Fractional Laplacian: explicit calculations and applications

Mateusz Kwasnicki
Wrocław University of Science and Technology
mateusz.kwasnicki@pwr.wroc.pl

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Outline

- Fractional Laplace operator
- Explicit formulae
- Eigenvalues in the unit ball
- Spectral theory in half-line

Based on joint work with:
- Bartłomiej Dyda (Wrocław)
- Alexey Kuznetsov (Toronto)
Question

How can one define $L = (-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2)$?

- Laplace operator: $-\Delta \hat{f}(\xi) = |\xi|^2 \hat{f}(\xi)$.
- Use spectral theorem!

Definition 1/10 (via Fourier transform)

Write $f \in \mathcal{D}_F$ if $f \in L^p$ and there is $Lf \in L^p$ such that

$$\hat{Lf}(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

- Here $p \in [1, 2]$ in order that $\hat{f}$ is well-defined.
• Can one relax the condition $p \in [1, 2]$?
• Use distribution theory!

**Definition 2/10 (weak formulation)**

Write $f \in \mathcal{D}_w$ if $f \in L^p$ and there is $Lf \in L^p$ such that

$$\int_{\mathbb{R}^d} Lf(x)g(x)dx = \int_{\mathbb{R}^d} f(x)Lg(x)dx$$

for $g \in C_c^\infty$.

• Works not only for $L^p$, but also $C_0$, $C_b$, $C_{bu}$, ...
• Is there a more explicit expression?

**Definition 3/10 (as a singular integral)**

Write \( f \in D_{pv} \) if \( f \in \mathcal{L}^p \) and the limit

\[
-Lf(x) = c \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B_\varepsilon} \frac{f(x+y) - f(x)}{|y|^{d+\alpha}} dy
\]

exists in \( \mathcal{L}^p \).

• Works in \( \mathcal{C}_0, \mathcal{C}_b, \mathcal{C}_{bu}, \ldots \)

• Allows for pointwise definition of \( Lf(x) \).

• Two common variants are less general:

\[
-Lf(x) = c \int_{\mathbb{R}^d} \frac{f(x+y) - f(x) - y \cdot \nabla f(x) 1_B(y)}{|y|^{d+\alpha}} dy,
\]

\[
-Lf(x) = c \int_{\mathbb{R}^d} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{d+\alpha}} dy.
\]
• Yet another variant is surprisingly useful!

**Definition 4/10 (as a Dynkin characteristic operator)**

Write \( f \in \mathcal{D}_\text{Dy} \) if \( f \in \mathcal{L}^p \) and the limit

\[
-Lf(x) = c \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B_\varepsilon} \frac{f(x + y) - f(x)}{|y|^d (|y|^2 - \varepsilon^2)^{\alpha/2}} \, dy
\]

exists in \( \mathcal{L}^p \).

• Works in \( \mathcal{C}_0, \mathcal{C}_b, \mathcal{C}_{\text{bu}} \), pointwise...

• To be discussed later!
- Caffarelli–Silvestre(–Molchanov–Ostrovski) extension technique is quite similar!

**Definition 5/10 (via harmonic extensions)**

Write $f \in D_h$ if $f \in L^p$ and the limit

$$-Lf(x) = c \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} \frac{f(x + y) - f(x)}{(\varepsilon^2 + |y|^2)^{(d+\alpha)/2}} dy$$

exists in $L^p$.

- Works in $C_0$, $C_b$, $C_{bu}$, pointwise...
- Originates as the Dirichlet-to-Neumann operator

$$\begin{cases} 
\Delta_x u(x, y) + cy^{2-2/\alpha} \frac{\partial^2 u}{\partial y^2}(x, y) = 0 & \text{for } y > 0 \\
u(x, 0) = f(x) \\
\partial_y u(x, 0) = Lf(x)
\end{cases}$$
• Semigroup definition is of the same kind!
• Let $\hat{p}_t(\xi) = \exp(-t|\xi|^\alpha)$.

**Definition 6/10 (as a generator of a $C_0$-semigroup)**

Write $f \in D_s$ if $f \in L^p$ and the limit

$$-L f(x) = c \lim_{t \to 0^+} \int_{\mathbb{R}^d} (f(x + y) - f(x)) p_t(y) dy$$

exists in $L^p$.

