Błądzenie losowe

mateusz.kwasnicki@pwr.wroc.pl

Określamy **ciąg losowy**

$$S_n = X_1 + X_2 + X_3 + \ldots + X_n.$$

$X_1, X_2, X_3, \ldots$ — wyniki **niezależnych** rzutów monetą,

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Niezależność:

$$\mathbf{P}(X_1 = a_1,\ X_2 = a_2,\ \ldots,\ X_n = a_n) = \frac{1}{2^n}.$$

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S_n przyjmuje wartości n, $n-2$, $n-4$, ..., $-n$;

$$\mathbf{P}(S_n = k) = \mathbf{P}\left(\text{w } n \text{ rzutach było } \frac{n+k}{2} \text{ sukcesów}\right).$$

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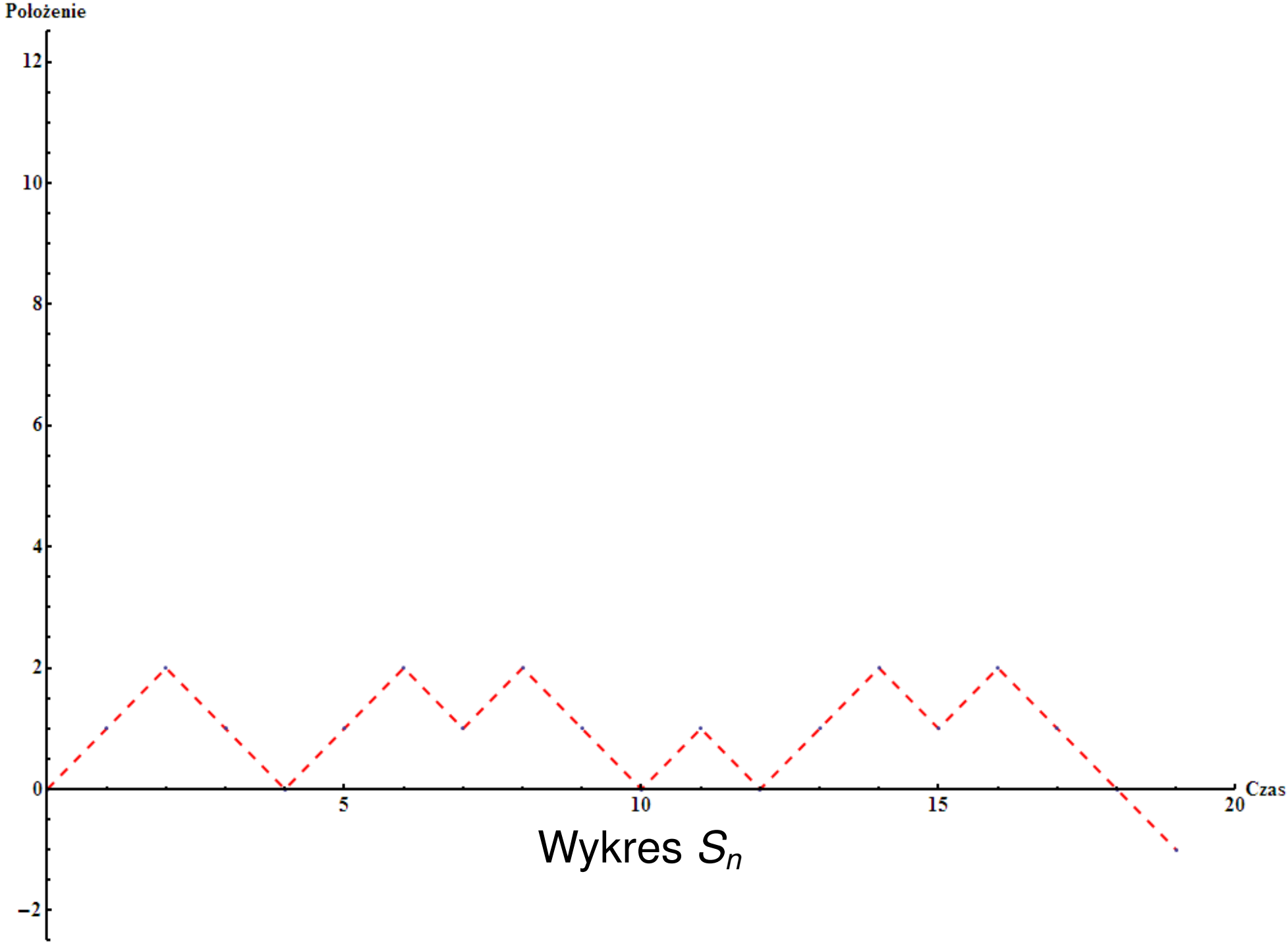
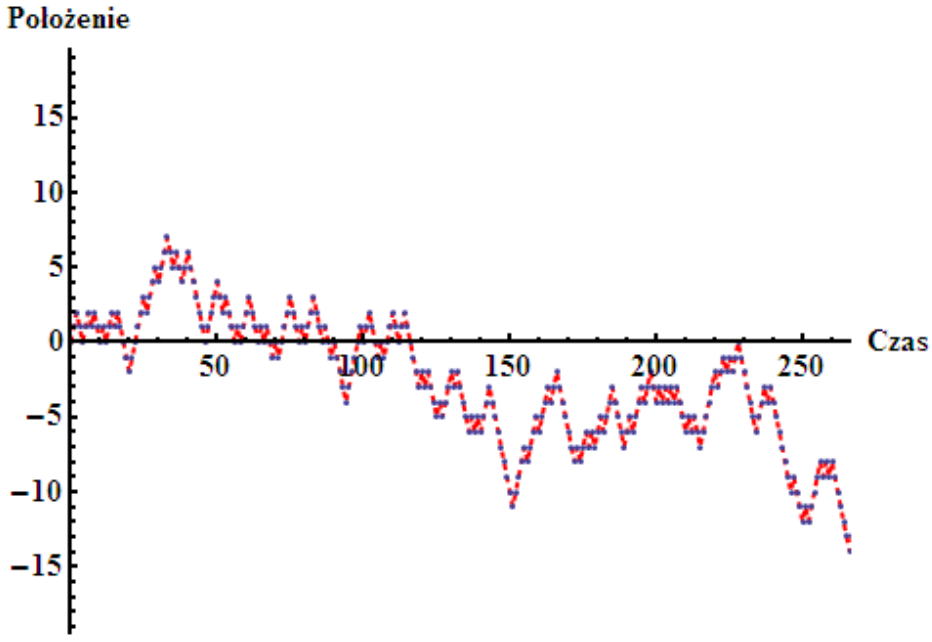
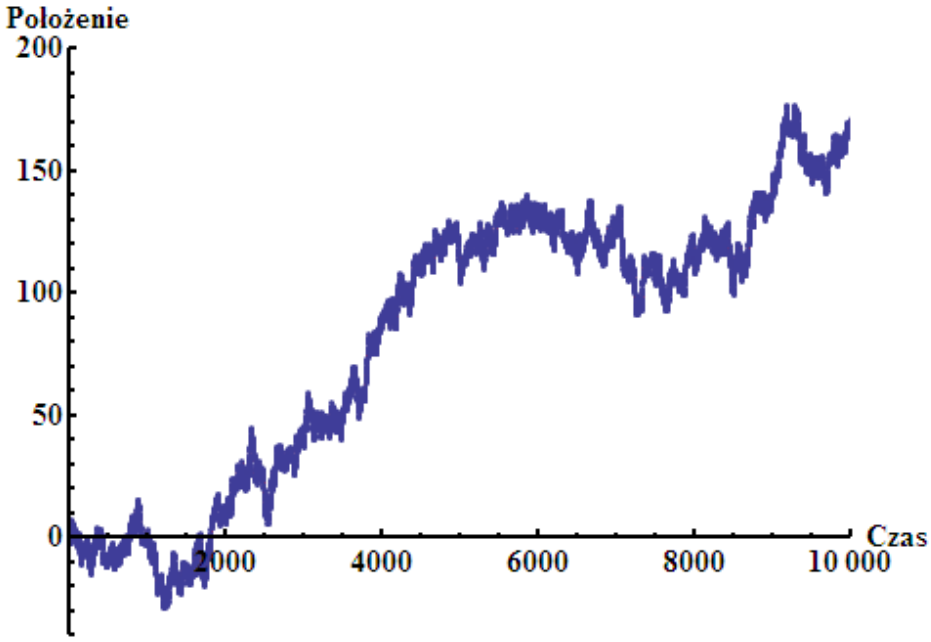
Pytanie: Co można powiedzieć o S_n dla dużych n?



Wykres S_n



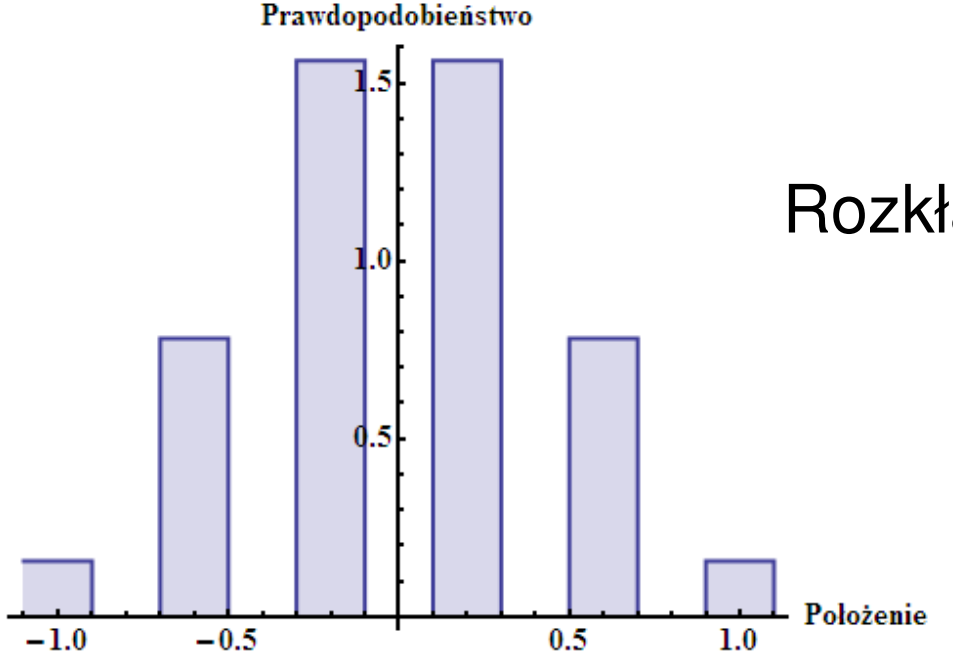


Obserwacje:

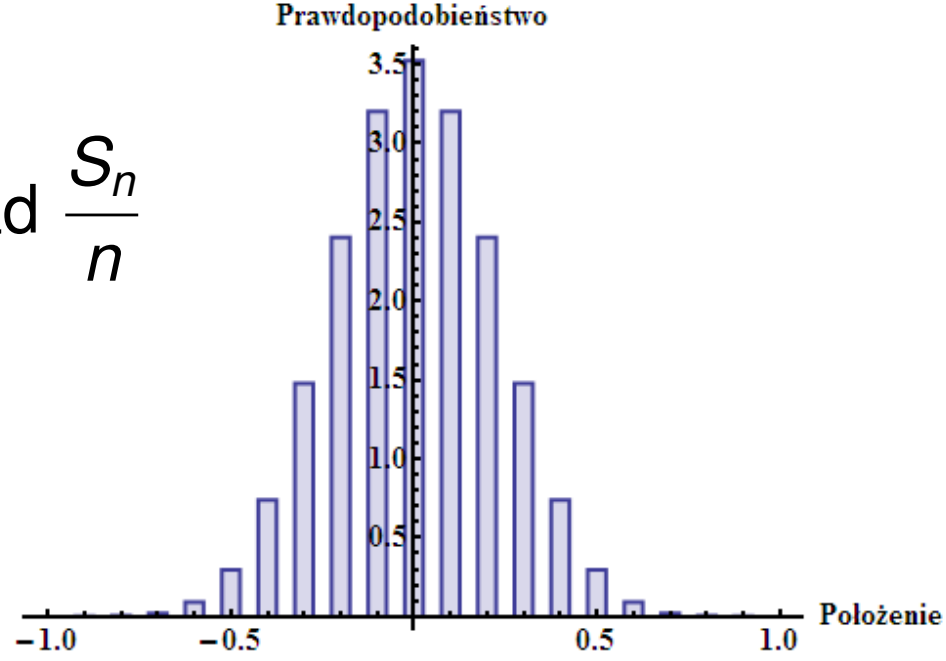
$$\frac{S_n}{n} \approx 0$$ dla dużych n.

$$\frac{S_n}{\sqrt{n}} \approx \frac{S_k}{\sqrt{k}}$$ dla dużych n, k.

