{p}\Big] = c\,{}_2F_1\big(\tfrac{d+\alpha}{2},\ \tfrac{\alpha}{2}-p;\tfrac{d}{2};|x|^2\big) \qquad \text{for } x \in B.$$

Theorem (M. Riesz; Bogdan–Żak)

The **Kelvin transform** is compatible with L:

$$L\left[\frac{1}{|x|^{d-\alpha}} f\left(\frac{x}{|x|^2}\right)\right] = \frac{1}{|x|^{d+\alpha}} Lf\left(\frac{x}{|x|^2}\right).$$

- **Translation invariance**: $L[f(x_0 + x)] = Lf(x_0 + x)$.
- **Scaling**: $L[f(rx)] = r^\alpha Lf(rx)$.
- This extends previous results to arbitrary balls, half-spaces or complements of balls.

- Full-space results are more rare!
- Fourier transform: $L[e^{i\xi x}] = |\xi|^\alpha e^{i\xi x}$.
- Composition of Riesz potentials: $L[|x|^{p-d}] = c|x|^{p-d-\alpha}$.

Theorem (Samko)

$$L[\exp(-|x|^2)] = c\,{}_1F_1(\tfrac{d+\alpha}{2}; \tfrac{d}{2}; -|x|^2);$$

$$L\left[\frac{1}{(1+|x|^2)^{(d+1)/2}}\right] = c\,{}_2F_1(\tfrac{d+1+\alpha}{2}, \tfrac{d+\alpha}{2}; \tfrac{d}{2}; -|x|^2);$$

$$L\left[\frac{1}{(1+|x|^2)^{(d-\alpha)/2+n}}\right] = c\,\frac{\{polynomial\}}{(1+|x|^2)^{(d+\alpha)/2+n}}.$$

Theorem (DKK)

$$L\left[{}_pF_q\left(\begin{matrix}\overbrace{a_1,\ldots,a_p}^{\mathbf{a}}\\ \underbrace{b_1,\ldots,b_{q-1}}_{\mathbf{b}},\ \frac{d}{2}\end{matrix};-|x|^2\right)\right]=$$

$$=c\,{}_pF_q\left(\begin{matrix}\mathbf{a}+\frac{\alpha}{2}\\ \mathbf{b}+\frac{\alpha}{2},\ \frac{d}{2}\end{matrix};-|x|^2\right).$$

- Note: $c=2^{\alpha}\dfrac{\Gamma(\mathbf{a}+\frac{\alpha}{2})\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})\Gamma(\mathbf{b}+\frac{\alpha}{2})}$.

B. Dyda, A. Kuznetsov, M. Kwaśnicki
Fractional Laplace operator and Meijer G-function
Constr. Approx., to appear

Theorem (DKK)

$$L\left[G^{m,n}_{p,q}\left(\begin{matrix}\overbrace{a_1,\ldots,a_p}^{\mathbf{a}}\\ \underbrace{b_1,\ldots,b_q}_{\mathbf{b}}\end{matrix};|x|^2\right)\right]=$$

$$=2^{\alpha}\,G^{m+1,n+1}_{p+2,q+2}\left(\begin{matrix}\frac{1-d-\alpha}{2}, & \mathbf{a}-\frac{\alpha}{2}, & -\frac{\alpha}{2}\\ 0, & \mathbf{b}-\frac{\alpha}{2}, & 1-\frac{d}{2}\end{matrix};|x|^2\right).$$

- The generalised hypergeometric function ${}_pF_q$ is already a complicated object.
- The Meijer G-function $G^{m,n}p,q$ is even worse.
- But it is perfectly compatible with L!

- **A lot of functions** can be expressed as $G^{m,n}_{p,q}$!
  (see a hundred-page-long table in Prudnikov's book)
- Full space:

$$(-\Delta)^{\alpha/2}\big[|x|^p(1+|x|^2)^{q/2}\big] = \frac{2^\alpha}{\Gamma(-\frac{q}{2})}\, G^{2,2}_{3,3}\left(\begin{matrix} 1-\frac{d+\alpha}{2},\ 1+\frac{p+q-\alpha}{2},\ -\frac{\alpha}{2} \\ 0,\ \frac{p-\alpha}{2},\ 1-\frac{d}{2} \end{matrix}; |x|^2\right).$$

- Unit ball:

$$(-\Delta)^{\alpha/2}\big[|x|^p(1-|x|^2)_+^{q/2}\big] = 2^\alpha\Gamma(1+\tfrac{q}{2})\, G^{2,1}_{3,3}\left(\begin{matrix} 1-\frac{d+\alpha}{2},\ 1+\frac{p+q-\alpha}{2},\ -\frac{\alpha}{2} \\ 0,\ \frac{p-\alpha}{2},\ 1-\frac{d}{2} \end{matrix}; |x|^2\right).$$

Theorem (DKK; follows from *Bochner's relation*)