• Works in $C_0$, $C_b$, $C_{bu}$, pointwise...
• General theory of $C_0$-semigroups is a powerful tool!
• The inverse of the generator is called potential
• The inverse of $L$ is the Riesz potential

**Definition 7/10 (as the inverse of a Riesz potential)**

Write $f \in D_R$ if $f \in L^p$ and there is $Lf \in L^p$ such that

$$f(x) = c \int_{\mathbb{R}^d} \frac{Lf(x + y)}{|y|^{d-\alpha}} dy.$$

• Requires $\alpha < d$ (when $d = 1$).
• The convolution is well-defined if $p \in [1, \frac{d}{\alpha})$. 
• The semigroup \( \exp(-tL) \) is subordinate (in the sense of Bochner) to the semigroup \( \exp(-t\Delta) \).

• \( \lambda^{\alpha/2} = \frac{1}{|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty (1 - e^{-t\lambda}) t^{-1-\alpha/2} dt \).

**Definition 8/10 (via Bochner’s subordination)**

Write \( f \in \mathcal{D}_{Bo} \) if \( f \in L^p \) and the integral

\[
Lf = \frac{1}{|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty (f - e^{t\Delta}f) t^{-1-\alpha/2} dt
\]

exists in \( L^p \).

• Works in \( C_0, C_b, C_{bu} \), pointwise...

• \( e^{t\Delta} \) is the convolution with Gauss–Weierstrass kernel.
A closely related idea is due to Balakrishnan.

\[ \lambda^{\alpha/2} = \frac{1}{\pi} \frac{\sin \frac{\alpha \pi}{2}}{s + \lambda} \int_0^\infty \Delta(s - \Delta)^{-1} f s^{\alpha/2 - 1} ds. \]

**Definition 9/10 (Balakrishnan’s definition)**

Write \( f \in \mathcal{D}_{B_\alpha} \) if \( f \in L^p \) and the integral

\[ Lf = \frac{1}{\pi} \frac{\sin \frac{\alpha \pi}{2}}{s + \lambda} \int_0^\infty \Delta(s - \Delta)^{-1} f s^{\alpha/2 - 1} ds \]

exists in \( L^p \).

- Works in \( C_0, C_b, C_{bu} \), pointwise...
- \( (s - \Delta)^{-1} \) is the s-resolvent for \( \Delta \); not very useful.
• Quadratic form is a natural approach in $\mathcal{L}^2$.

• Define

$$E(f, g) = c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} \, dx \, dy$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^\alpha \hat{f}(\xi) \hat{g}(\xi) \, d\xi.$$

**Definition 10/10 (via quadratic forms)**

Write $f \in \mathcal{D}_q$ if $f \in \mathcal{L}^2$ and there is $Lf \in \mathcal{L}^2$ such that

$$E(f, g) = - \int_{\mathbb{R}^d} Lf(x)g(x) \, dx.$$
Theorem *(many authors)*

The above ten definitions are all equivalent:

\[ D_F = D_w = D_{pv} = D_{Dy} = D_h = D_s = D_R = D_{Bo} = D_{Ba} = D_q \]

in \( L^p, p \in [1, \infty) \) (whenever meaningful).

Norm convergence implies a.e. convergence in four definitions:

\[ pv, Dy, h, s. \]

- Very well-known for smooth functions.
- Some parts are very general (e.g. \( D_s = D_{Bo} = D_{Ba} \)).
- Some pieces were apparently missing.
Theorem \textit{(many authors)}

Seven out of ten definitions are equivalent:

\[ D_w = D_{pv} = D_{Dy} = D_h = D_s = D_{Bo} = D_{Ba} \]

in $C_0$ and $C_{bu}$.

Uniform convergence is \textit{equivalent} to pointwise convergence with a limit in $C_0$ or $C_{bu}$ in five definitions:

$pv, Dy, h, s, Bo$.

- The remaining three definitions (F, R, q) are meaningless for $C_0$ or $C_{bu}$.

M. Kwaśnicki

\textit{Ten equivalent definitions of the fractional Laplace operator}

arXiv:1507.07356
• The study of the fractional Laplacian was initiated by **Marcel Riesz** in 1938.
• His seminal article contains a lot of results!
• Some of them are often attributed to other authors.