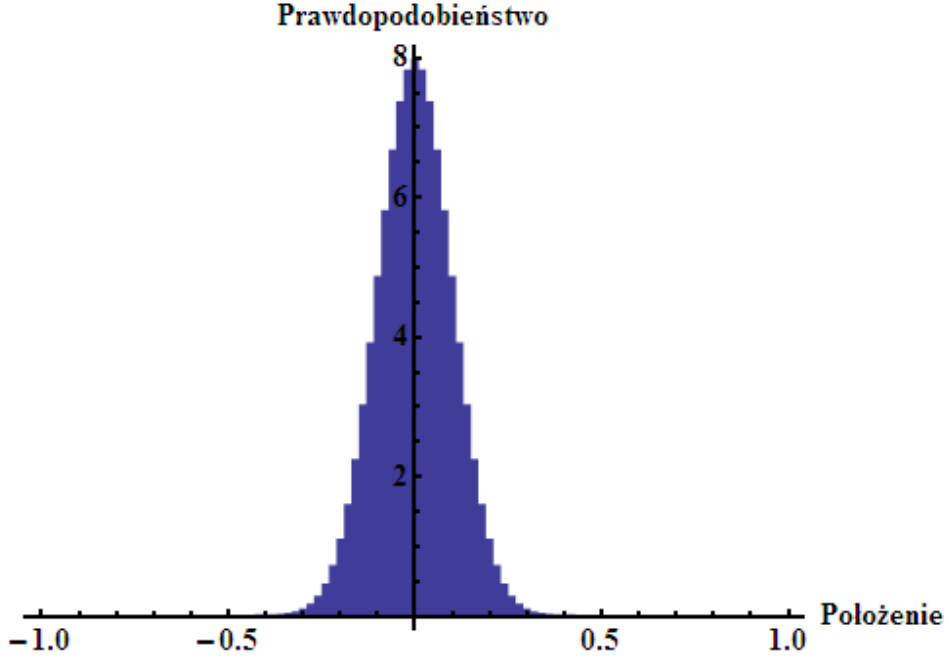
Rozkład $\frac{S_n}{n}$



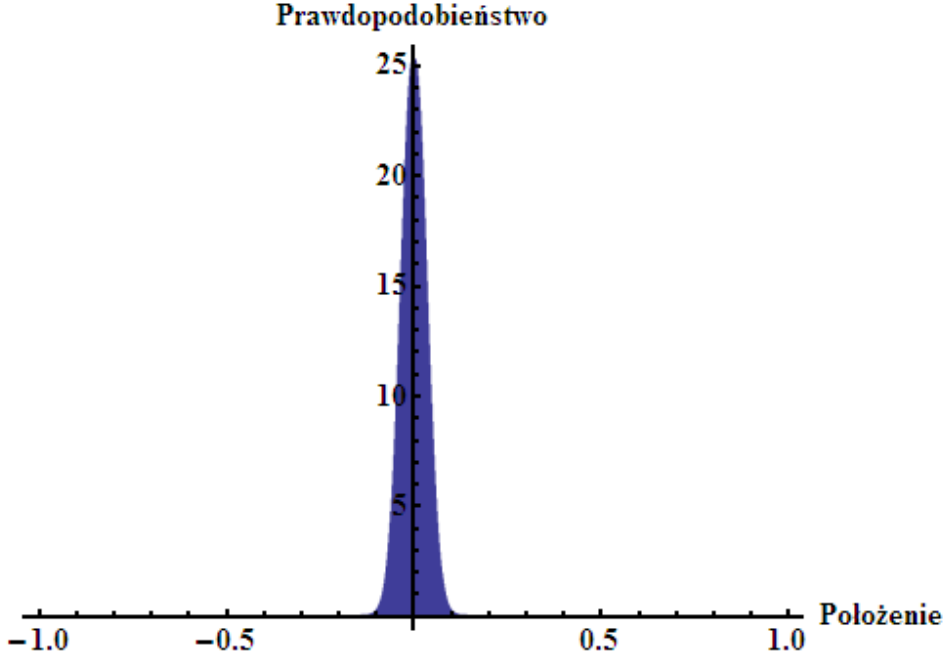
$n = 5$



$n = 20$

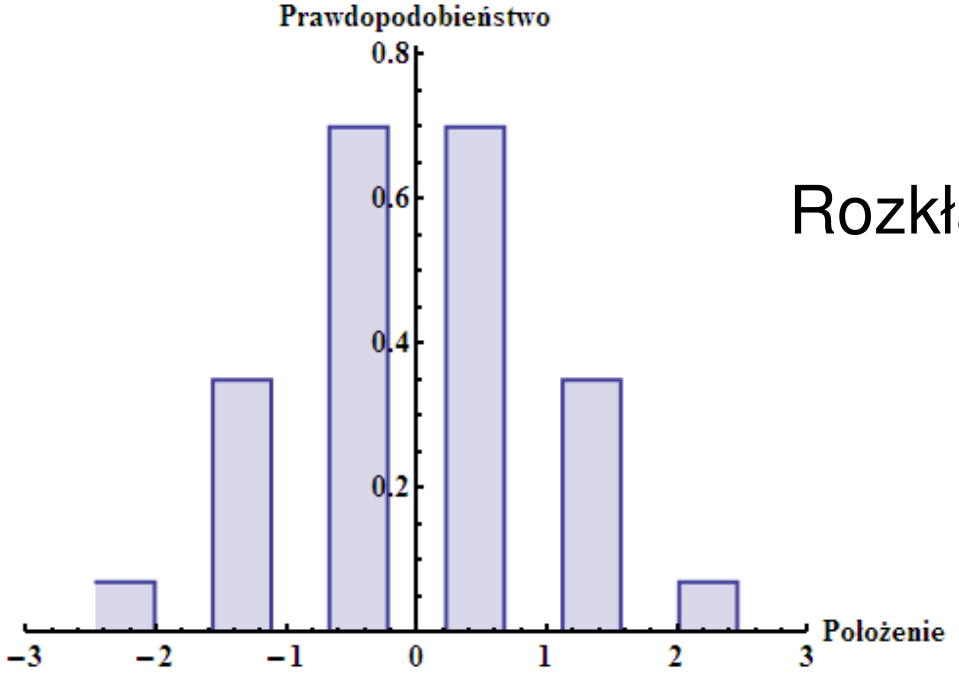


$n = 100$



$n = 1000$

Rozkład $\frac{S_n}{\sqrt{n}}$



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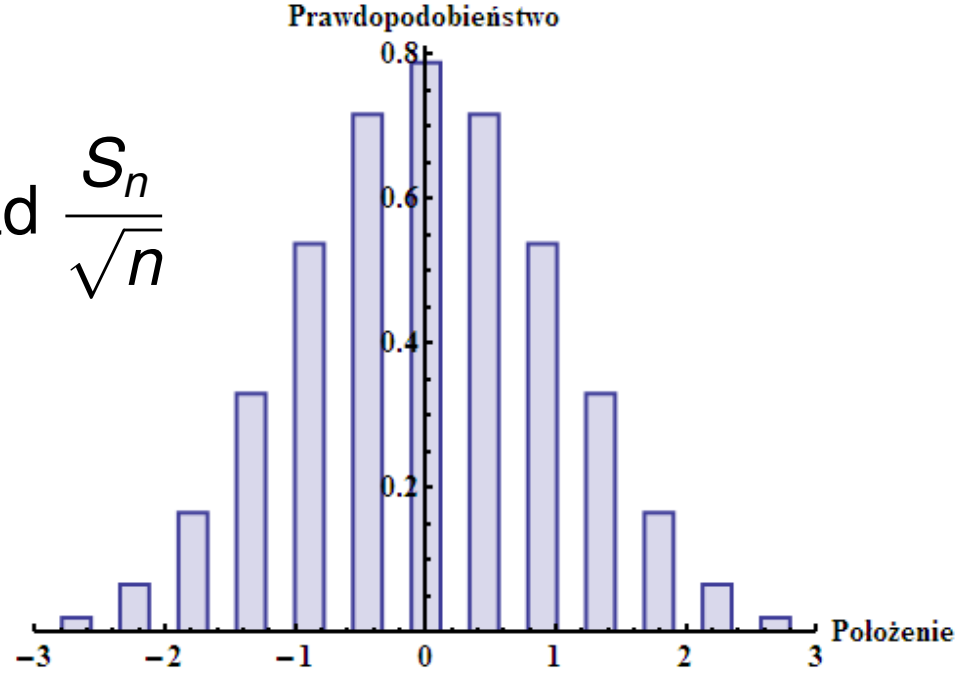
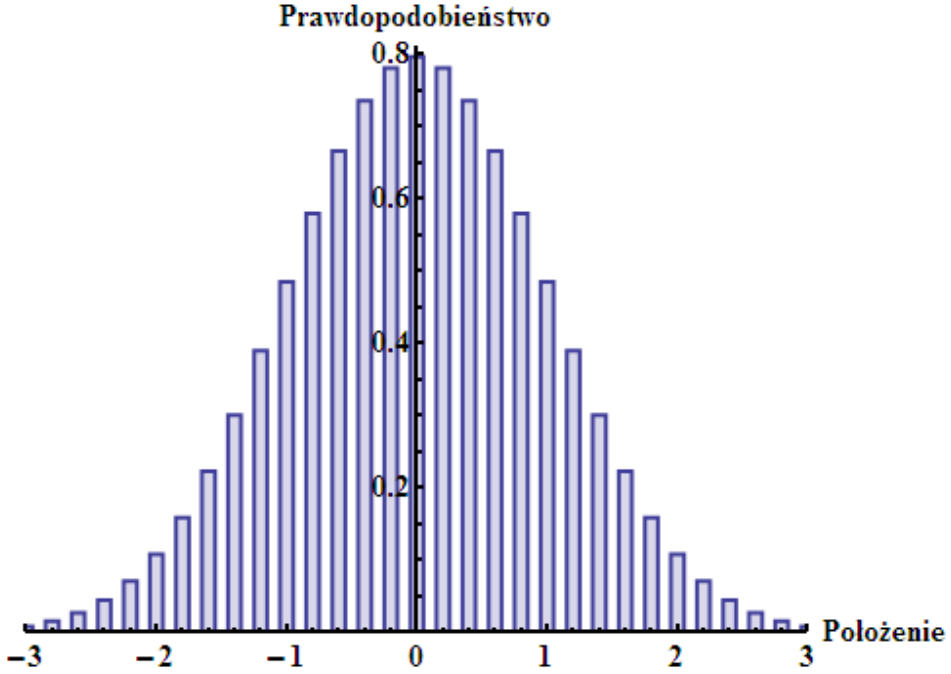


$n = 20$



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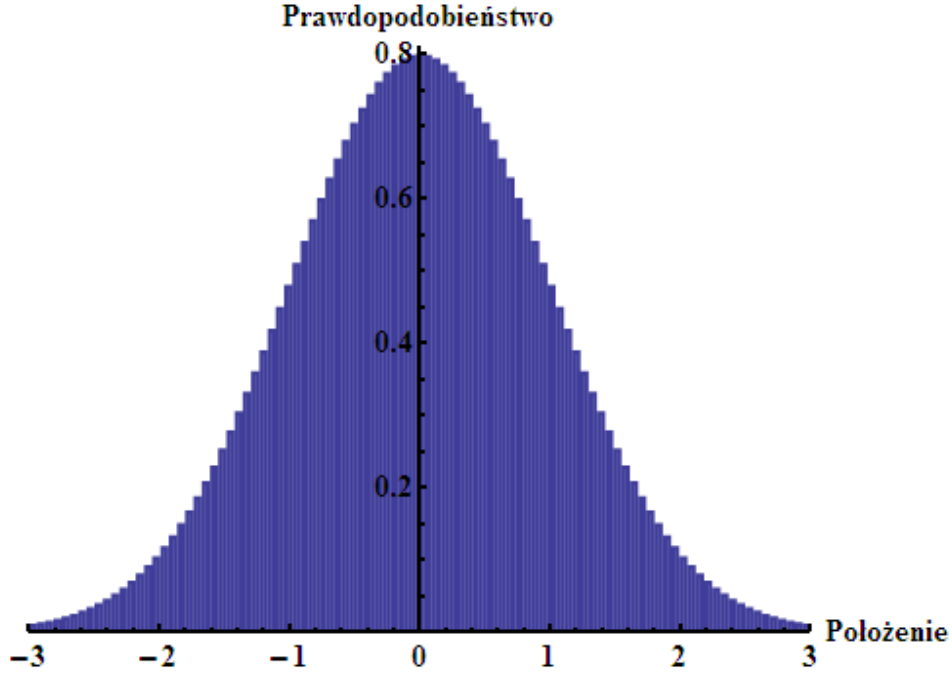