Let $V(x)$ be a **solid harmonic polynomial** of degree ℓ. Then:

$$L[V(x)\,f(|x|)] = V(x)\,g(|x|) \qquad \text{in } \mathbf{R}^d$$

if and only if

$$L[f(|y|)] = g(|y|) \qquad \text{in } \mathbf{R}^{d+2\ell}.$$

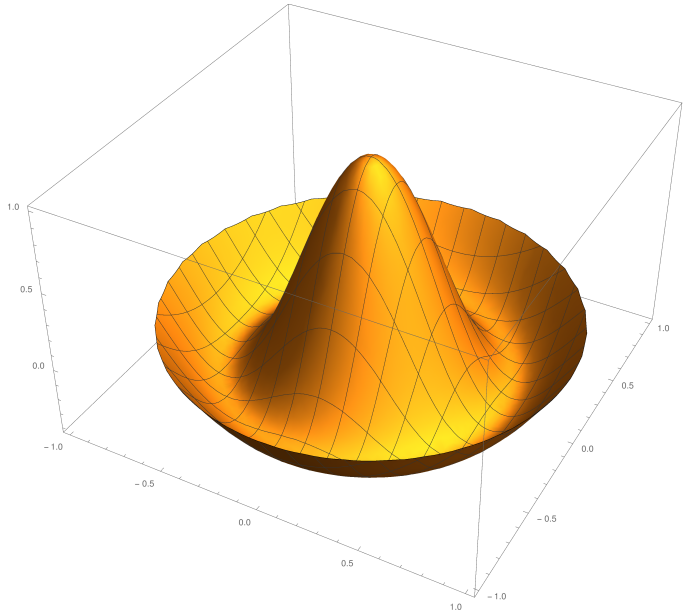
- Here 'solid' = 'homogeneous'.
- Examples of $V(x)$: 1, x_1, x_1x_2, $x_1x_2\ldots x_d$, $x_1^2 - x_2^2$.
- Solid harmonic polynomials span $\mathscr{L}^2(\partial B)$.
- Extends to arbitrary convolution operators with isotropic kernels.

Eigenvalue problem

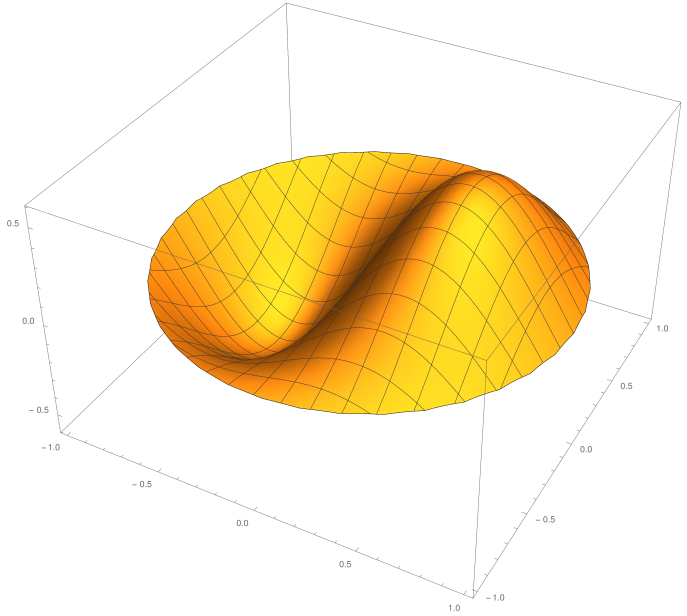
$$\begin{cases} (-\Delta)^{\alpha/2}\varphi_n(x) = \lambda_n \varphi_n(x) & \text{for } x \in B \\ \varphi_n(x) = 0 & \text{otherwise} \end{cases}$$

Question

We know that φ_1 is radial. Is φ_2 radial or antisymmetric?



or

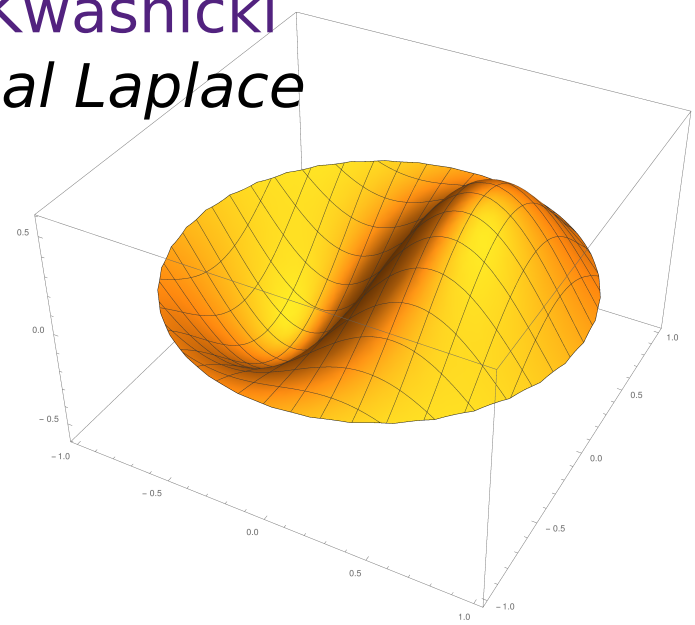


- This is still an **open problem**!
- We have a partial answer.
- Plus strong numerical evidence in the general case.