**M. Riesz**

*Intégrales de Riemann–Liouville et potentiels*


**M. Riesz**

*Rectification au travail “Intégrales de Riemann–Liouville et potentiels”*

Theorem (M. Riesz)

\[ L \left[ (1 - |x|^2)^{\alpha/2} \right] = c \quad \text{for} \ x \in B. \]

Theorem (M. Riesz; Kac)

The Poisson kernel for \( L \) in \( B \) is given by

\[ P_B(x, z) = c \left( \frac{1 - |x|^2}{|z|^2 - 1} \right)^{\alpha/2} \frac{1}{|x - z|^d}, \]

where \( x \in B, z \in \mathbb{R}^d \setminus B \).

Theorem (M. Riesz; Kac; Blumenthal–Getoor–Ray)

The Green function for \( L \) in \( B \) is given by

\[ G_B(x, y) = \frac{c}{|y - z|^{d-\alpha}} \int_0^1 \frac{(r^2 - |y|^2)(r^2 - |z|^2)}{r^2|y - z|^2} s^{\alpha/2 - 1} \frac{1}{(1 + s)^{d/2}} ds, \]

where \( x, y \in B \).
**Theorem (Hmissi, Bogdan)**

If \( z \in \partial B \), then

\[
L \left[ \frac{(1 - |x|^2)^{\alpha/2}}{|x - z|^d} \right] = 0 \quad \text{for } x \in B;
\]

that is:

\[
\Delta f = 0 \text{ in } B \iff L \left[ (1 - |x|^2)^{\alpha/2} f(x) \right] = 0 \text{ in } B.
\]

In particular,

\[
L \left[ (1 - |x|^2)^{\alpha/2-1} \right] = 0 \quad \text{for } x \in B.
\]

**Theorem (Biler–Imbert–Karch; Dyda)**

\[
L \left[ (1 - |x|^2)^{\beta} \right] = c_2 F_1 \left( \frac{d+\alpha}{2}, \frac{\alpha}{2} - \beta; \frac{d}{2}; |x|^2 \right) \quad \text{for } x \in B.
\]
Theorem (M. Riesz; Bogdan–Żak)

The Kelvin transform is compatible with $L$:

$$L\left[\frac{1}{|x|^{d-\alpha}} f\left(\frac{x}{|x|^2}\right)\right] = \frac{1}{|x|^{d+\alpha}} Lf\left(\frac{x}{|x|^2}\right).$$

- **Translation invariance**: $L[f(x_0 + x)] = Lf(x_0 + x)$.
- **Scaling**: $L[f(rx)] = r^\alpha Lf(rx)$.
- This extends previous results to arbitrary balls, half-spaces or complements of balls.
• Full-space results are more rare!
• Fourier transform: $L[e^{i\xi x}] = |\xi|^\alpha e^{i\xi x}$.
• Composition of Riesz potentials: $L[|x|^{p-d}] = c|x|^{p-d-\alpha}$.

**Theorem (Samko)**

\[
\begin{align*}
L[\exp(-|x|^2)] &= c_1 F_1 \left( \frac{d+\alpha}{2}; \frac{d}{2}; -|x|^2 \right); \\
L \left[ \frac{1}{(1 + |x|^2)^{(d+1)/2}} \right] &= c_2 F_1 \left( \frac{d+1+\alpha}{2}, \frac{d+\alpha}{2}; \frac{d}{2}; -|x|^2 \right); \\
L \left[ \frac{1}{(1 + |x|^2)^{(d-\alpha)/2+n}} \right] &= c \frac{\{\text{polynomial}\}}{(1 + |x|^2)^{(d+\alpha)/2+n}}.
\end{align*}
\]
Theorem (DKK)

\[
L \left[ \begin{array}{c}
\begin{array}{c}
\alpha \\
\vdots \\
a_p \\
b_1, \ldots, b_{q-1}, \frac{d}{2}
\end{array}
\end{array} \right] = \]

\[
= c \begin{array}{c}
\begin{array}{c}
\alpha \\
\frac{d}{2}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\alpha \\
\frac{d}{2}
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\alpha \\
\frac{d}{2}
\end{array}
\end{array}
\frac{\Gamma(a + \frac{\alpha}{2}) \Gamma(b)}{\Gamma(a) \Gamma(b + \frac{\alpha}{2})}.
\]

Note: \( c = 2^\alpha \frac{\Gamma(a + \frac{\alpha}{2}) \Gamma(b)}{\Gamma(a) \Gamma(b + \frac{\alpha}{2})} \).