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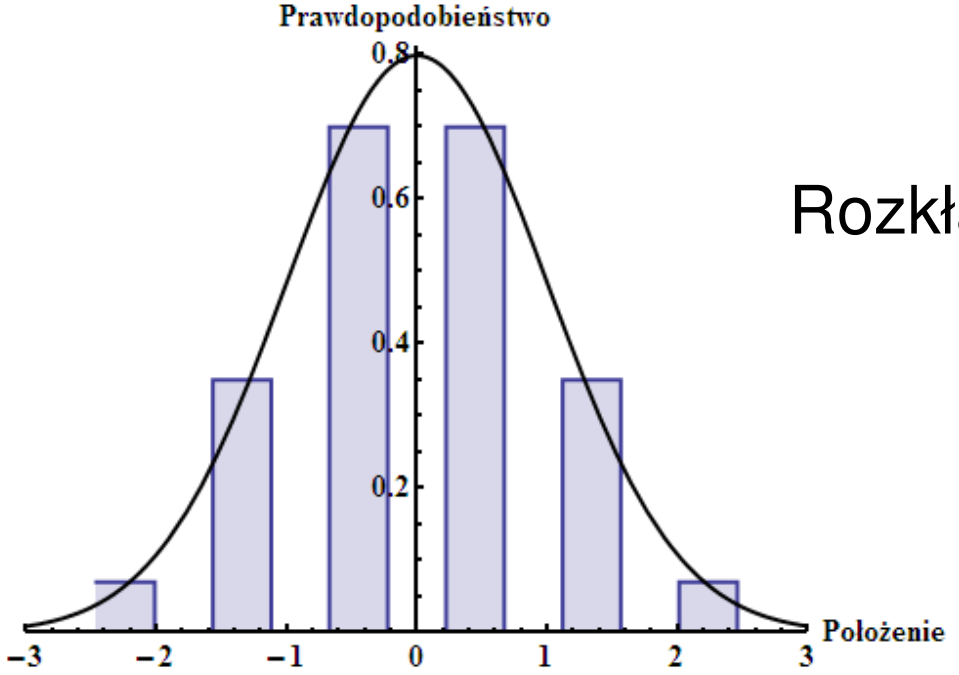

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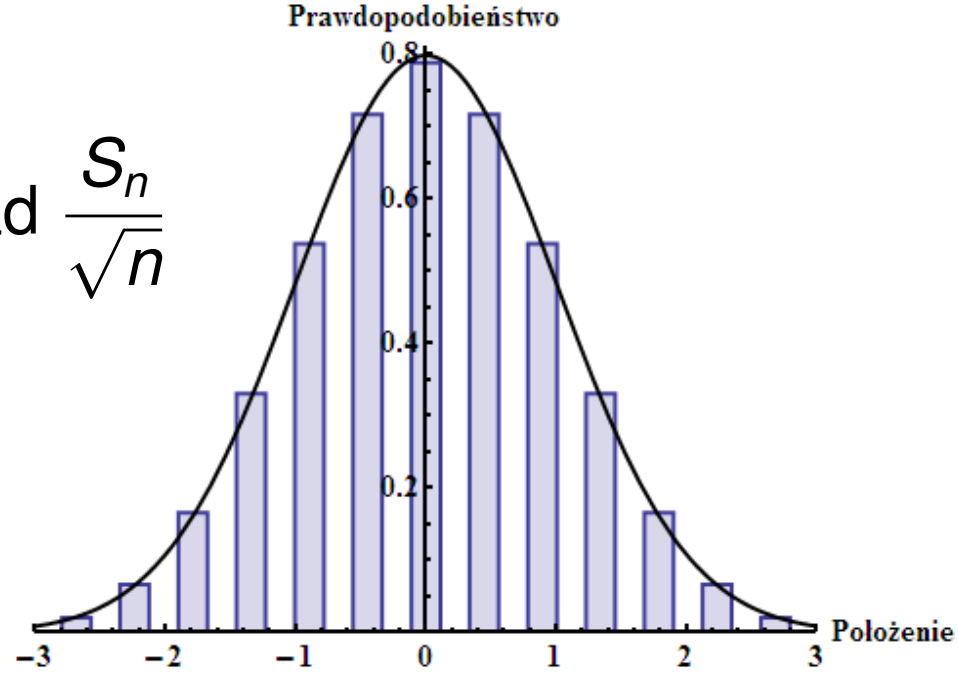

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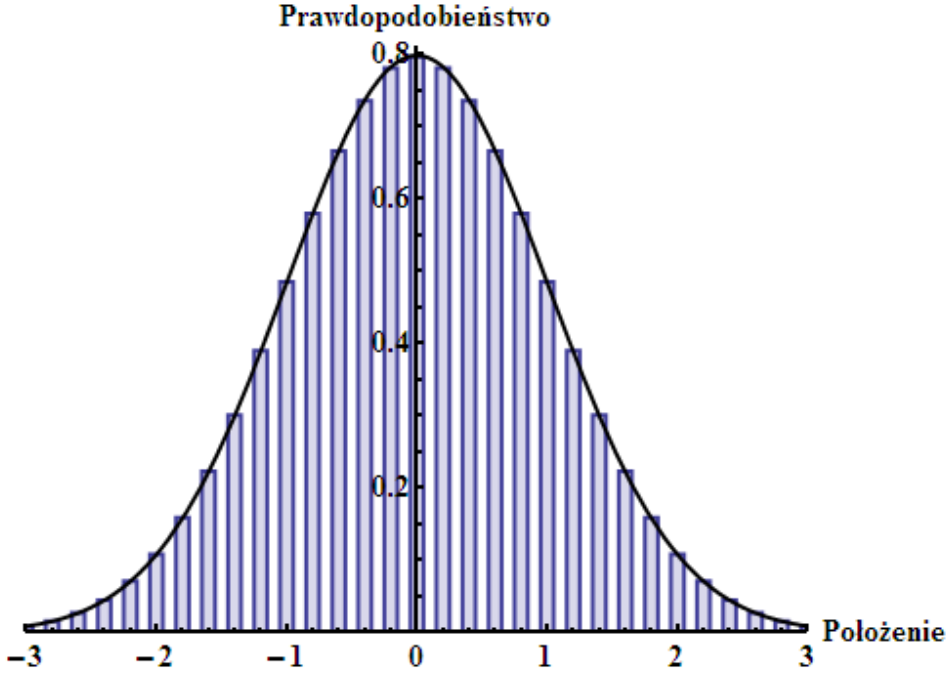

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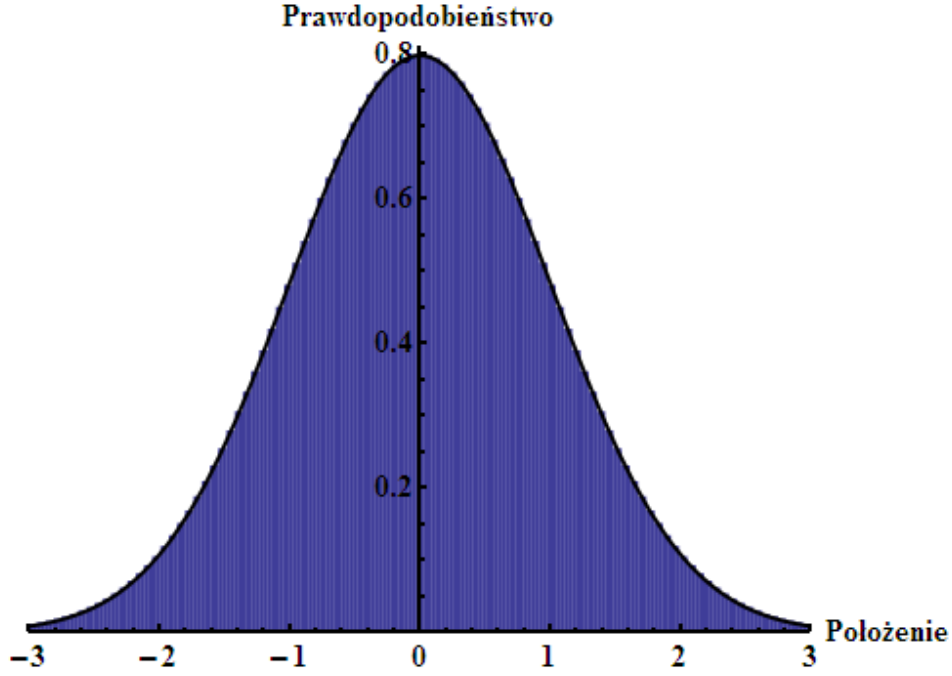

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Twierdzenie: Zachodzi

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{S_n}{\sqrt{n}} \in (a,b)\right) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

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Dowód: Dla dużych n,

$$\mathbf{P}(S_n = k) \approx \frac{2}{\sqrt{2\pi n}} e^{-\frac{k^2}{2n}} = \frac{2}{\sqrt{n}} f\left(\frac{k}{\sqrt{n}}\right)$$

(o ile $2 \big| k + n$, w przeciwnym razie $\mathbf{P}(S_n = k) = 0$).

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$$\left[n! \approx \sqrt{2\pi n}\,\frac{n^n}{e^n}\,, \qquad \binom{n}{\frac{n+k}{2}} \frac{1}{2^n} \approx \frac{1}{\sqrt{2\pi n\left(1-\frac{k^2}{n^2}\right)}} \frac{1}{\left(1+\frac{k}{n}\right)^k \left(1-\frac{k^2}{n^2}\right)^{\frac{n-k}{2}}} \right]$$

$$\begin{aligned}
\mathbf{P}\left(\frac{S_n}{\sqrt{n}} \in (a,b)\right) &= \mathbf{P}\left(S_n \in (a\sqrt{n}, b\sqrt{n})\right) \\
&= \sum_{k \in (a\sqrt{n}, b\sqrt{n})} \mathbf{P}(S_n = k) \\
&\approx \sum_{\substack{k \in (a\sqrt{n}, b\sqrt{n}) \\ 2 \mid k+n}} \frac{2}{\sqrt{n}} f\left(\frac{k}{\sqrt{n}}\right).
\end{aligned}$$

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— suma częściowa całki Riemanna

$$\int_a^b f(x)dx$$

Pytanie: A co jeśli X_1, X_2, X_3, ... są inne?

(X_k — skoki)