Theorem (DKK)

φ_2 is antisymmetric if $d \leq 2$, or if $\alpha = 1$ and $d \leq 9$.

B. Dyda, A. Kuznetsov, M. Kwaśnicki
Eigenvalues of the fractional Laplace operator in the unit ball
arXiv:1509.08533.



- Upper bounds: **Rayleigh–Ritz variational method**.
- Lower bounds: **Weinstein–Aronszajn method** of intermediate problems.
- These are numerical methods!
- We use them analytically for small (2×2) matrices.
- As a side-result, we get an extremely efficient numerical scheme for finding λ_n in a ball B.
- Requires **explicit expressions** for $(-\Delta)^{\alpha/2} f(x)$.

Theorem (DKK)

Let $V(x)$ be a **solid harmonic polynomial** of degree ℓ. Let $P_n^{(\alpha,\beta)}(r)$ be the **Jacobi polynomial**. Then:

$$L\left[(1-|x|^2)_+^{\alpha/2}\, V(x)\, P_n^{(\frac{\alpha}{2},\frac{d+2\ell}{2}-1)}(2|x|^2-1)\right] =$$

$$= c\, V(x)\, P_n^{(\frac{\alpha}{2},\frac{d+2\ell}{2}-1)}(2|x|^2-1) \qquad \text{for } x \in B.$$

- Here $c = 2^\alpha \dfrac{\Gamma(\frac{\alpha}{2}+n+1)\Gamma(\frac{d+2\ell+\alpha}{2}+n)}{n!\,\Gamma(\frac{d+2\ell}{2}+n)}$.
- For $\ell = 0$, c is an upper bound for radial eigenvalues!

Theorem (K, KMR, KK)

For $x > 0$ let

$$F(x) = \sin\left(x + \tfrac{(2-\alpha)\pi}{8}\right) - \int_0^\infty e^{-xs}\,\Phi(s)\,ds,$$

where

$$\Phi(s) = \frac{\sqrt{2\alpha}\,\sin\frac{\alpha\pi}{2}}{2\pi}\,\frac{s^\alpha}{1+s^{2\alpha}-2s^\alpha\cos\frac{\alpha\pi}{2}}$$

$$\times\exp\left(\frac{1}{\pi}\int_0^\infty \frac{1}{1+r^2}\log\frac{1-s^2r^2}{1-s^\alpha r^\alpha}\,dr\right)$$

$$= \frac{\sqrt{\alpha}\,S_2(-\frac{\alpha}{2})}{4\pi}\,s^{\alpha/4-1/2}\left|S_2\left(\alpha;1+\alpha+\tfrac{\alpha}{4}+\tfrac{i\alpha\log s}{2\pi};\alpha\right)\right|^2,$$

and $F(x) = 0$ for $x \le 0$. Then $LF(x) = F(x)$ for $x > 0$.

- Due to scaling, $L[F(\lambda x)] = \lambda^\alpha F(\lambda x)$.
- S_2 is the Koyama–Kurokawa's double sine function.

M. Kwaśnicki
Spectral analysis of subordinate Brownian motions on the half-line
Studia Math. 206(3) (2011): 211–271

M. Kwaśnicki, J. Małecki, M. Ryznar
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Stoch. Proc. Appl. 123 (2013): 1820–1850

A. Kuznetsov, M. Kwaśnicki
Spectral analysis of stable processes on the positive half-line
arXiv:1509.06435

- Extends to general symmetric operators with completely monotone kernels.
- Extends to **non-symmetric** fractional derivatives!

- In the **non-symmetric** case F has exponential decay or growth:

$$F(x) = e^{ax}\sin(bx+\theta) + \int_0^\infty e^{-xs}\,\Phi(s)\,ds,$$

where $a = \cos(\pi\rho)$, $b = \sin(\pi\rho)$, $\theta = \frac{1}{2}\pi\rho(1-\alpha+\alpha\rho)$,

$$\Phi(s) = s^{\alpha\rho/2-1/2}|S_2(\alpha; 1+\tfrac{3\alpha}{2}-\tfrac{\alpha\rho}{2}+\tfrac{i\alpha\log s}{2\pi}; \alpha)|^2,$$

- Still, it gives rise to a **Fourier-type transform**!
- Describes the spectral resolution for L in $(0,\infty)$: e.g. heat kernel of L in $(0,\infty)$ can be written in terms of F.
- Application: explicit expression for the supremum of a stable Lévy process.

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