B. Dyda, A. Kuznetsov, M. Kwaśnicki

Fractional Laplace operator and Meijer G-function

Constr. Approx., to appear
Theorem (DKK)

\[ L \left[ G_{p,q}^{m,n} \left( a_1, \ldots, a_p \middle| b_1, \ldots, b_q \right) ; |x|^2 \right] = \]

\[ = 2^\alpha G_{p+2,q+2}^{m+1,n+1} \left( \frac{1-d-\alpha}{2}, a-\frac{\alpha}{2}, -\frac{\alpha}{2} \middle| 0, b-\frac{\alpha}{2}, 1-d \frac{d}{2}; |x|^2 \right). \]

- The generalised hypergeometric function \( pF_q \) is already a complicated object.
- The Meijer G-function \( G_{p,q}^{m,n} \) is even worse.
- But it is perfectly compatible with \( L \)!
• **A lot of functions** can be expressed as $G_{p,q}^{m,n}!$
  (see a hundred-page-long table in Prudnikov’s book)

• **Full space**:
  \[
  (-\Delta)^{\alpha/2} [ |x|^p (1 + |x|^2)^{q/2} ] = \frac{2^{\alpha}}{\Gamma(-\frac{q}{2})} G_{3,3}^{2,2} \left( \frac{1 - \frac{d+\alpha}{2}, 1 + \frac{p+q-\alpha}{2}, -\alpha}{0, \frac{p-\alpha}{2}, 1 - \frac{d}{2}} ; |x|^2 \right).
  \]

• **Unit ball**:
  \[
  (-\Delta)^{\alpha/2} [ |x|^p (1 - |x|^2)^{q/2} ] = 2^{\alpha} \Gamma(1 + \frac{q}{2}) G_{3,3}^{2,1} \left( \frac{1 - \frac{d+\alpha}{2}, 1 + \frac{p+q-\alpha}{2}, -\alpha}{0, \frac{p-\alpha}{2}, 1 - \frac{d}{2}} ; |x|^2 \right).
  \]
Theorem (DKK; follows from Bochner’s relation)

Let $V(x)$ be a **solid harmonic polynomial** of degree $\ell$. Then:

$$L[V(x)f(|x|)] = V(x)g(|x|) \quad \text{in } \mathbb{R}^d$$

if and only if

$$L[f(|y|)] = g(|y|) \quad \text{in } \mathbb{R}^{d+2\ell}.$$

- Here ‘solid’ = ‘homogeneous’.
- Examples of $V(x)$: $1, x_1, x_1 x_2, x_1 x_2 \ldots x_d, x_1^2 - x_2^2$.
- Solid harmonic polynomials span $L^2(\partial B)$.
- Extends to arbitrary convolution operators with isotropic kernels.
Eigenvalue problem

\[
\begin{cases}
(−\Delta)^{\alpha/2} \varphi_n(x) = \lambda_n \varphi_n(x) & \text{for } x \in B \\
\varphi_n(x) = 0 & \text{otherwise}
\end{cases}
\]

Question

We know that $\varphi_1$ is radial. Is $\varphi_2$ radial or antisymmetric?
• This is still an **open problem**!
• We have a partial answer.
• Plus strong numerical evidence in the general case.

**Theorem (DKK)**

φ₂ is antisymmetric if \( d \leq 2 \), or if \( \alpha = 1 \) and \( d \leq 9 \).