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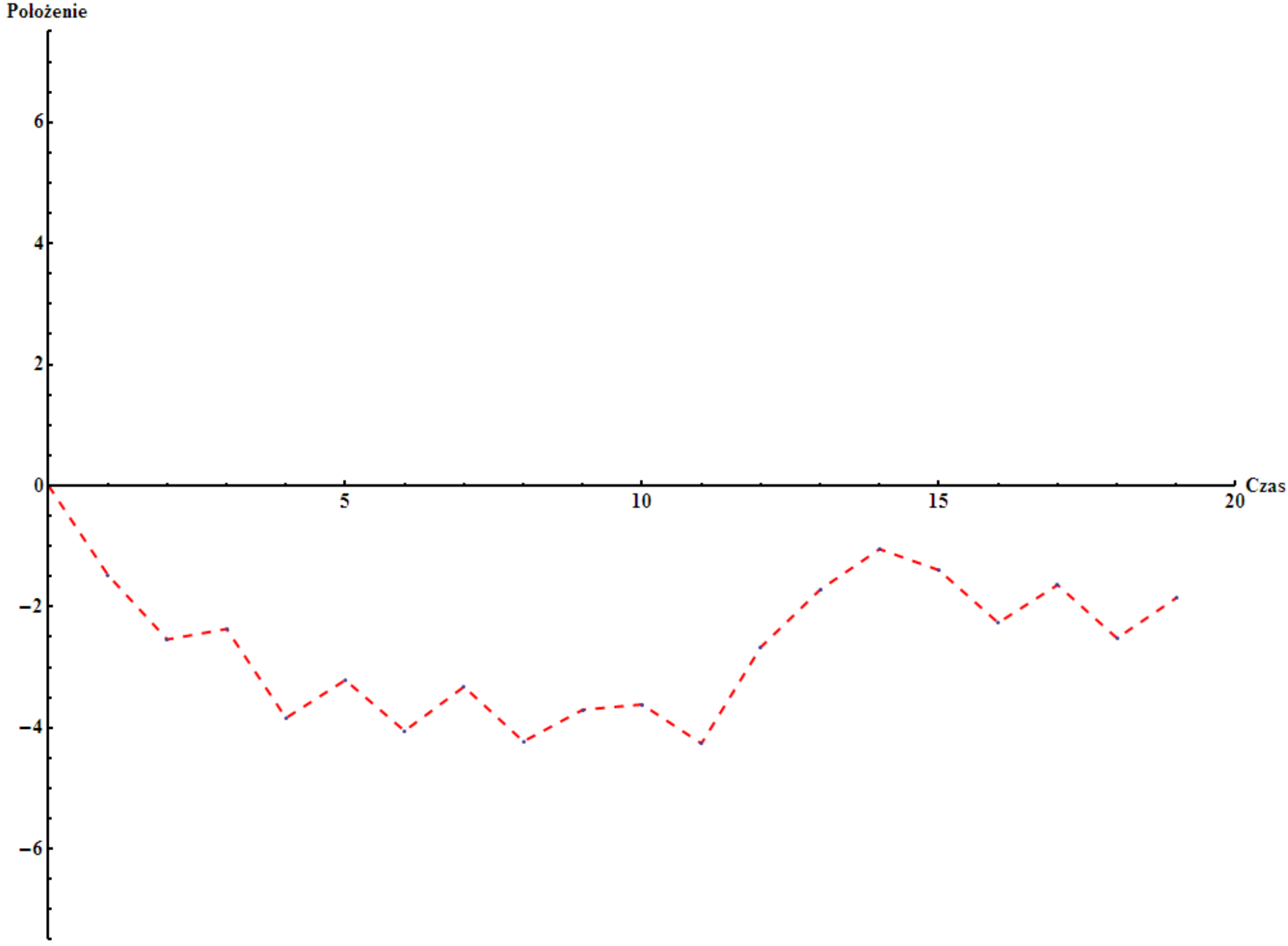
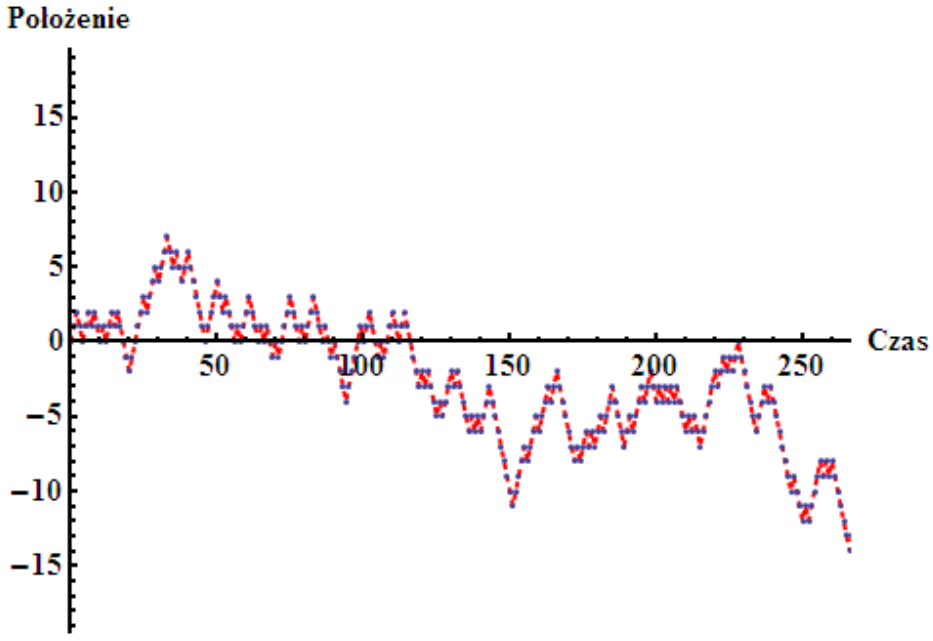
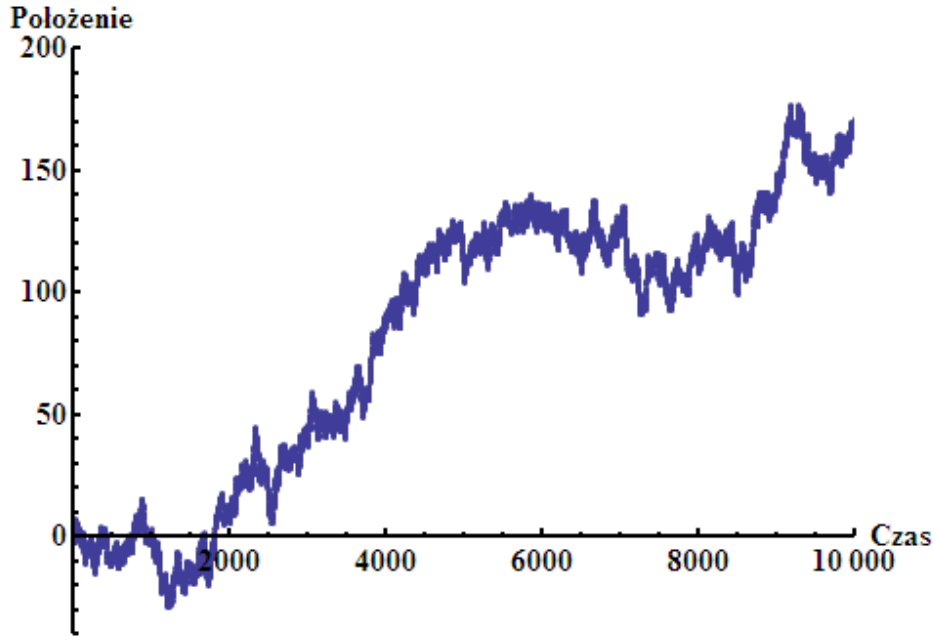
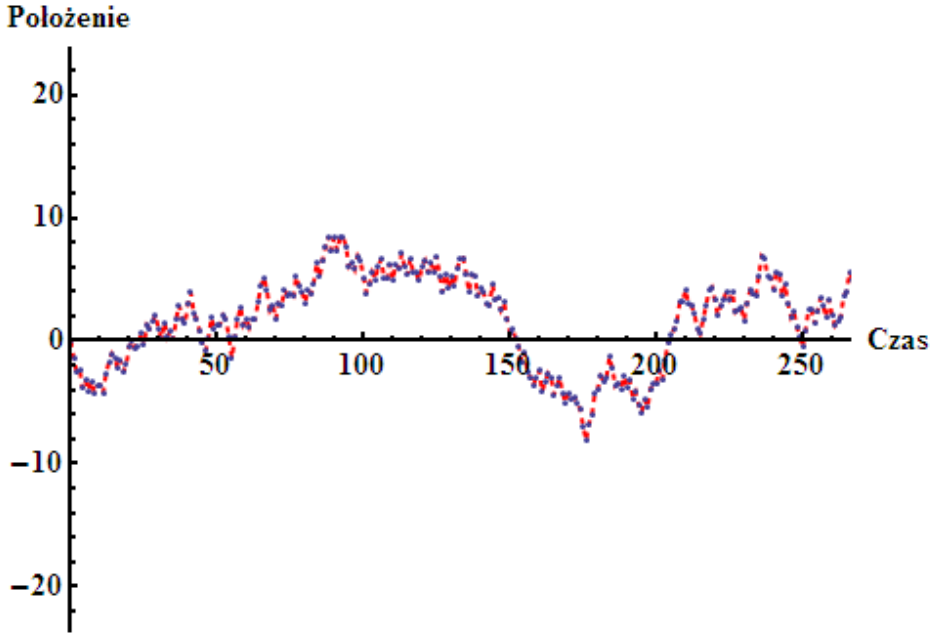
Na przykład: X_k są wybierane losowo z przedziału $[-1, 1]$.

(albo lepiej $[-\sqrt{3}, \sqrt{3}]$)


Położenie
6
4
2
0
−2
−4
−6
5
10
15
20
Czas


Położenie
15
10
5
0
−5
−10
−15
50
100
150
200
250
Czas


Położenie
200
150
100
50
0
2000
4000
6000
8000
10 000
Czas


Położenie
20
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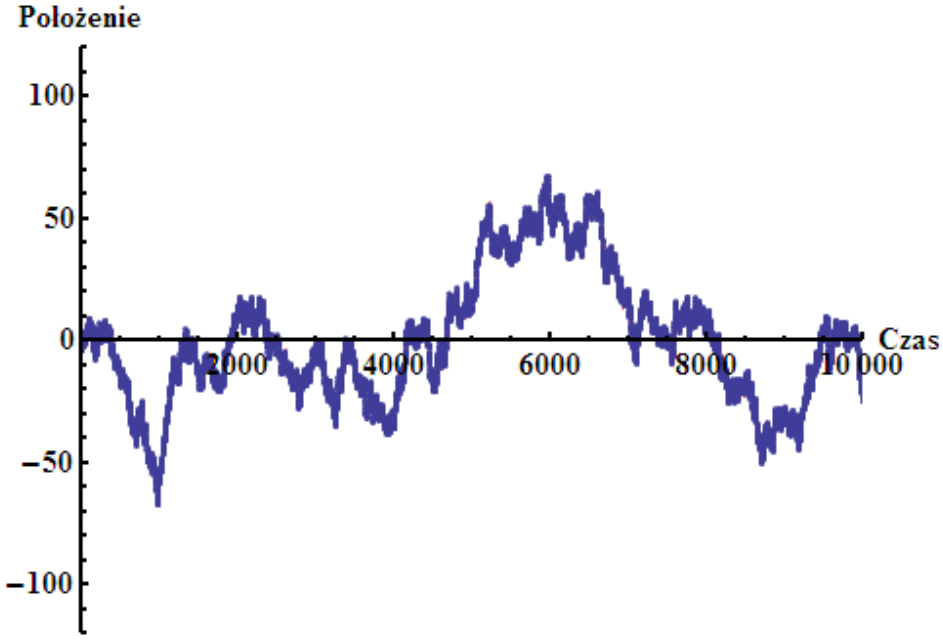
150
200
250
Czas


Położenie
100
50
0
−50
−100
2000
4000
6000
8000
10 000
Czas

Pytanie: Czy granica nie zależy od X_1, X_2, X_3, ...?

(X_k — skoki)

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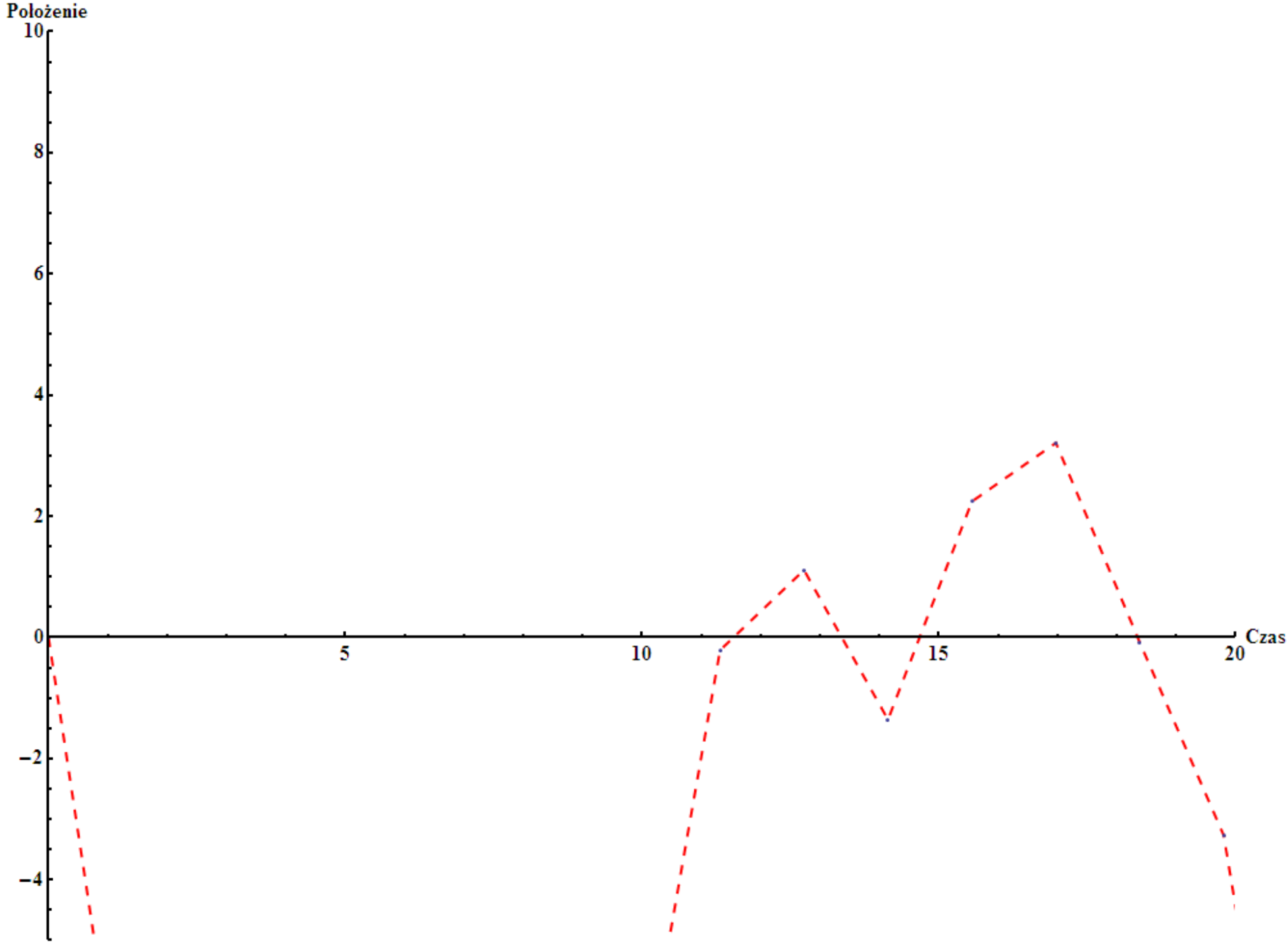
Przykład: Y_k są wybierane losowo z przedziału $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$X_k = \operatorname{tg} Y_k.$$