B. Dyda, A. Kuznetsov, M. Kwaśnicki

*Eigenvalues of the fractional Laplace operator in the unit ball*

• Upper bounds: **Rayleigh–Ritz variational method**.
• Lower bounds: **Weinstein–Aronszajn method** of intermediate problems.
• These are numerical methods!
• We use them analytically for small (2 × 2) matrices.
• As a side-result, we get an extremely efficient numerical scheme for finding \( \lambda_n \) in a ball \( B \).
• Requires **explicit expressions** for \((-\Delta)^{\alpha/2}f(x)\).
Theorem (DKK)

Let $V(x)$ be a solid harmonic polynomial of degree $\ell$. Let $P_n^{(\alpha,\beta)}(r)$ be the Jacobi polynomial. Then:

$$L \left[ (1 - |x|^2)^{\alpha/2} V(x) P_n^{\left(\frac{\alpha}{2}, \frac{d+2\ell}{2} - 1\right)}(2|x|^2 - 1) \right] =$$

$$= c V(x) P_n^{\left(\frac{\alpha}{2}, \frac{d+2\ell}{2} - 1\right)}(2|x|^2 - 1) \quad \text{for } x \in B.$$

- Here $c = 2^\alpha \frac{\Gamma\left(\frac{\alpha}{2} + n + 1\right)\Gamma\left(\frac{d+2\ell+\alpha}{2} + n\right)}{n! \Gamma\left(\frac{d+2\ell}{2} + n\right)}$.
- For $\ell = 0$, $c$ is an upper bound for radial eigenvalues!
Theorem (K, KMR, KK)

For $x > 0$ let

$$F(x) = \sin(x + \frac{(2-\alpha)\pi}{8}) - \int_0^\infty e^{-xs} \Phi(s) \, ds,$$

where

$$\Phi(s) = \frac{\sqrt{2\alpha} \sin \frac{\alpha\pi}{2}}{2\pi} \frac{s^\alpha}{1 + s^{2\alpha} - 2s^\alpha \cos \frac{\alpha\pi}{2}}$$

$$\times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{1}{1 + r^2} \log \frac{1 - s^2r^2}{1 - s^\alpha r^\alpha} \, dr \right)$$

$$= \frac{\sqrt{\alpha} S_2(-\frac{\alpha}{2})}{4\pi} s^{\alpha/4-1/2} |S_2(\alpha; 1 + \alpha + \frac{\alpha}{4} + \frac{i\alpha \log s}{2\pi}; \alpha)|^2,$$

and $F(x) = 0$ for $x \leq 0$. Then $LF(x) = F(x)$ for $x > 0$.

- Due to scaling, $L[F(\lambda x)] = \lambda^\alpha F(\lambda x)$.
- $S_2$ is the Koyama–Kurokawa’s double sine function.
M. Kwaśnicki
*Spectral analysis of subordinate Brownian motions on the half-line*

M. Kwaśnicki, J. Małecki, M. Ryznar
*First passage times for subordinate Brownian motions*

A. Kuznetsov, M. Kwaśnicki
*Spectral analysis of stable processes on the positive half-line*
arXiv:1509.06435

- Extends to general symmetric operators with completely monotone kernels.
- Extends to **non-symmetric** fractional derivatives!
• In the non-symmetric case $F$ has exponential decay or growth:

$$F(x) = e^{ax} \sin(bx + \theta) + \int_0^\infty e^{-xs} \Phi(s) \, ds,$$

where $a = \cos(\pi \rho)$, $b = \sin(\pi \rho)$, $\theta = \frac{1}{2} \pi \rho (1 - \alpha + \alpha \rho)$,

$$\Phi(s) = s^{\alpha \rho / 2 - 1 / 2} |S_2(\alpha; 1 + \frac{3\alpha}{2} - \frac{\alpha \rho}{2} + \frac{i\alpha \log s}{2\pi}; \alpha)|^2,$$

• Still, it gives rise to a Fourier-type transform!

• Describes the spectral resolution for $L$ in $(0, \infty)$: e.g. heat kernel of $L$ in $(0, \infty)$ can be written in terms of $F$.

• Application: explicit expression for the supremum of a stable Lévy process.
P. Biler, C. Imbert, G. Karch
Barenblatt profiles for a nonlocal porous medium equation

R. M. Blumenthal, R. K. Getoor, D. B. Ray
On the distribution of first hits for the symmetric stable processes

K. Bogdan, T. Žak
On Kelvin transformation

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Fractional calculus for power functions and eigenvalues of the fractional Laplacian
F. Hmissi
*Fonctions harmoniques pour les potentiels de Riesz sur la boule unité*

M. Kac
*Some remarks on stable processes*

S. Samko
*Hypersingular Integrals and their applications*

A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev
*Integrals and Series, Vol. 3: More Special Functions*
Gordon and Breach, 1989