Położenie
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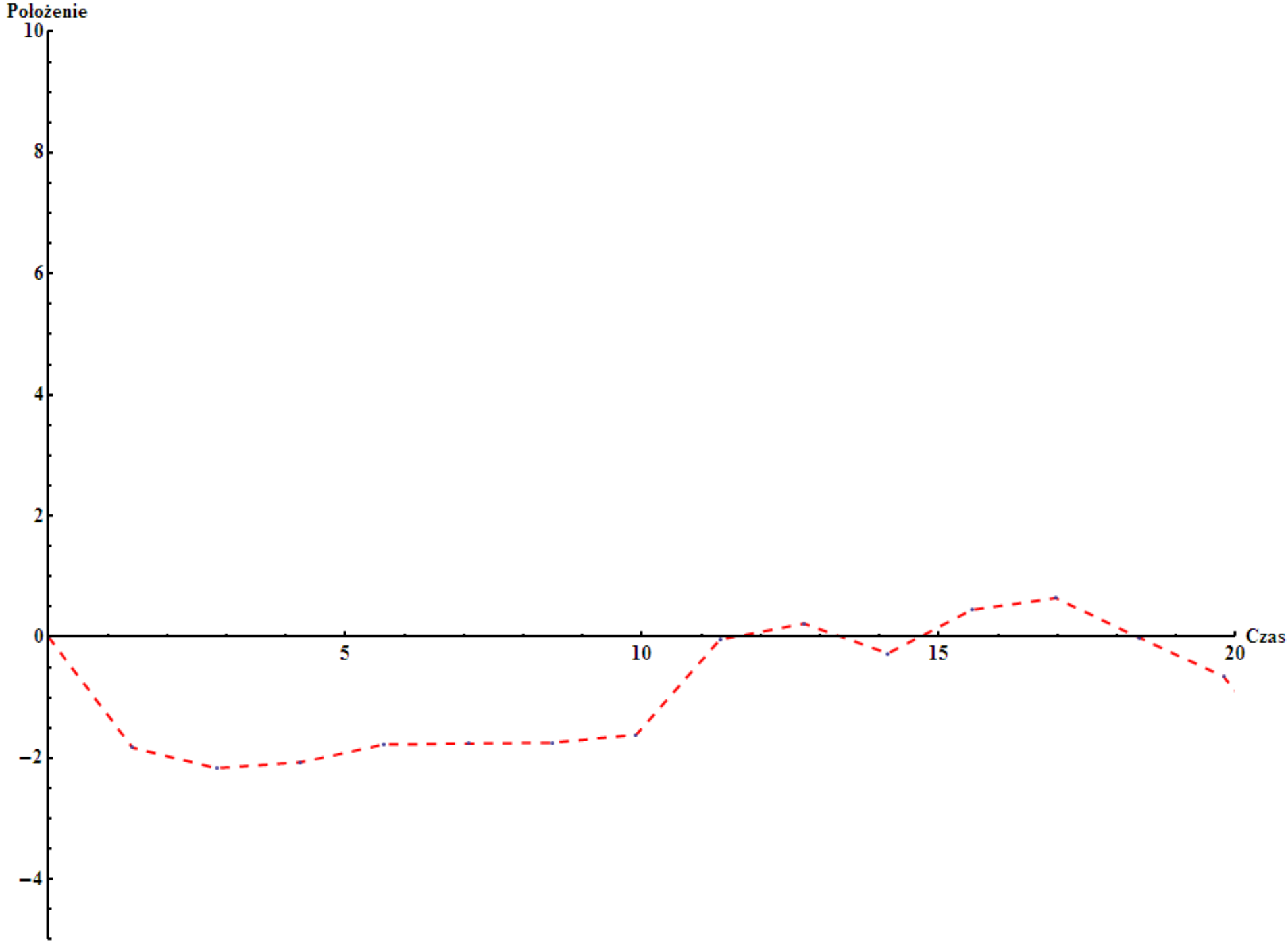
8
6
4
2
0
−2
−4
5
10
15
20
Czas


Położenie
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−4
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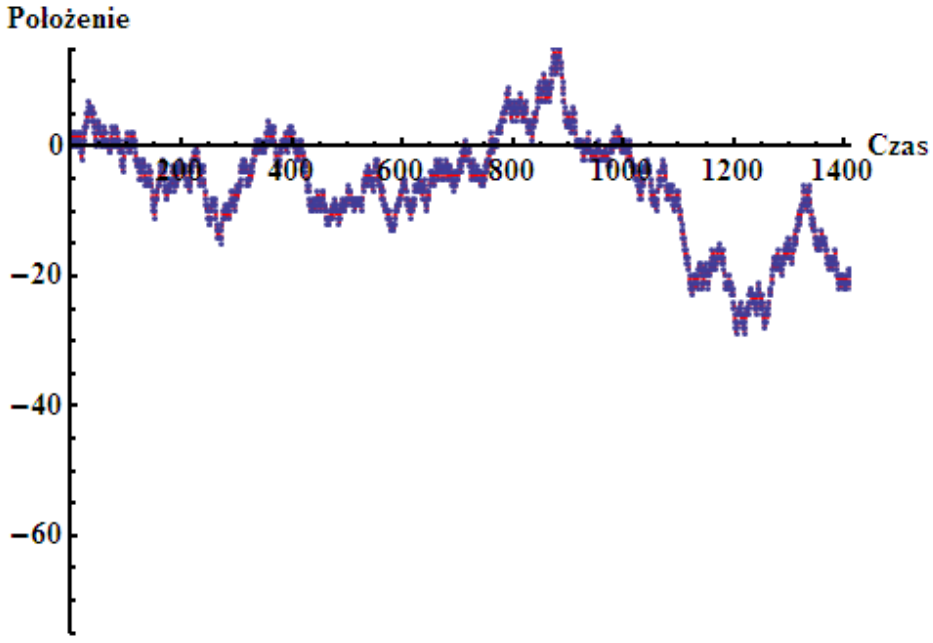
15
20
Czas


Położenie
Czas
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200
400
600
800
1000
1200
1400
−20
−40
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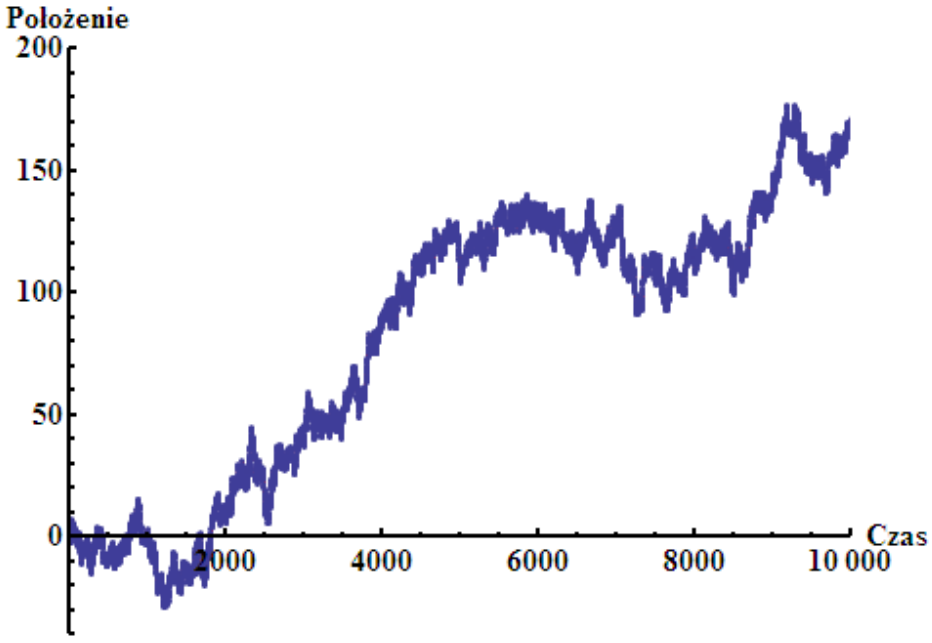
Położenie
Czas
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8000
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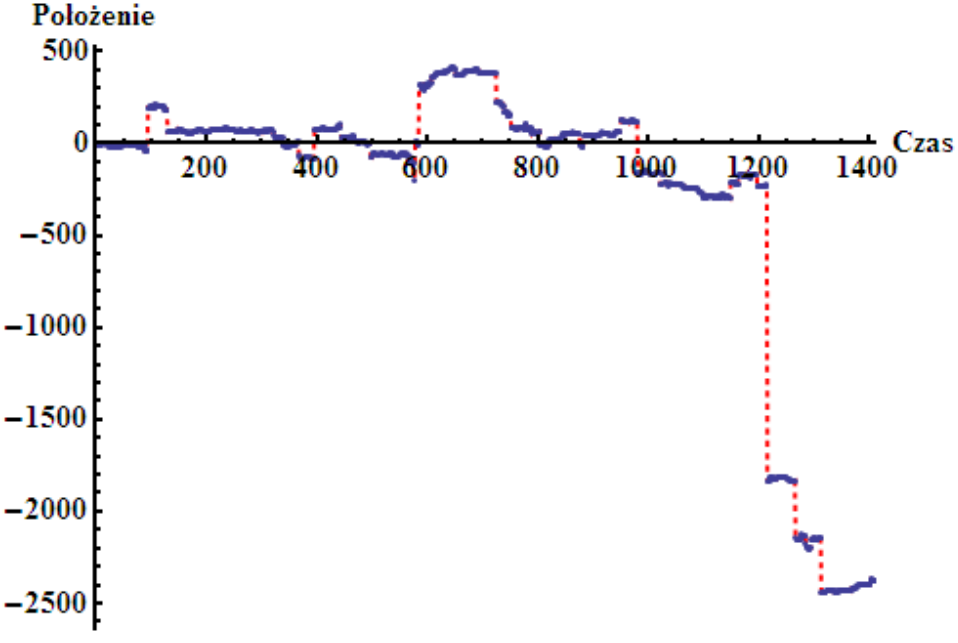
Położenie
Czas
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800
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1400
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−2000
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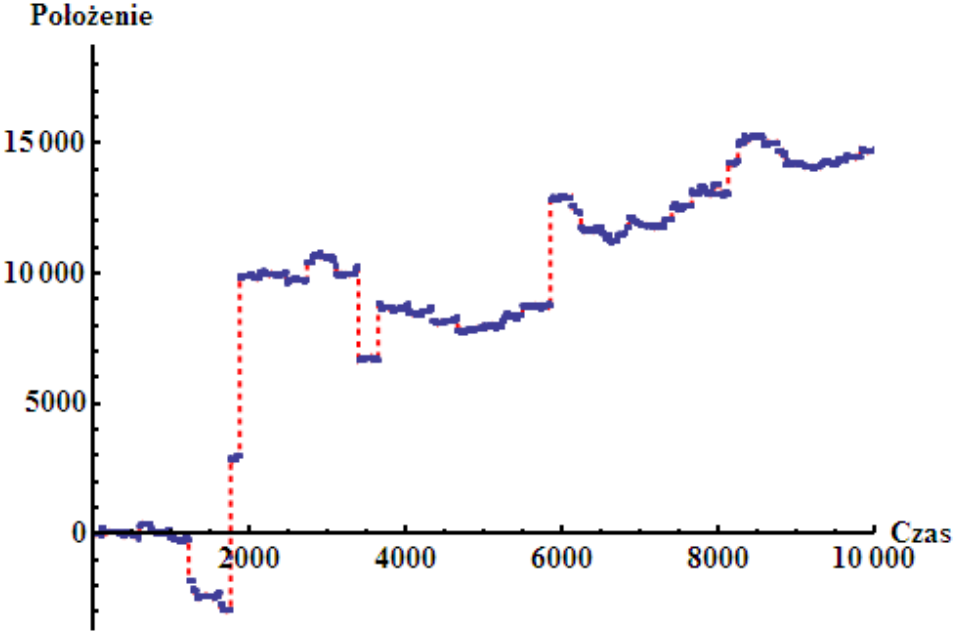
Położenie
Czas
15 000
10 000
5000
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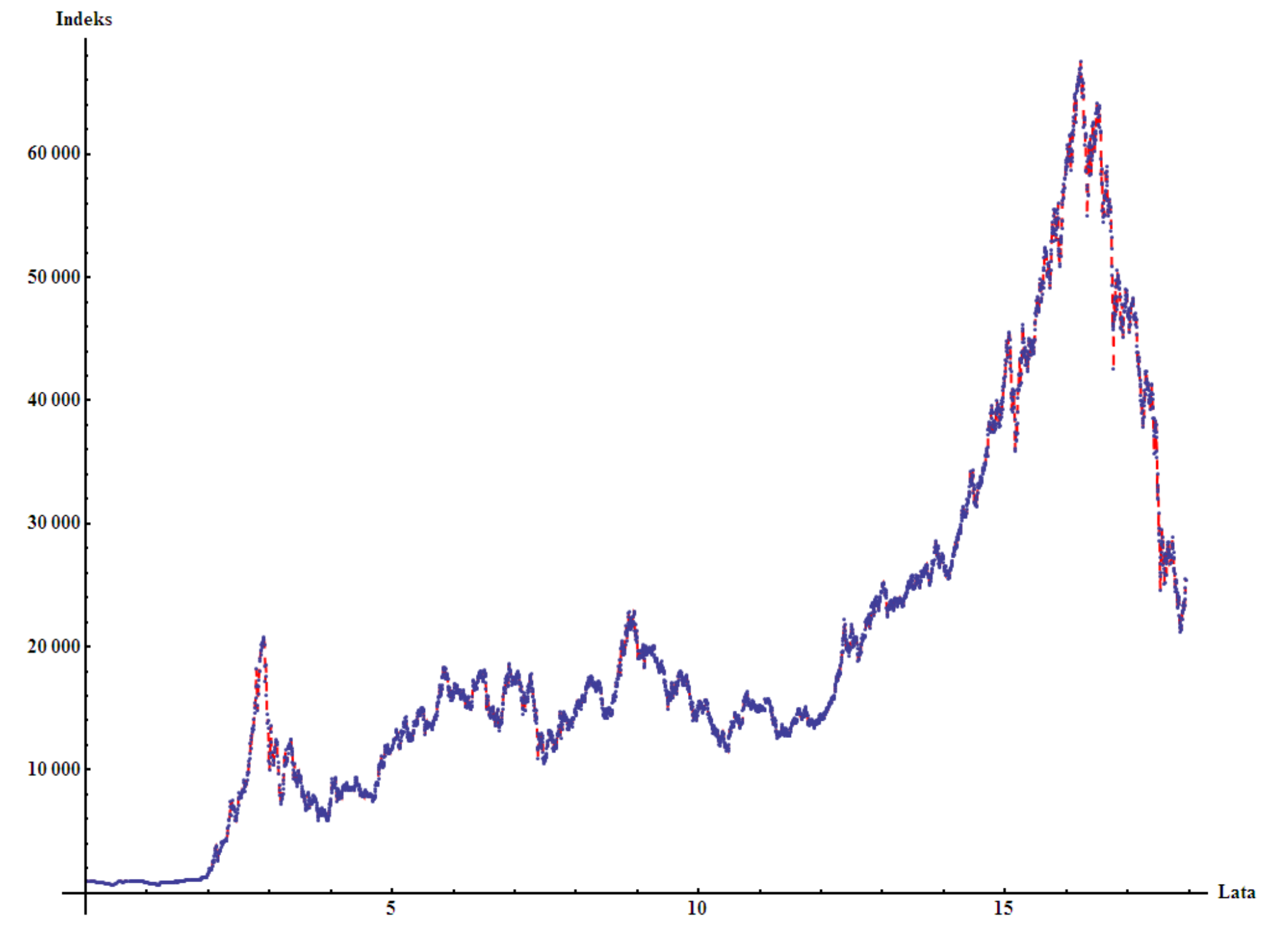
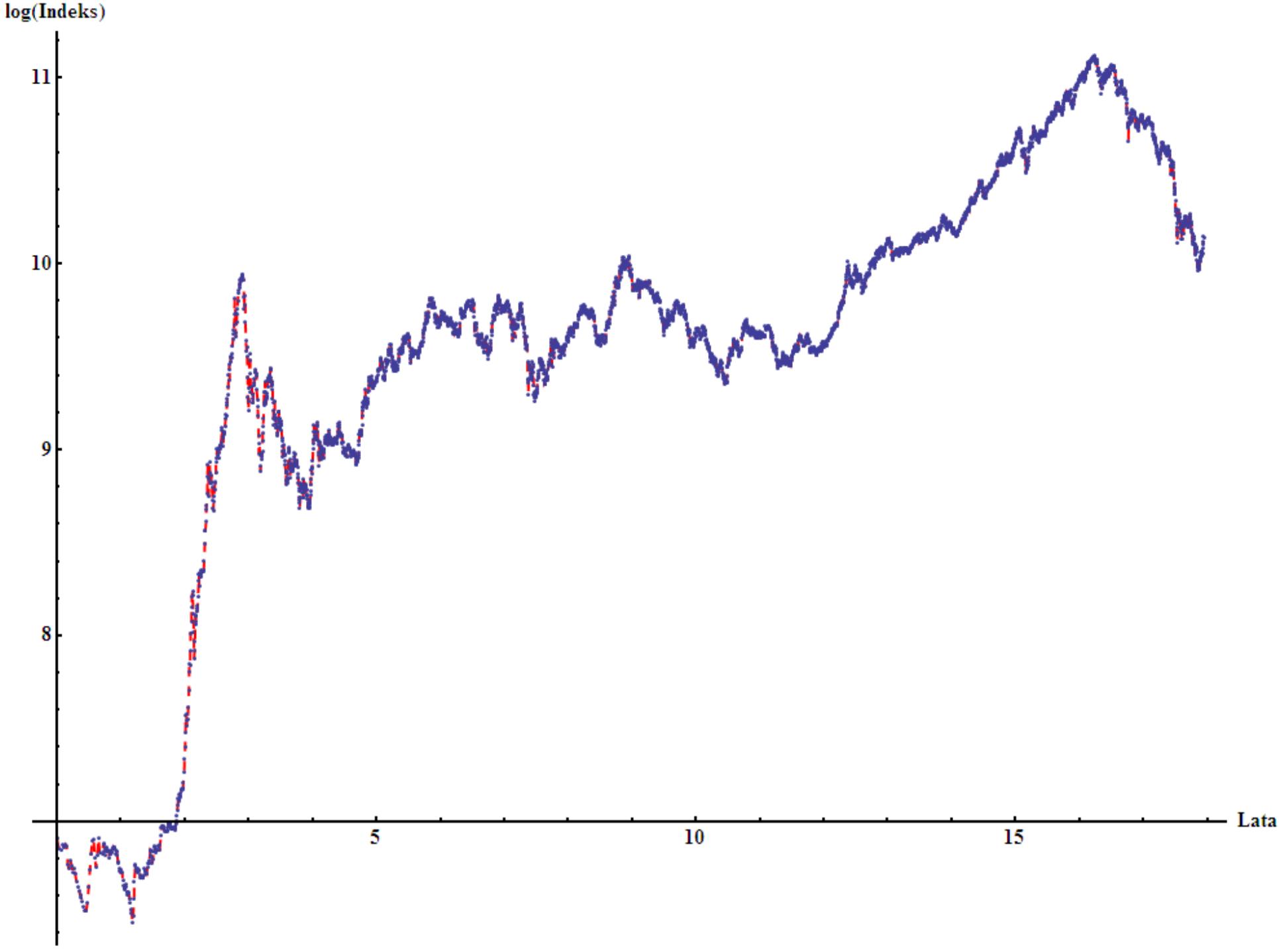
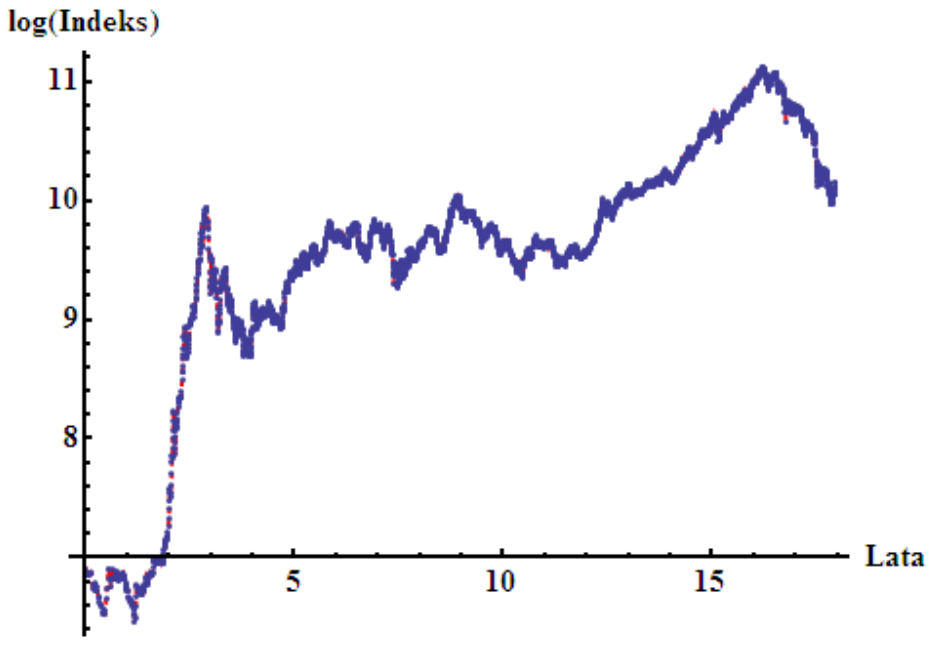
2000
4000
6000
8000
10 000


Indeks
60 000
50 000
40 000
30 000
20 000
10 000
5
10
15
Lata


log(Indeks)
11
10
9
8
5
10
15
Lata


log(Indeks)
11
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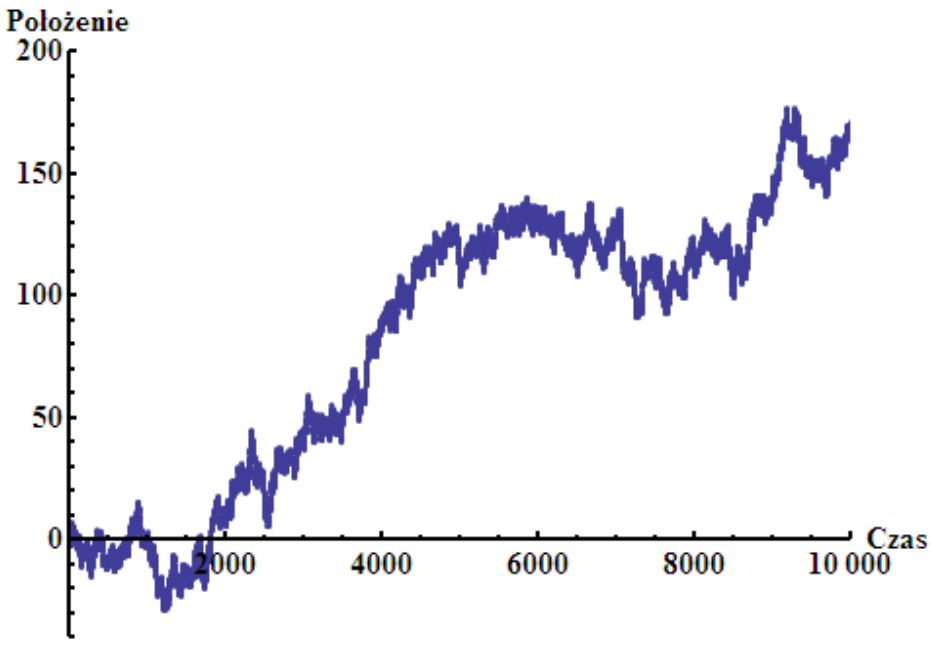
Lata


Położenie
200
150
100
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0
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6000
8000
10 000
Czas

y
6
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2
−2
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−6
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−5
5
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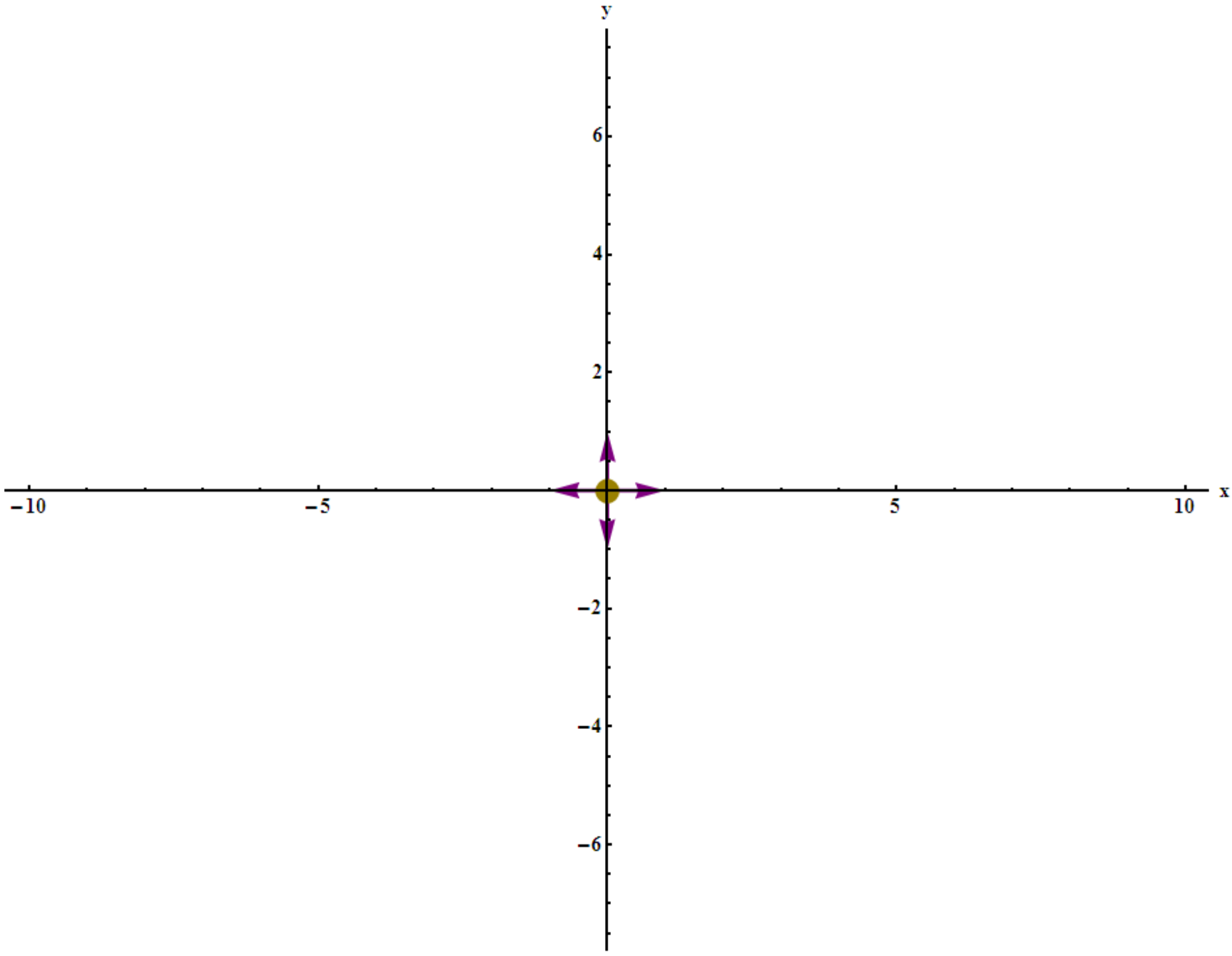
x


y
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−6
−10
−5
5
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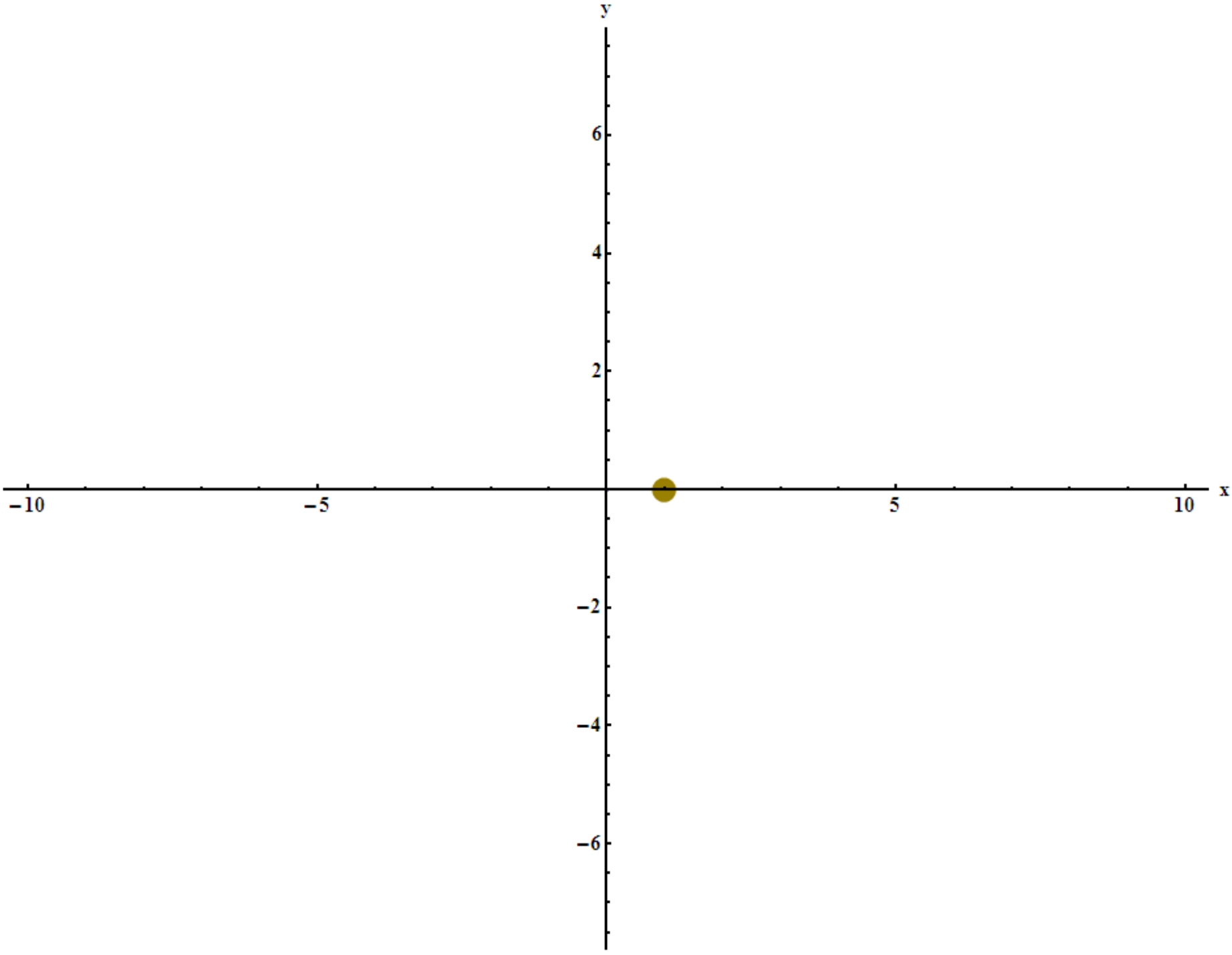
x


y
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2
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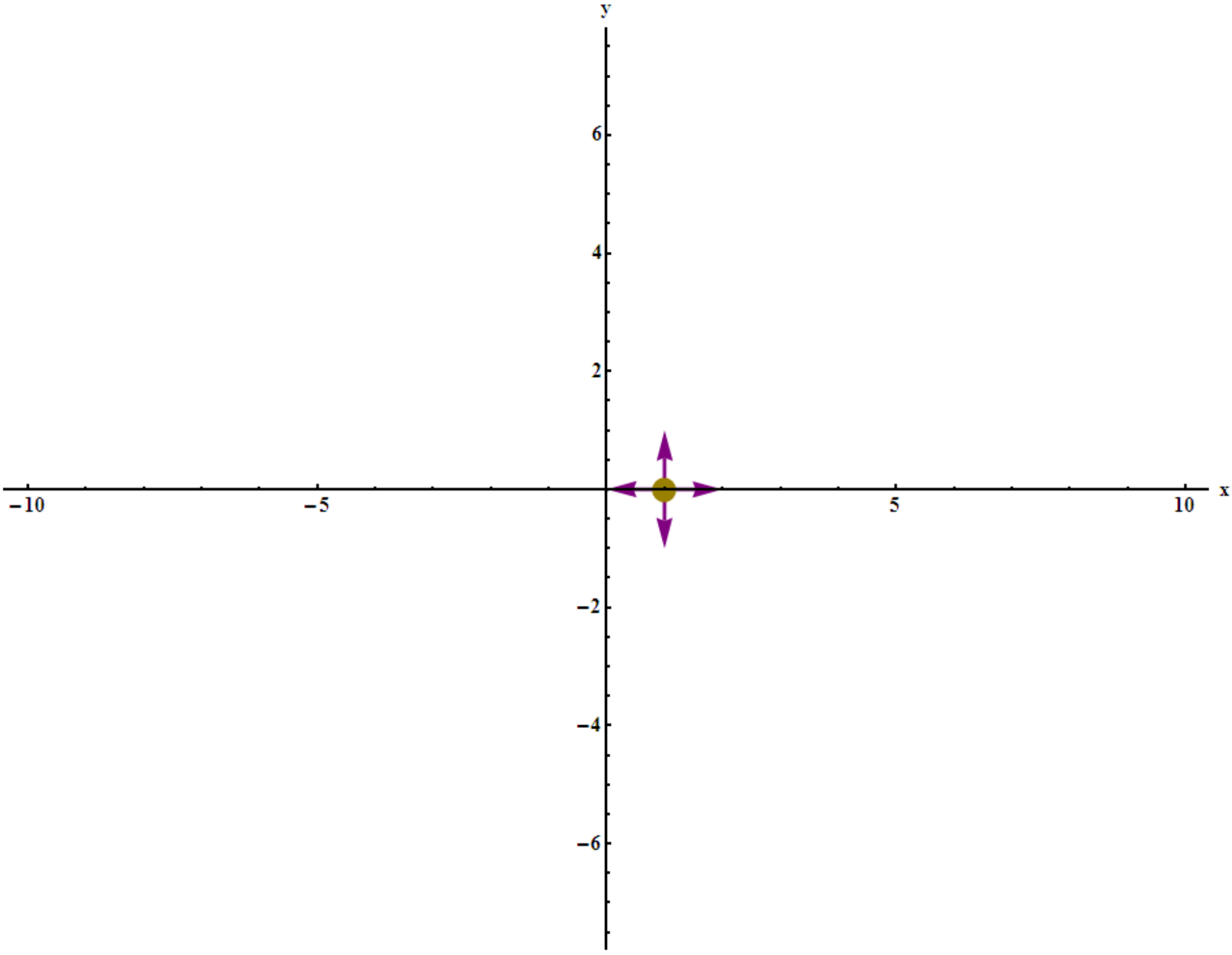
10
x


y
6
4
2
−2
−4
−6
−10
−5
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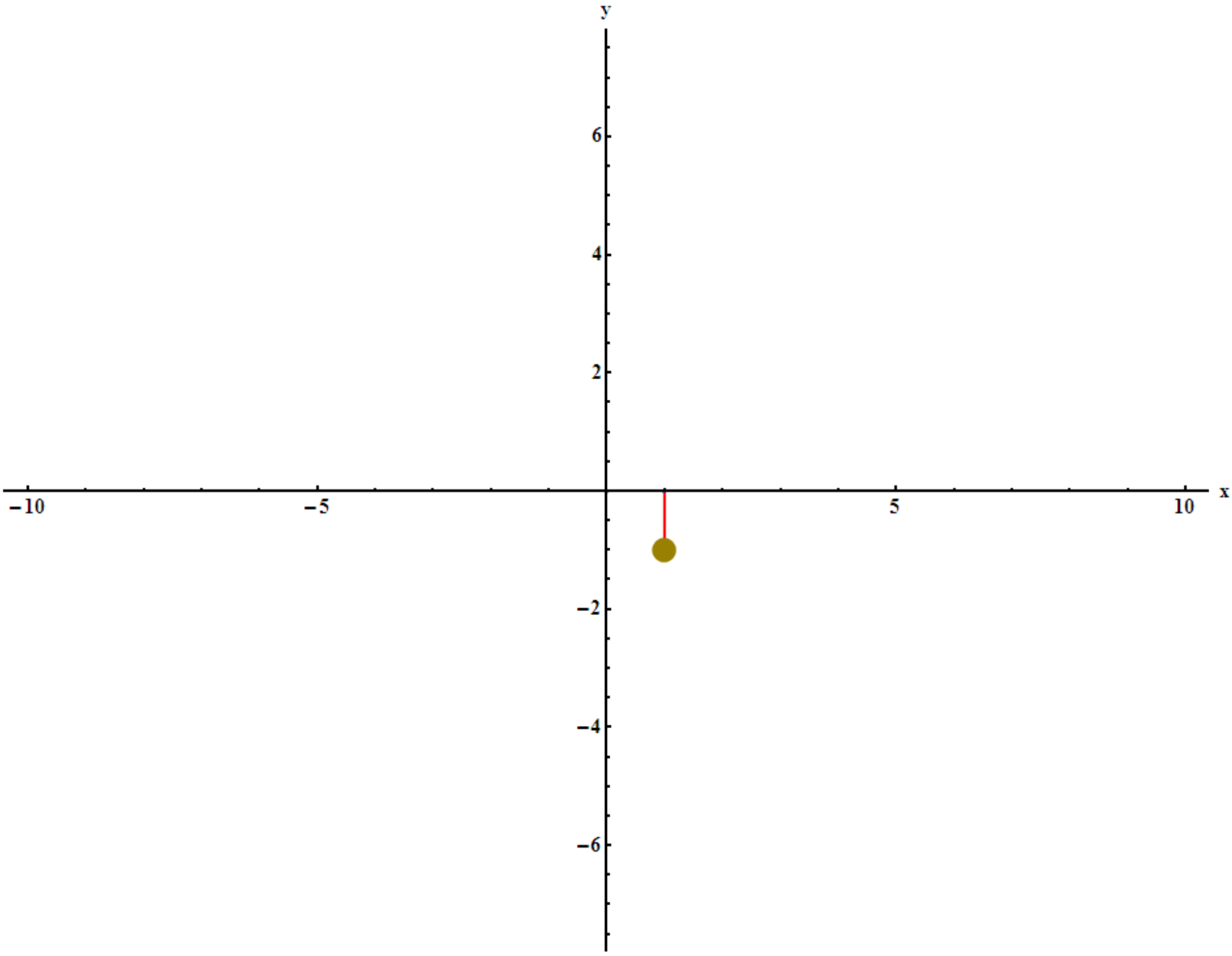
x


y
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−2
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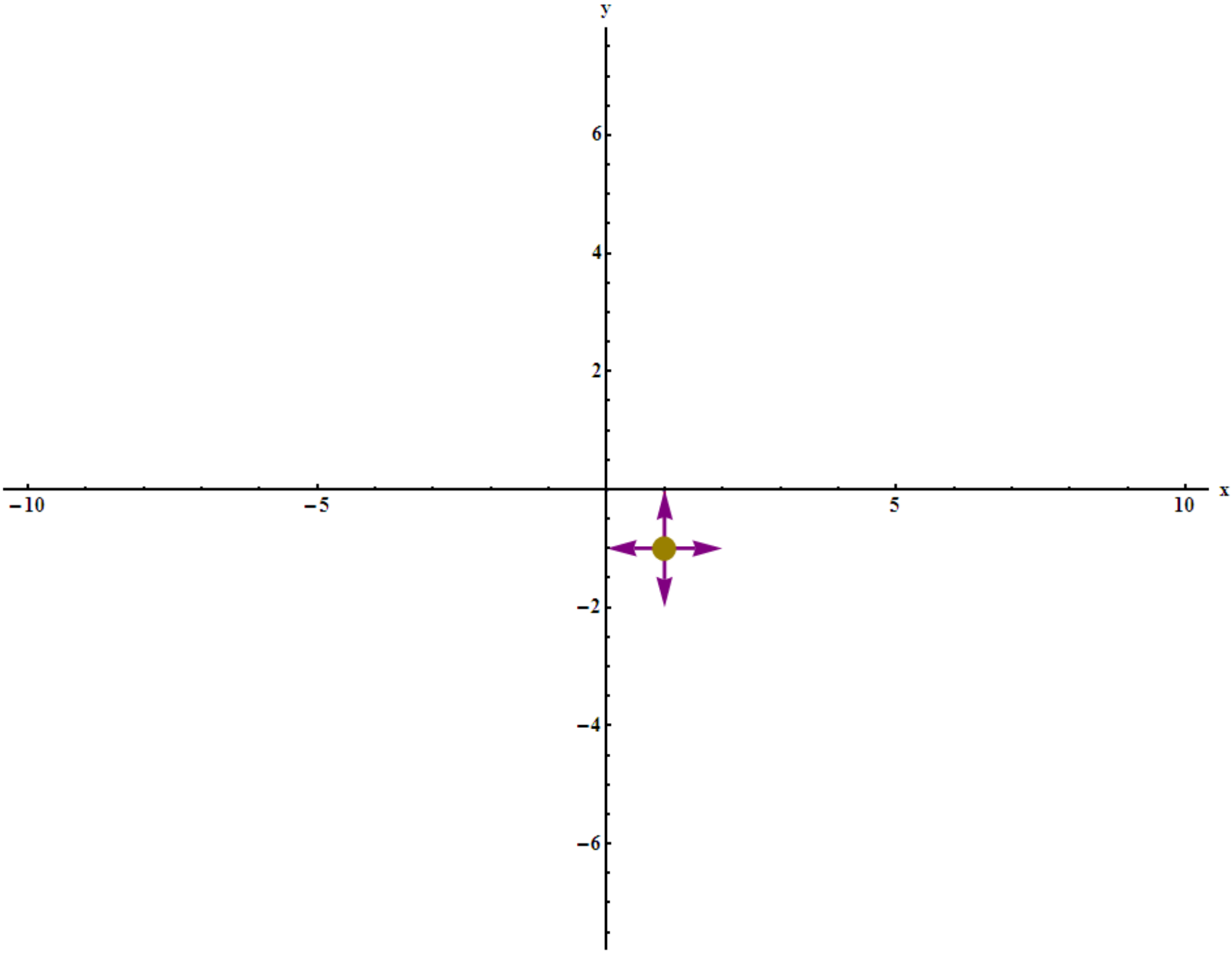
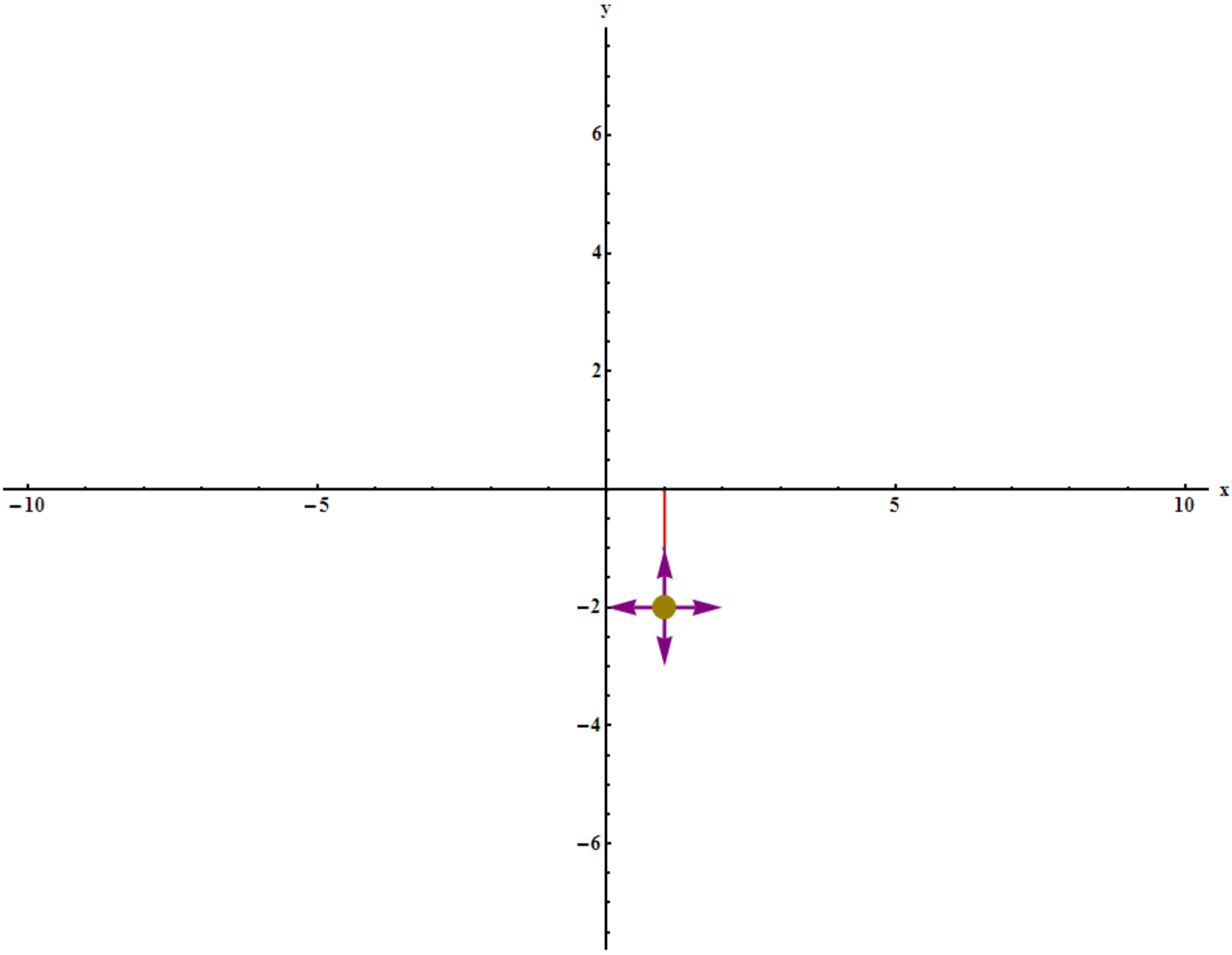
−10
−5
5
10
x


y
6
4
2
−10
−5
5
10
x
−2
−4
−6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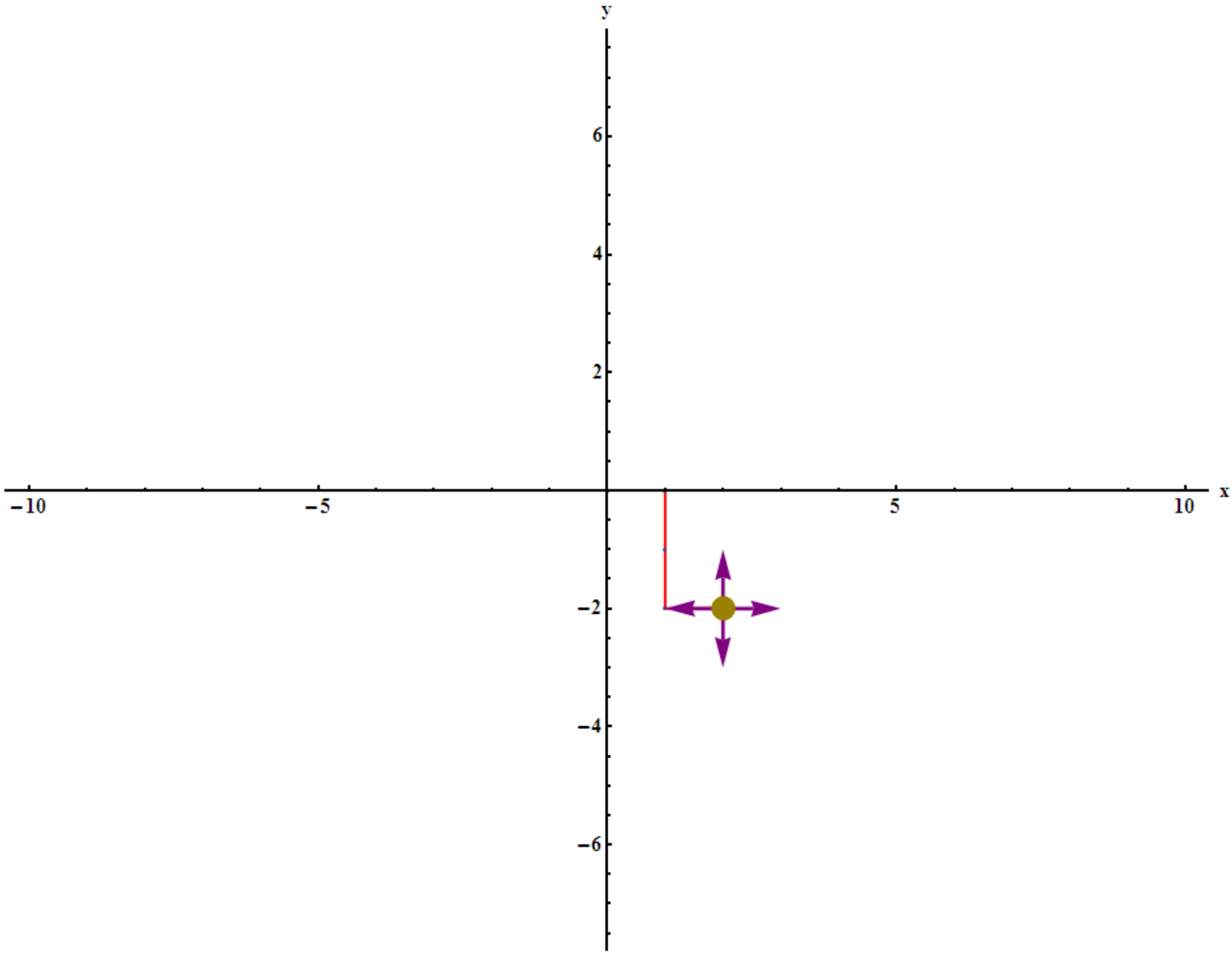
6
4
2
−10
−5
5
10
x
−2
−4
−6


y
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4
2
−10
−5
5
10
x
−2
−4
−6